

Breakdown of Lüscher Formalism near Left Hand Cuts

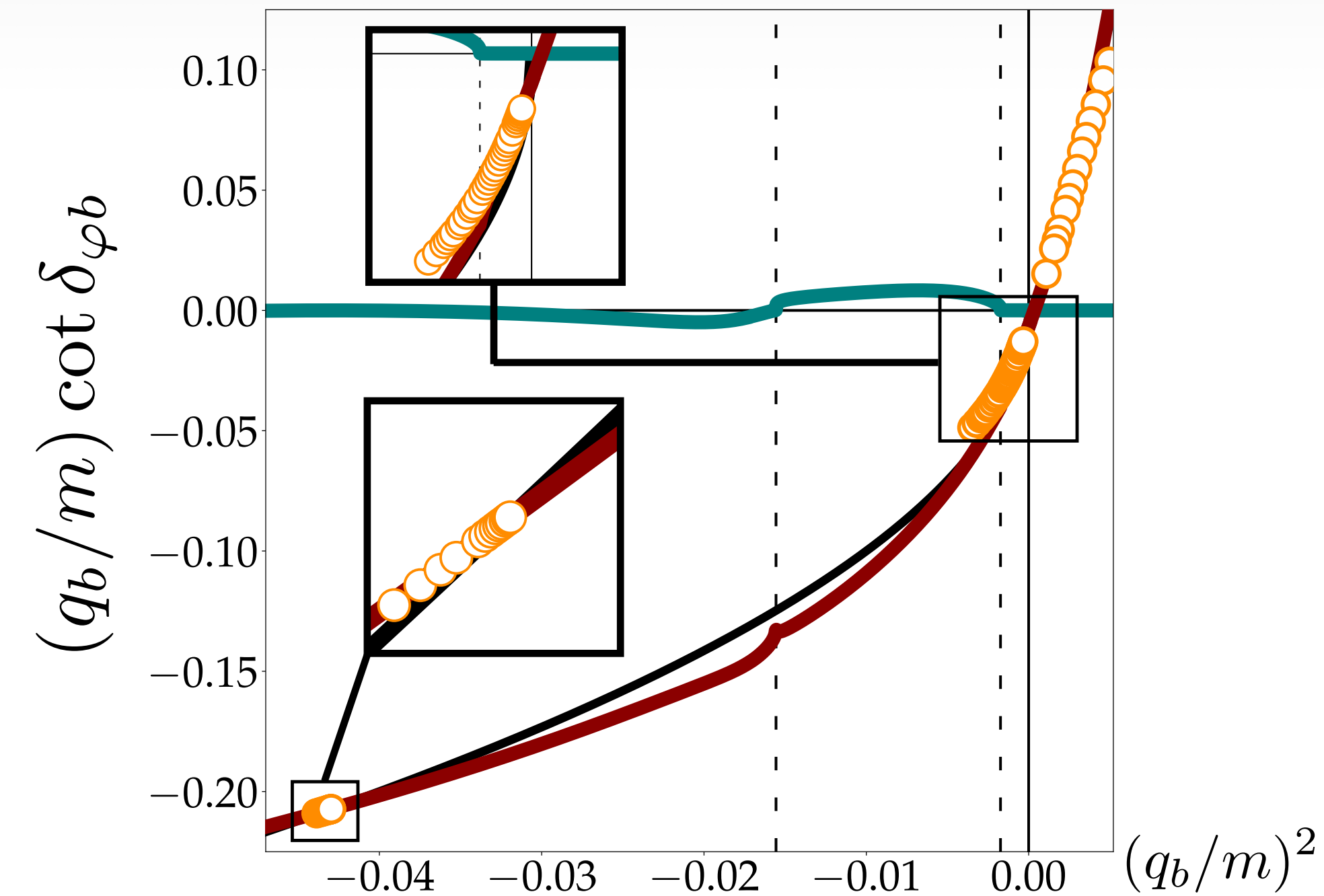
