LATTICE 2023 Fermilab

The Lüscher scattering formalism on the left-hand cut

CENTRE FOR THEORETICAL PHYSICS · Higgs
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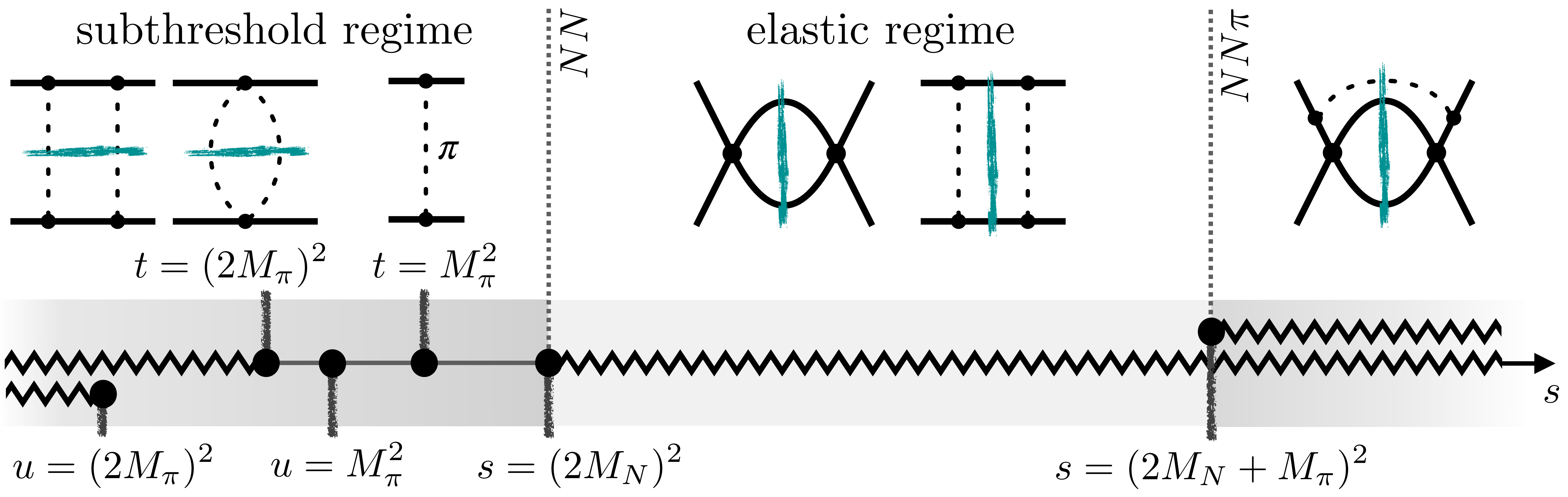
Motivation

- Lüscher formalism is widely used for extraction of 2-to-2 scattering amplitudes from finite-volume energies
- baryon-baryon scattering investigations have highlighted limitations of formalism in the presence of left-hand cuts [Green, Hanlon, Junarkar, Wittig 2021]

- left-hand cuts in the AM-projected scattering amplitudes associated to light meson exchanges
- also relevant for heavy flavor systems (e.g. DD* [Meng-Lin Du et al, 2023]).
- extension of existing formalism needed for resolving these issues



$NN \to NN$ amplitude at fixed CM scattering angle

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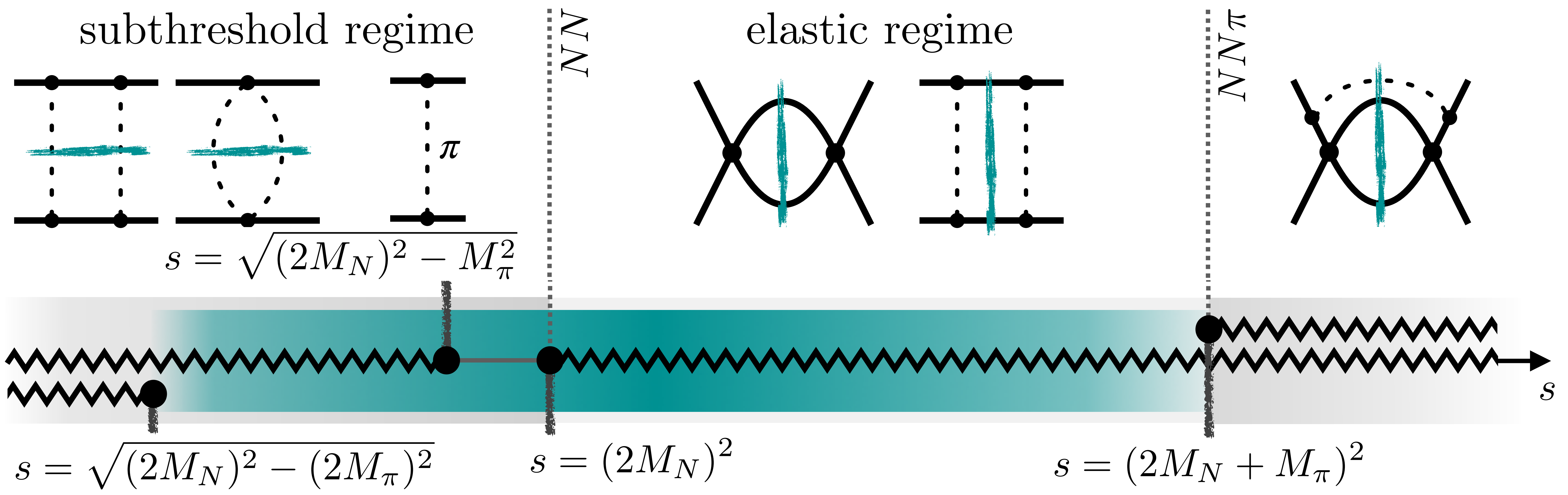
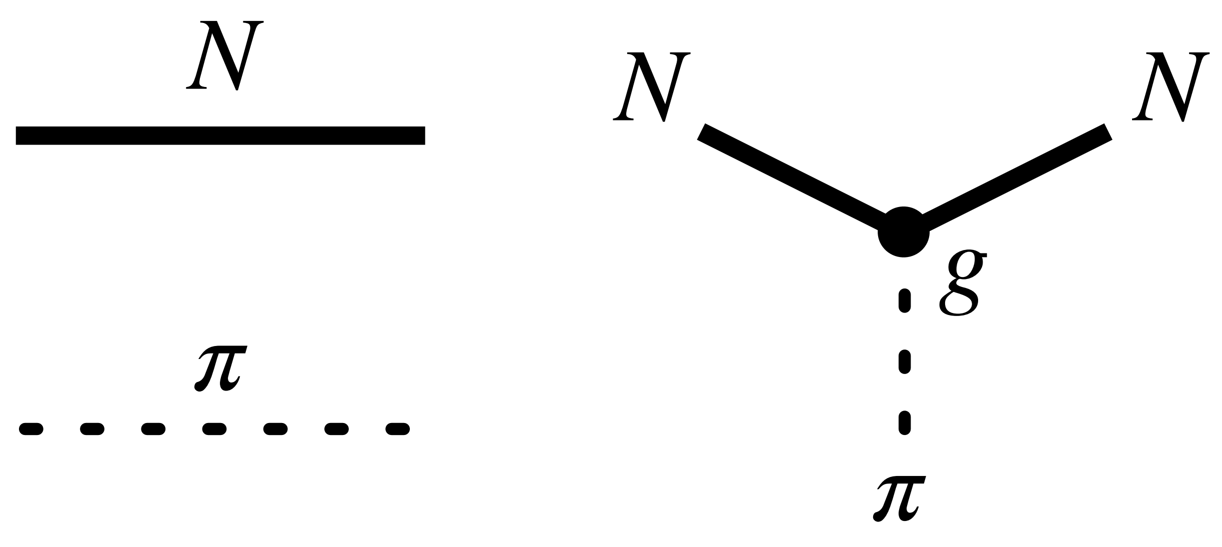


$NN \to NN$ angular-momentum-projected amplitude

Introductory details

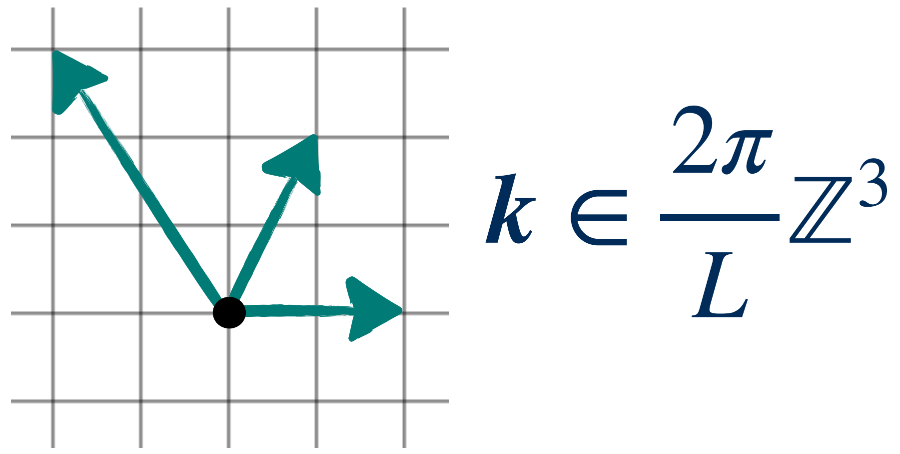
Theoretical setup:

- generic low-energy EFT with "nucleons" N and lighter "pions" π (masses M_N and M_π)
- N and π with arbitrary spins
- generic interactions, including $N\overline{N}\pi$ vertex with coupling g

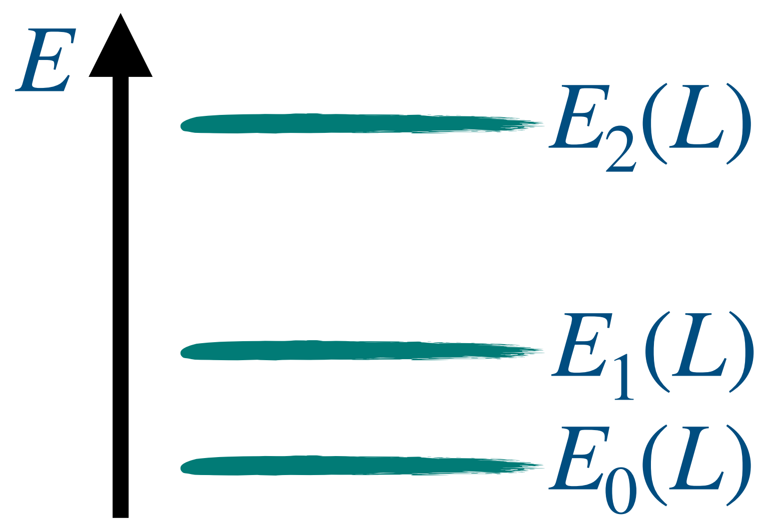


Finite volume setup:

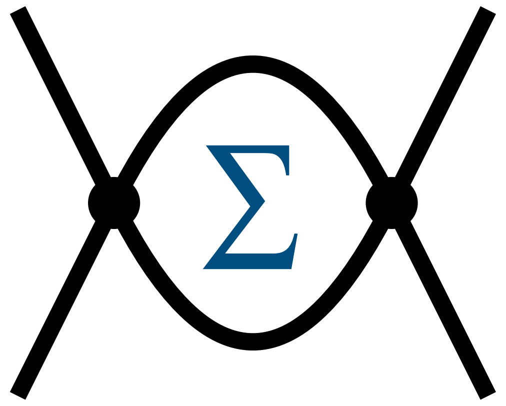
- periodic cubic spatial volume of side L, finite time extent T
- L large enough to neglect $\mathcal{O}\left(e^{-M_\pi L}\right)$ effects



discretized momenta



discretized spectrum

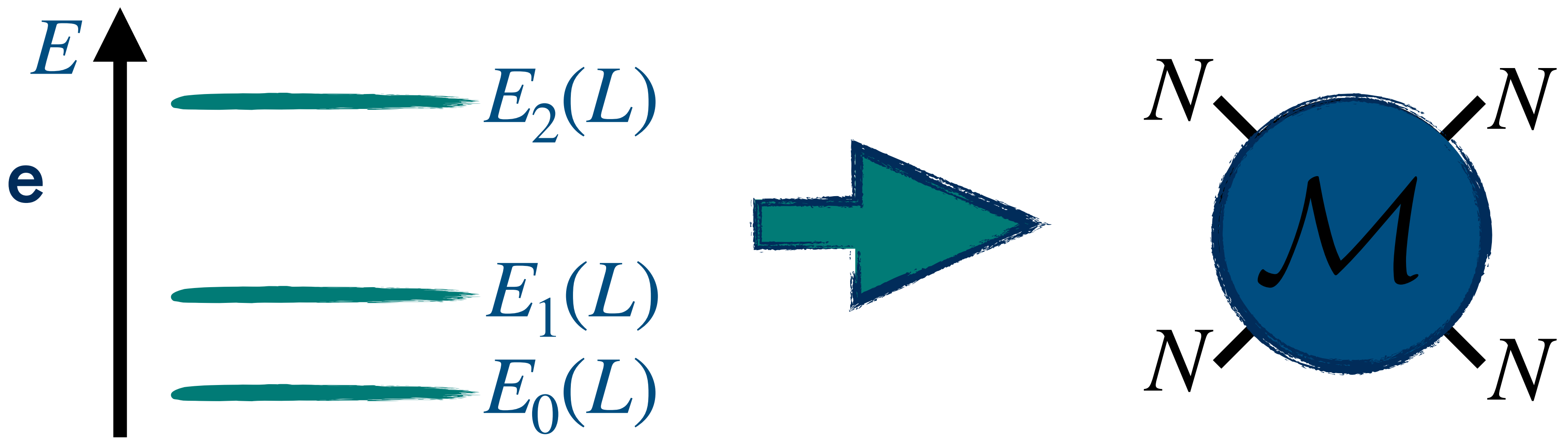


spatial loop
momenta summed

Finite-volume scattering formalism

What we want from a **scattering formalism:**

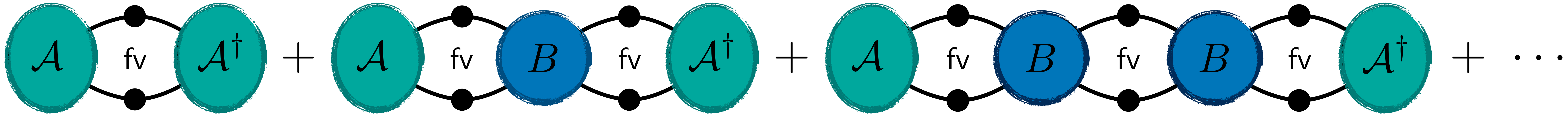
finite-volume spectrum



infinite-volume scattering amplitude

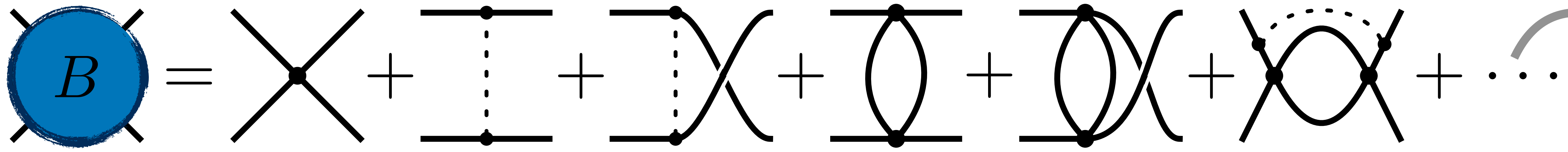
Consider **finite-volume correlator** – has poles at the finite-volume energy levels

$C_L(P) \ =$



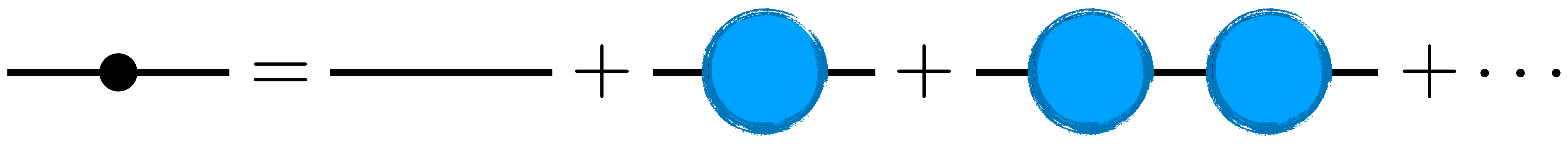
[Kim, Sachrajda, Sharpe 2005]

Bethe-Salpeter kernel



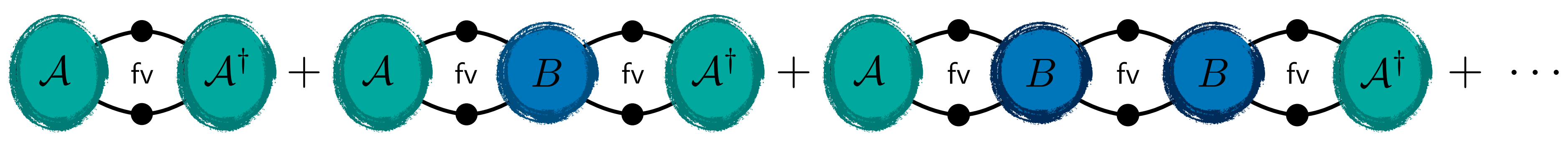
all other diagrams which are 2-particle irreducible in the s-channel

dressed N propagator

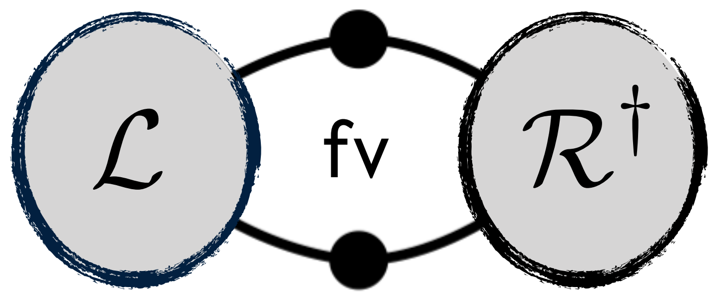


1-particle irreducible diagrams

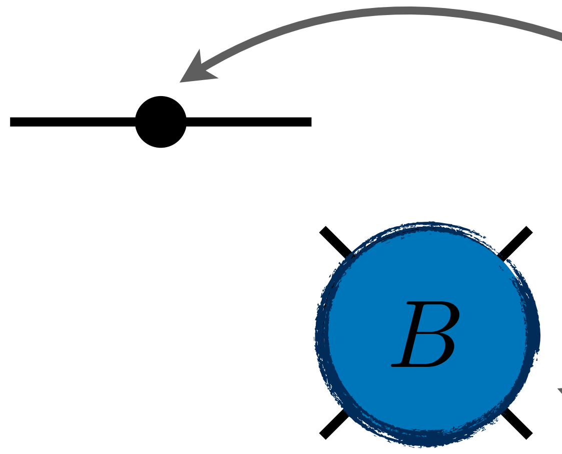
The standard derivation

$C_L(P) =$ 

Analysis of finite-volume loops in elastic regime $(2M_N)^2 < s < (2M_N + M_\pi)^2$:

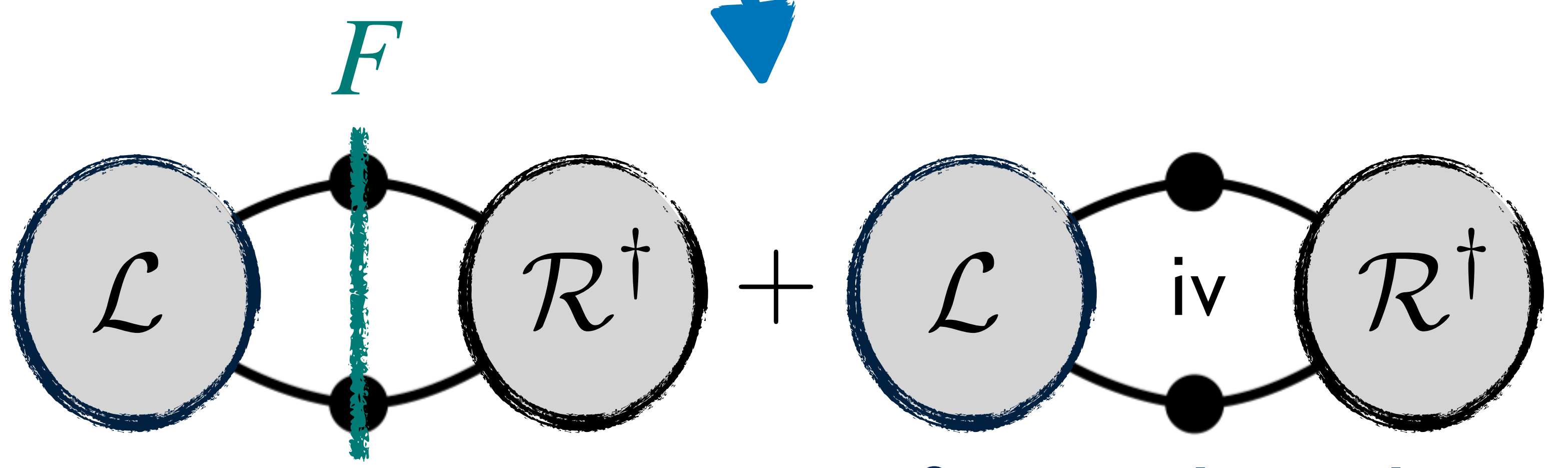


2-particle loops lead to $\mathcal{O}\left(L^{-n}\right)$ effects



other **fv** loops can be replaced by **iv** loops up to $\mathcal{O}\left(e^{-M_\pi L}\right)$ corrections

[Lüscher 1986]
[Kim, Sachrajda, Sharpe 2005]



infinite-volume loop

F cut $\sum_k - \int_k$

- intermediate two-particle (NN) state dominates
- left and right functions set to on-shell kinematics

infinite-volume correlator

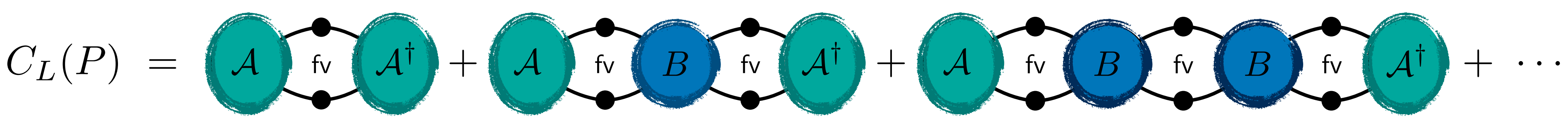
$$C_L(P) = C_\infty(P) + A(P) \frac{i}{F(P,L)^{-1} + \mathcal{K}(P)} A(P)^\dagger$$

matrix of known functions K-matrix operator "overlaps"

$$\det\left[F(P,L)^{-1} + \mathcal{K}(P)\right] = 0 \quad \text{at fv energy levels}$$

Lüscher quantization condition

The standard derivation



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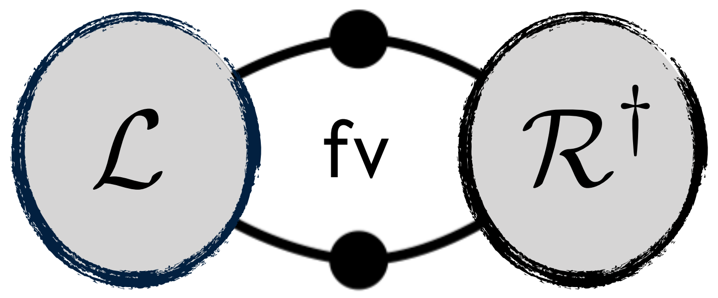

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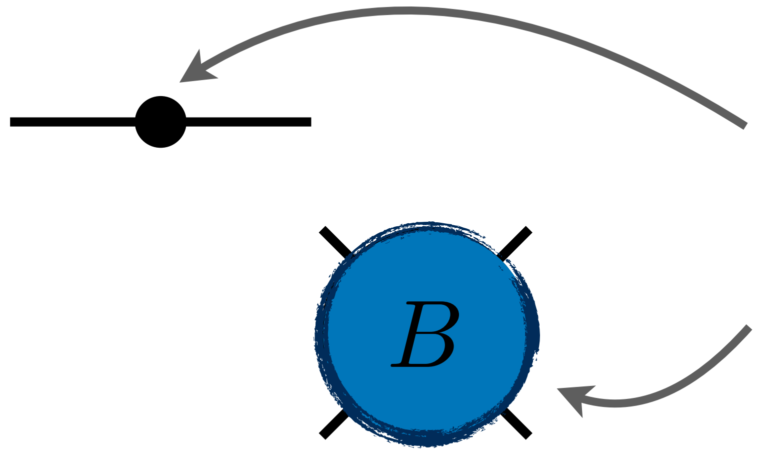

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[Lüscher 1986]
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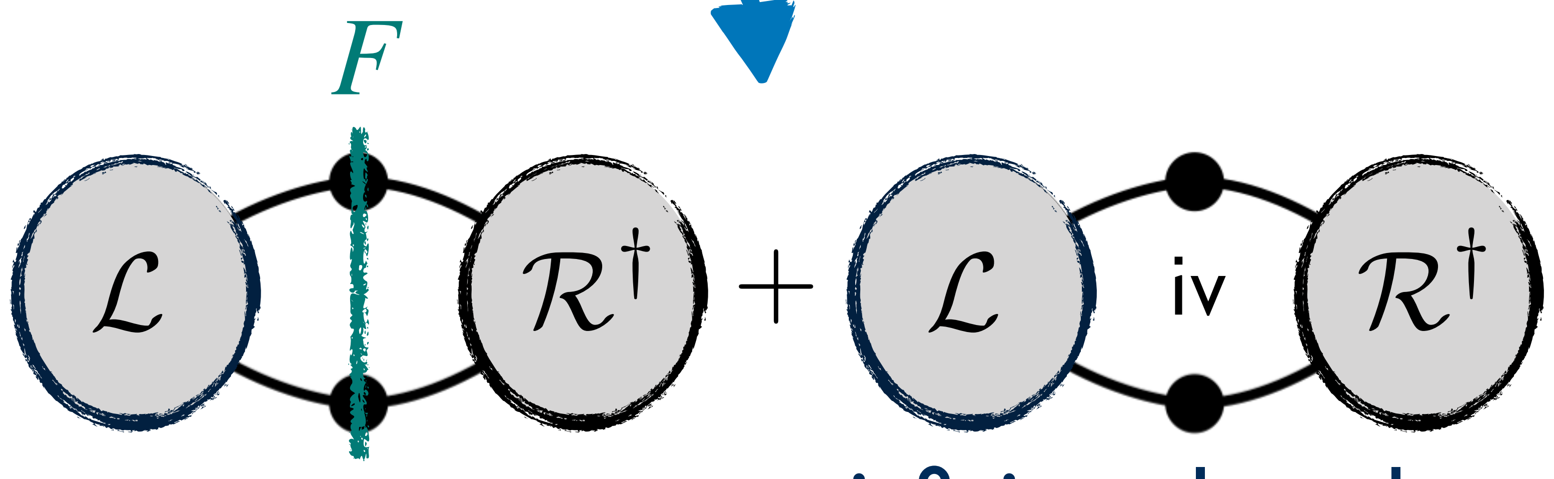

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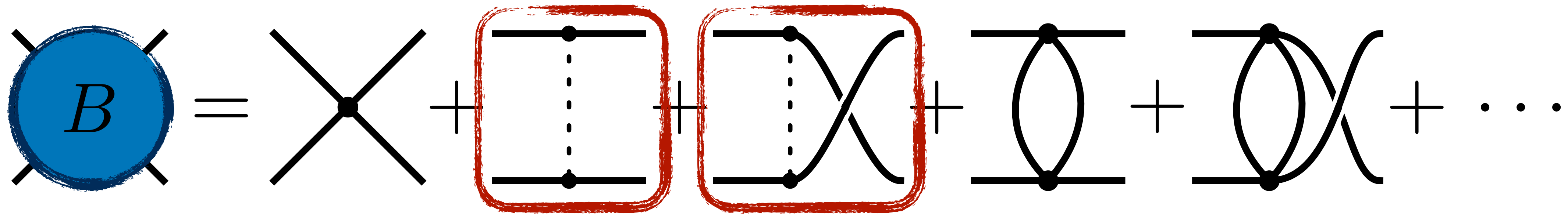
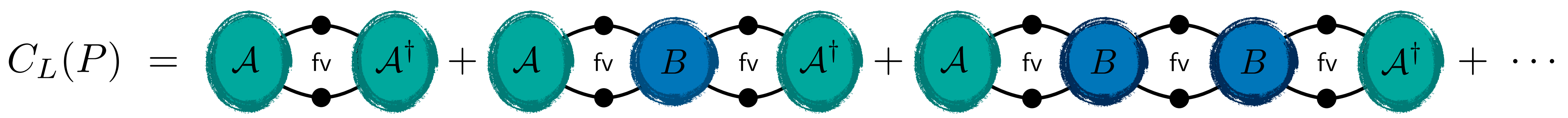
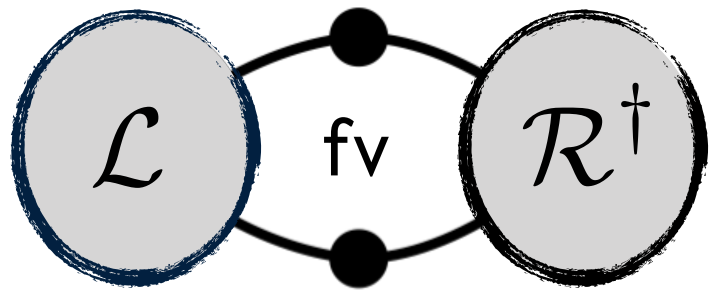


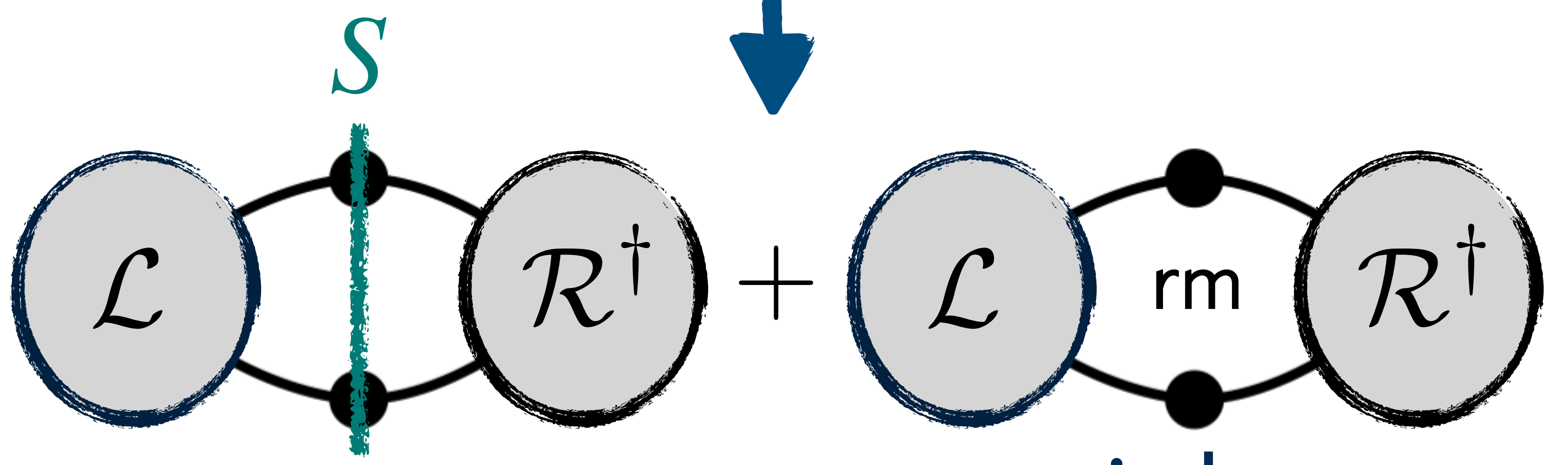
- placing BS kernels on shell introduces singularities and left-hand cuts below threshold – not present in the correlator
- cut near threshold arises from the π exchanges shown
- invalidates next steps in derivation

Proposed formalism



We propose the following instead:

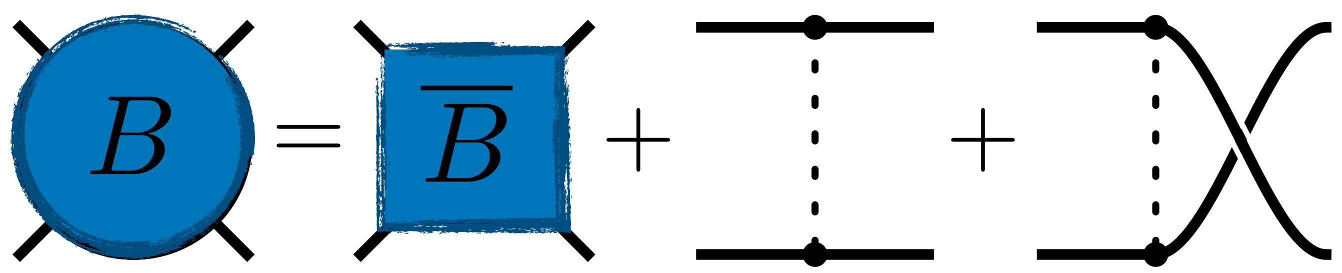




S cut:
keep the sum over momenta

remainder term

key step



split BS kernel: remove π exchanges

dangerous π exchanges are never put fully on shell, only the safe kernels $\overline{B}$

remainder — endcap vectors

$$C_L(P) = \mathcal{I}(P) + \tilde{A}(P)\xi \frac{i}{S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}(P)\xi + 2g^2 T(P)} \xi^\dagger \tilde{A}(P)^\dagger$$

$$\det\left[S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}(P)\xi + 2g^2 T(P)\right] = 0$$

at finite-volume energy levels

modified quantization condition

- quantities in QC live in angular momentum plus discrete spatial momentum index space: $\boldsymbol{k}^\star \ell m; \boldsymbol{k}^{\star\prime} \ell' m'$ with $\boldsymbol{k}, \boldsymbol{k}' \in 2\pi\mathbb{Z}^3/L$
- determinant taken over this full space
  (similarity to 3-particle RFT formalism [Hansen, Sharpe 2014])

Quantization condition

$$\det\left[S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}(P)\xi + 2g^2 T(P)\right] = 0$$

constrains the K-bar $\overline{\mathcal{K}}(P)$ (and coupling g) from the finite-volume spectrum

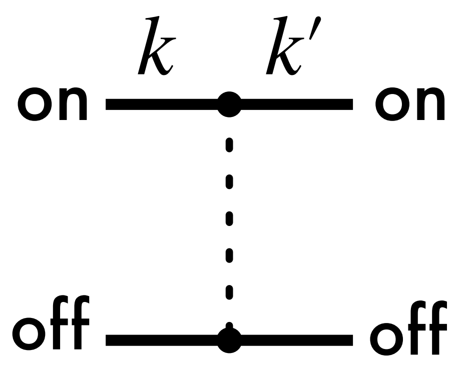
- **S-cut matrix:**

$$S_{\boldsymbol{k}^\star \ell m, \boldsymbol{k}'^\star \ell' m'}(P,L) = \frac{1}{2L^3} \frac{4\pi\, Y_{\ell m}(\hat{\boldsymbol{k}}^\star)\, Y^*_{\ell' m'}(\hat{\boldsymbol{k}}^\star)\, \delta_{\boldsymbol{k}^\star \boldsymbol{k}'^\star}\, |\boldsymbol{k}^\star|^{\ell+\ell'}\, H(\boldsymbol{k}^\star)}{4\omega_N(\boldsymbol{k})\left[(k^\star_{\text{os}})^2 - (\boldsymbol{k}^\star)^2\right]}$$

regulator function

on-shell CM momentum magnitude $(k^\star_{\text{os}})^2 = s/4 - M_N^2$

- **T matrix:** partial wave projections of partially off-shell t-channel diagram



e.g. *S-wave:*

$$\mathcal{T}^{\ell=0}_{\boldsymbol{k}^\star, \boldsymbol{k}^{\star\prime}}(P) = \frac{1}{2|\boldsymbol{k}^\star||\boldsymbol{k}'^\star|} \log\left(\frac{2\omega_N(\boldsymbol{k}^\star)\omega_N(\boldsymbol{k}'^\star) + 2|\boldsymbol{k}^\star||\boldsymbol{k}'^\star| - 2M_N^2 + M_\pi^2 - i\epsilon}{2\omega_N(\boldsymbol{k}^\star)\omega_N(\boldsymbol{k}'^\star) - 2|\boldsymbol{k}^\star||\boldsymbol{k}'^\star| - 2M_N^2 + M_\pi^2 - i\epsilon}\right)$$

$(\omega(\boldsymbol{k}) = \sqrt{\boldsymbol{k}^2 + M_N^2})$

- **Trivial projectors ξ, $\xi^\dagger$:** $\xi_{\boldsymbol{k}^\star} = 1$

- **$\overline{\mathcal{K}}(P)$ matrix:** matrix in AM index space, projections of a Lorentz scalar $\overline{\mathcal{K}}(s)$

Particles with nonzero spin taken into account by incorporating spin state indices into the above quantities.

Workflow to obtain amplitude

An extra step is needed to connect K-bar to the amplitude:

We need to solve integral equations of the type

$$\mathcal{M}^{\text{aux}}(P,p,p') = \overline{\mathcal{K}}(P,p,p') + 2g^2\mathcal{T}(P,p,p') - \frac{1}{2}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{H(\boldsymbol{k}^\star)\mathcal{M}^{\text{aux}}(P,p,k)\big(\overline{\mathcal{K}}(P,k,p') + 2g^2\mathcal{T}(P,k,p')\big)}{4\omega(\boldsymbol{k})\left[(k^\star_{\text{os}})^2 - (\boldsymbol{k}^\star)^2 + i\epsilon\right]}$$

solve for auxiliary amplitude

$$\mathcal{M}(P,p,p') = \frac{1}{2}\left[\mathcal{M}^{\text{aux}}(P,p,p') + \mathcal{M}^{\text{aux}}(P,p,P-p')\right]$$

symmetrize to get amplitude (bosons)

can profit from work done on solving integral equations resulting from the three-particle RFT formalism

[Romero-Lopez et al. 2019], [Jackura et al. 2021], [Dawid et al. 2023]

Equivalence to the standard formalism

Explicit equivalence to usual Lüscher method in the $g=0$ case:

$$C_L(P) = \mathcal{I}(P) + \sum_{n=0}^{\infty} \tilde{A}(P)\xi\, iS(P,L)\left[(\xi^\dagger\, i\overline{\mathcal{K}}(P)\,\xi + \cancel{2ig^2}T)\, iS(P,L)\right]^n \xi^\dagger \tilde{A}(P)^\dagger$$

$$\sum_{n=0}^{\infty} \tilde{A}(P)\xi\, iS(P,L)\left[\xi^\dagger\, i\overline{\mathcal{K}}(P)\,\xi\, iS(P,L)\right]^n \xi^\dagger \tilde{A}(P)^\dagger \quad \xrightarrow{\cdots} \quad \det\left[S(P,L)^{-1} + \xi^\dagger \overline{\mathcal{K}}(P)\xi\right] = 0$$

[objects in $\boldsymbol{k}^\star \ell m; \boldsymbol{k}^{\star\prime} \ell' m'$ space]

$$\sum_{n=0}^{\infty} \tilde{A}(P)\xi\, iS(P,L)\xi^\dagger\left[i\overline{\mathcal{K}}(P)\,\xi\, iS(P,L)\xi^\dagger\right]^n \tilde{A}(P)^\dagger$$

[objects in $\ell m; \ell' m'$ space]

use: $\xi\, iS(P,L)\xi^\dagger = F(P,L) + I(P)$

Lüscher F — integral term

$$\longrightarrow \sum_{n=0}^{\infty} A(P)\, iF(P,L)\left[i\mathcal{K}(P)\, iF(P,L)\right]^n A(P)^\dagger \quad \xrightarrow{\cdots} \quad \det\left[F(P,L)^{-1} + \mathcal{K}(P)\right] = 0$$

where we use the relation: $\mathcal{K}(P)^{-1} = \overline{\mathcal{K}}(P)^{-1} + I(P)$

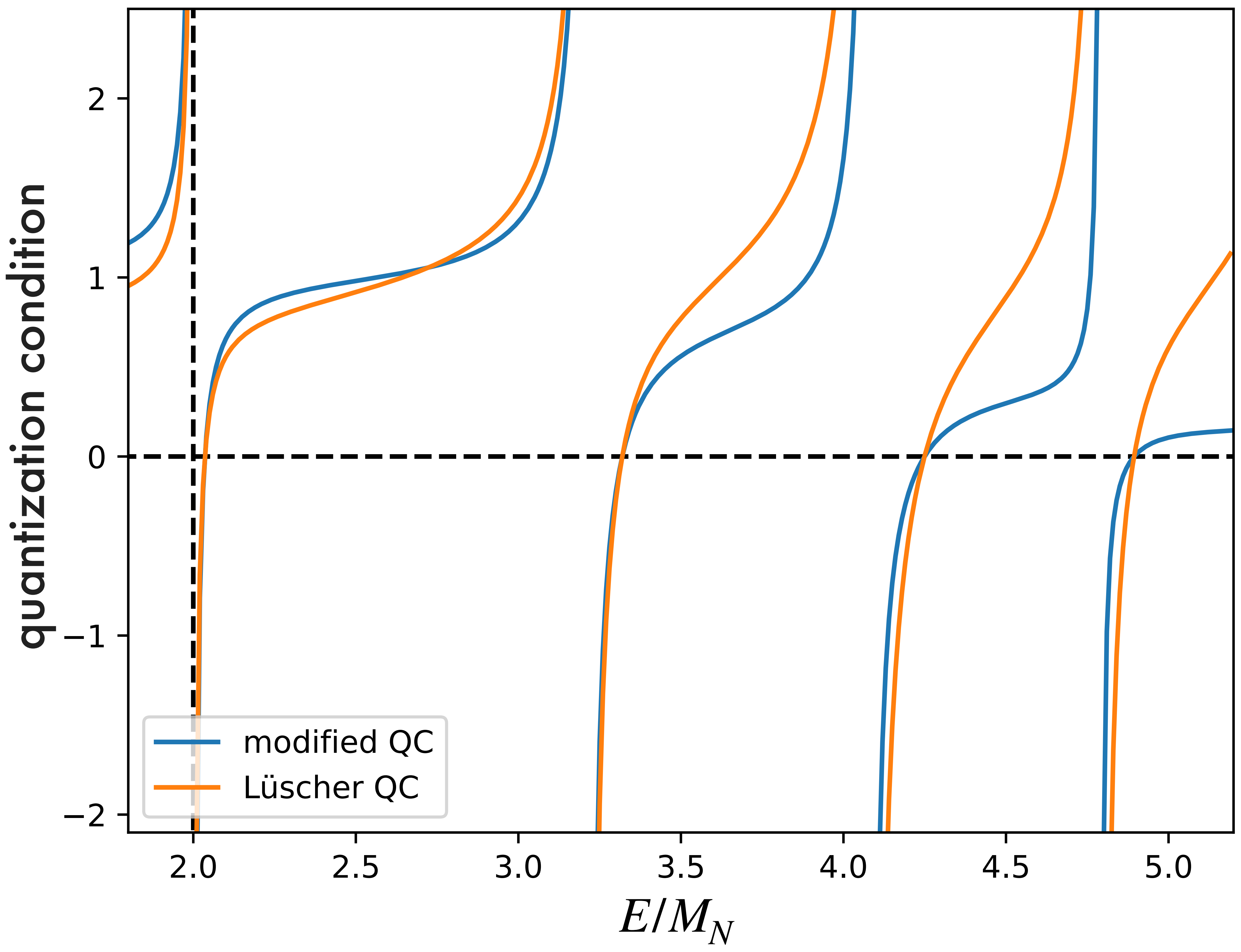
(can be shown using integral equations linking the standard K-matrix and K-bar)

Equivalence to the standard formalism

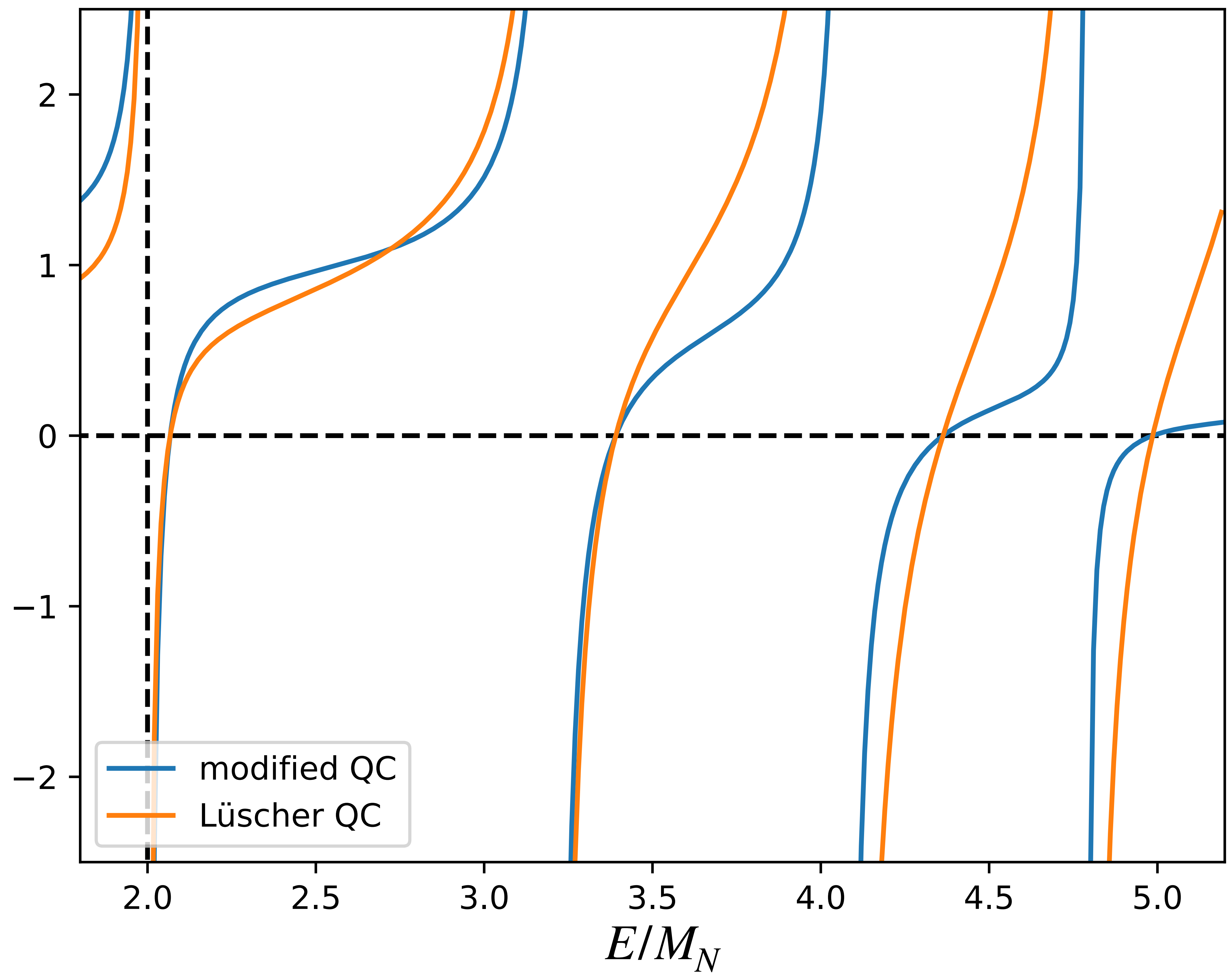
Plotting the modified and Lüscher QCs in S-wave in the $g = 0$ case with K-matrix:

$$\mathcal{K}(s) = 16\pi\sqrt{s}\frac{1}{p\cot\delta}\,, \qquad p\cot\delta = -\frac{1}{a} + \frac{1}{2}rp^2 + \mathcal{O}(p^4)$$

$a = 0.3/M_N\,,\ r = 0$



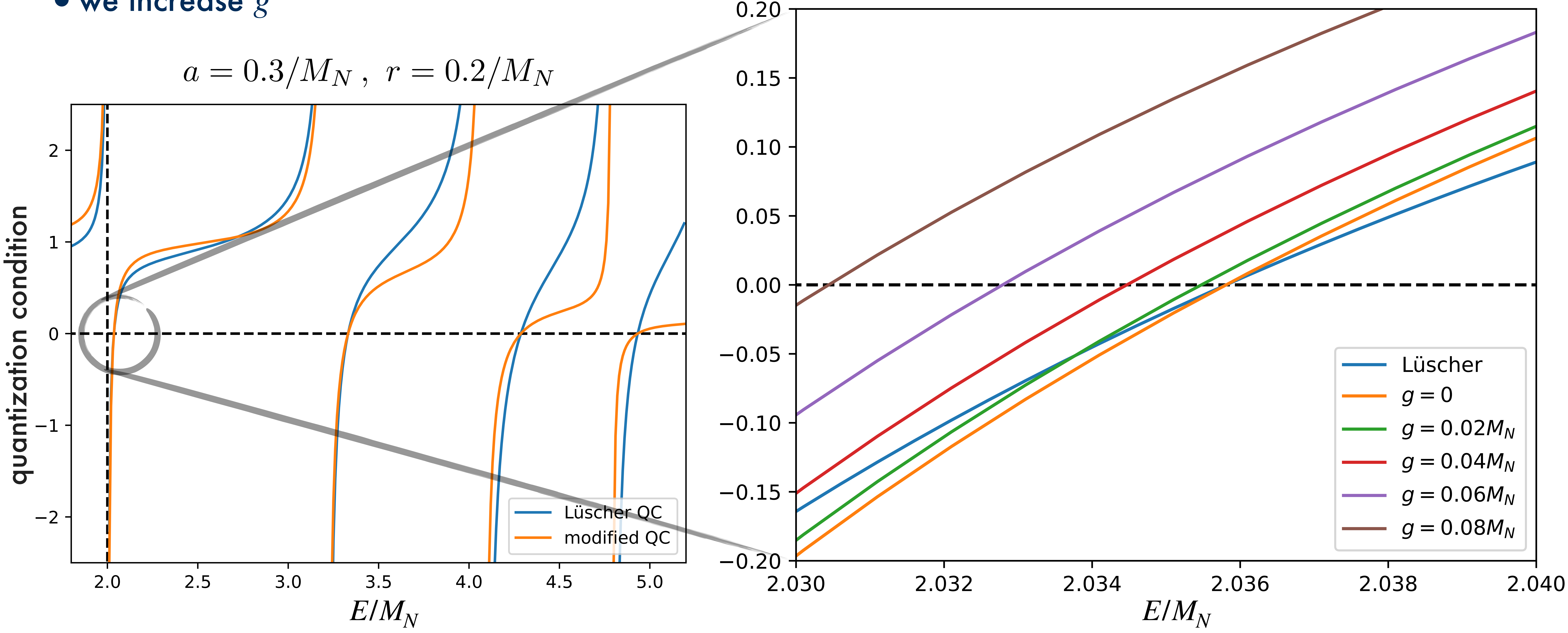
$a = 0.5/M_N\,,\ r = 0.1/M_N$



Turning on the g coupling

- K-matrix is fixed ERE plus t-channel pole
- K-bar matrix is found from K-matrix by solving integral equation
- we increase g

$$a = 0.3/M_N\,,\ r = 0.2/M_N$$



Conclusions & Outlook

- We have presented a method that extends the Lüscher formalism to the left-hand cut, now accounting also for spin and both t- and u-channels
- Full workflow including the solving of integral equations allows extraction of the amplitude
- Modified procedure has been shown to be equivalent to standard Lüscher method when the latter is applicable
- Paper with full details will be on the arXiv very soon!

- Extensions of formalism (e.g. non-identical particles, different masses, lower energy range) currently being investigated
- Taking advantage of progress in solving integral equations in the three-particle RFT formalism
- Clarifying and exploring connections and consistency to three-particle formalism (e.g. this method as a limiting case?)

Breakdown of Lüscher Formalism near Left Hand Cuts — *Md Habib E Islam*
Curia II, WH2SW — 13:50 - 14:10

Resolving the left-hand-cut problem in lattice studies of the doubly-charmed tetraquark — *Steve Sharpe*
Curia II, WH2SW — 14:10 - 14:30

the left-hand cut
from the perspective of the
three-particle formalism

Thank you for your attention!

Any questions?