Presented by : Md Habib E Islam (Digonto)

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With Sebastian M. Dawid (U Washington)

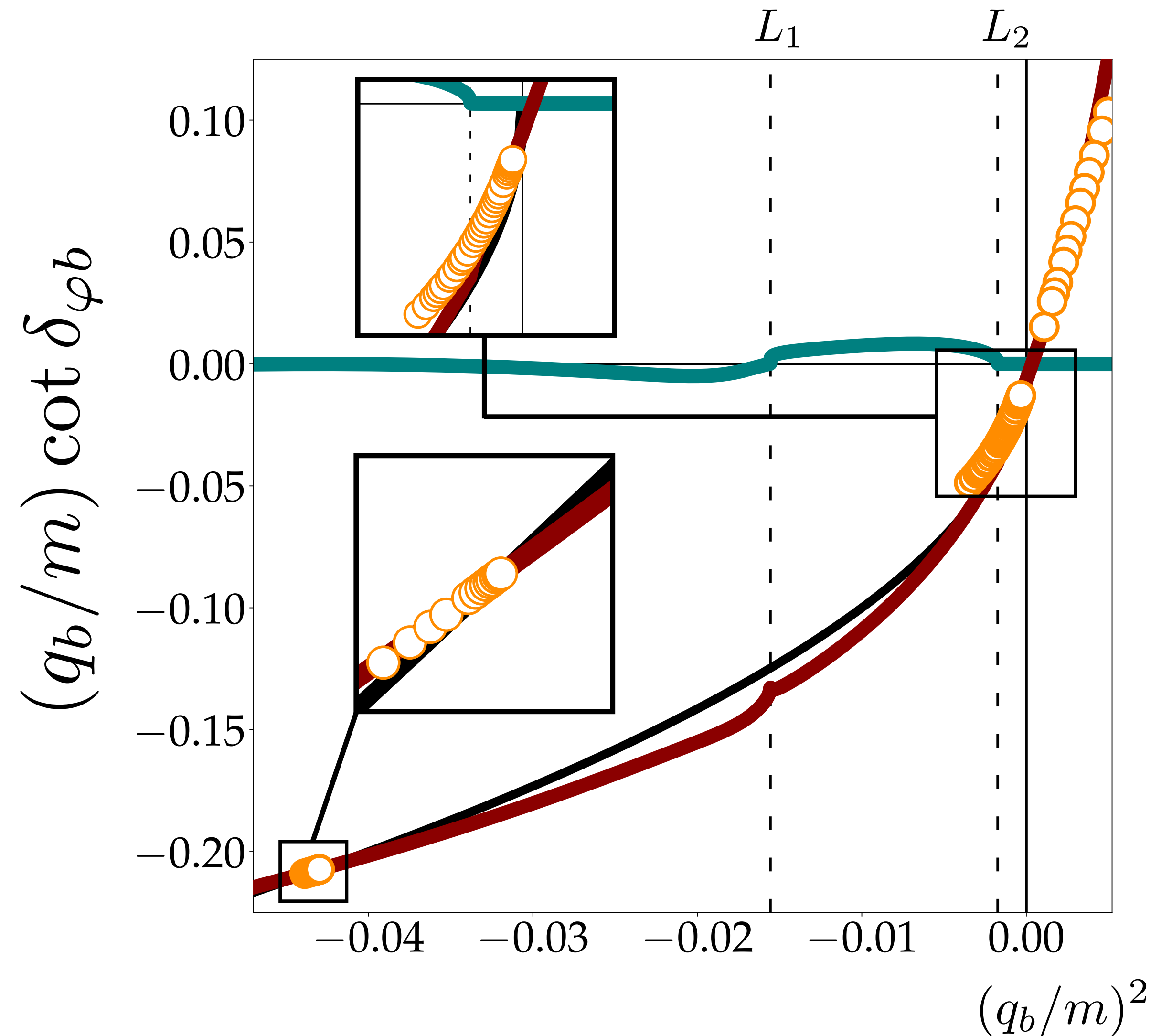
and

Raúl Briceño (UC Berkeley)



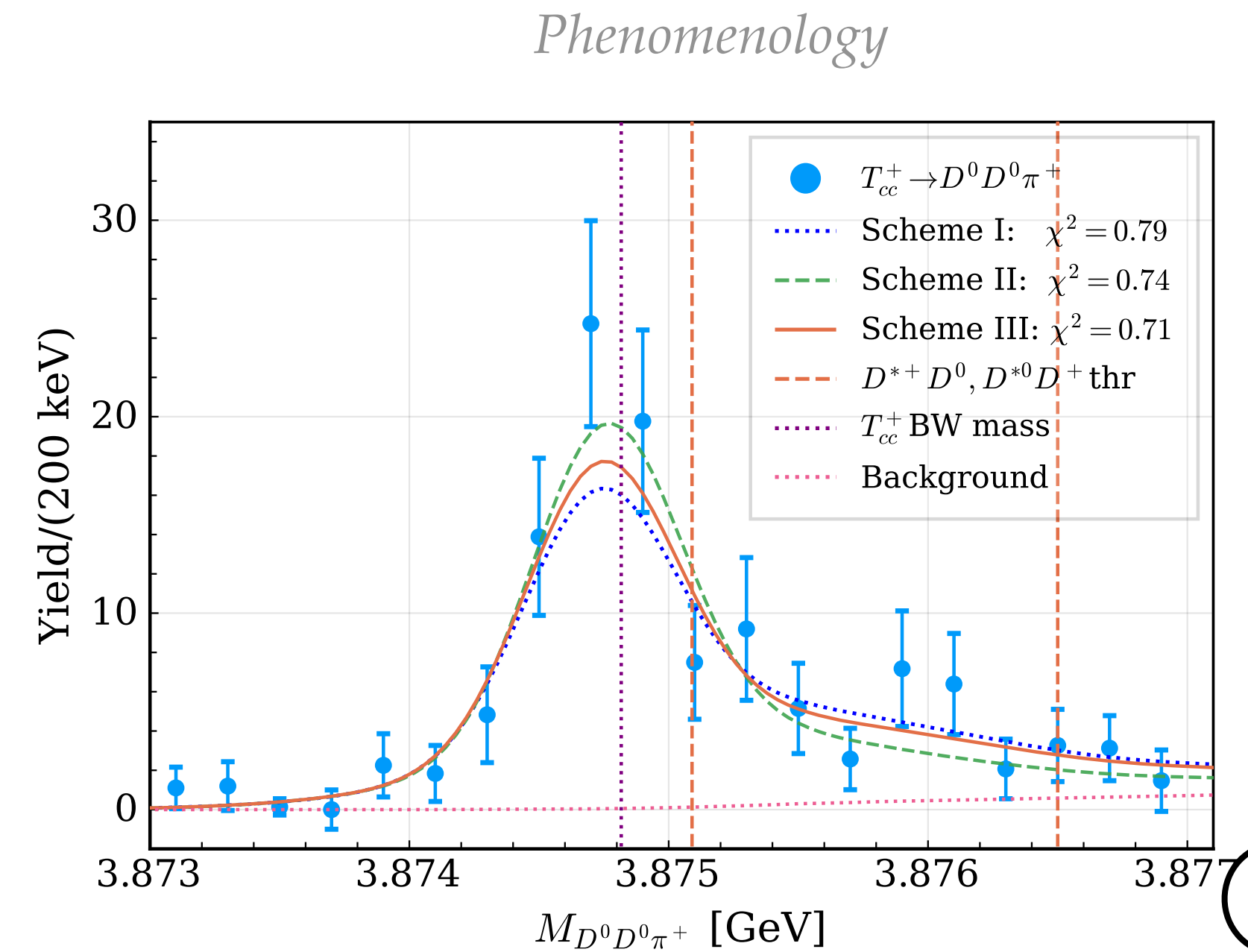
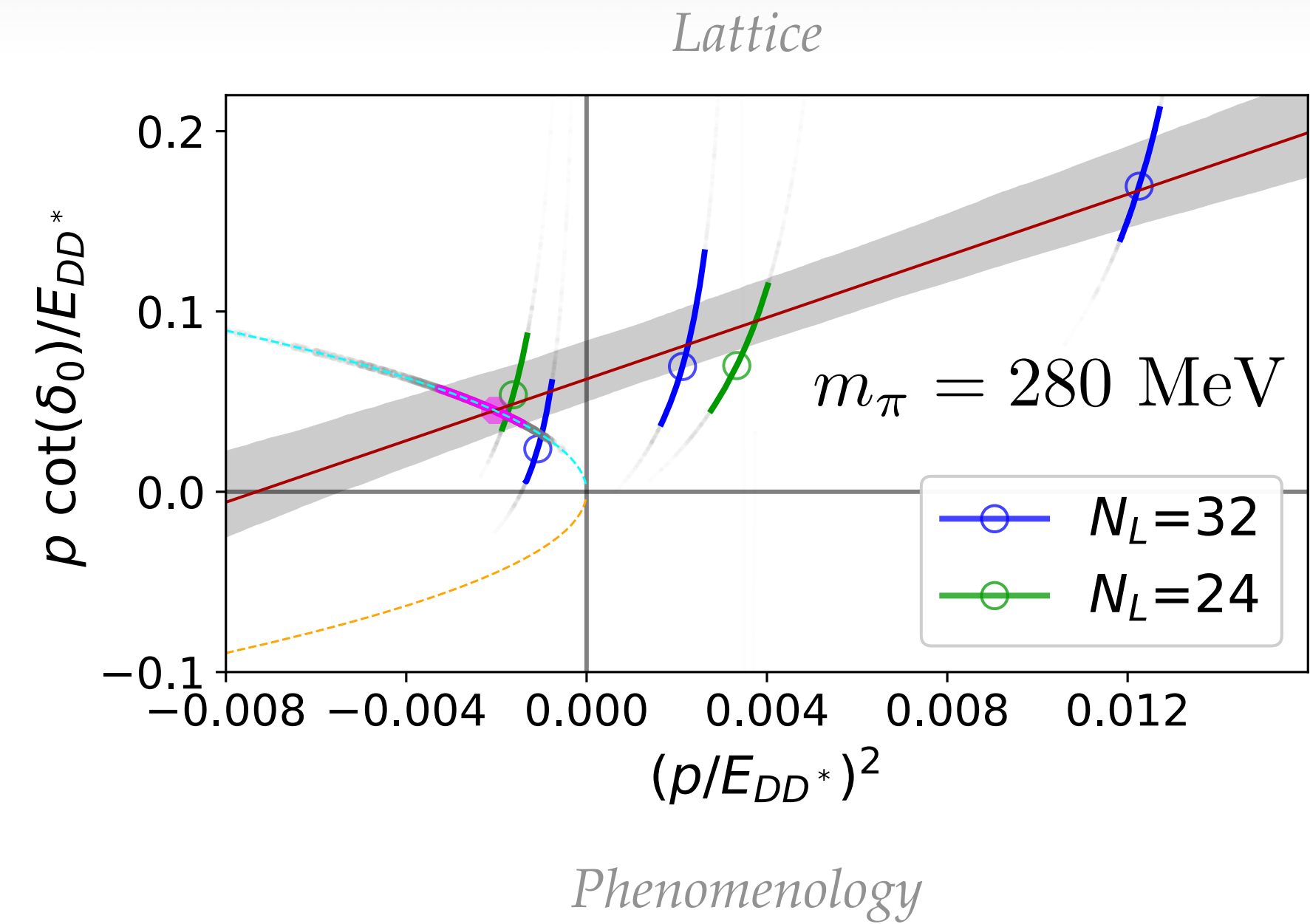
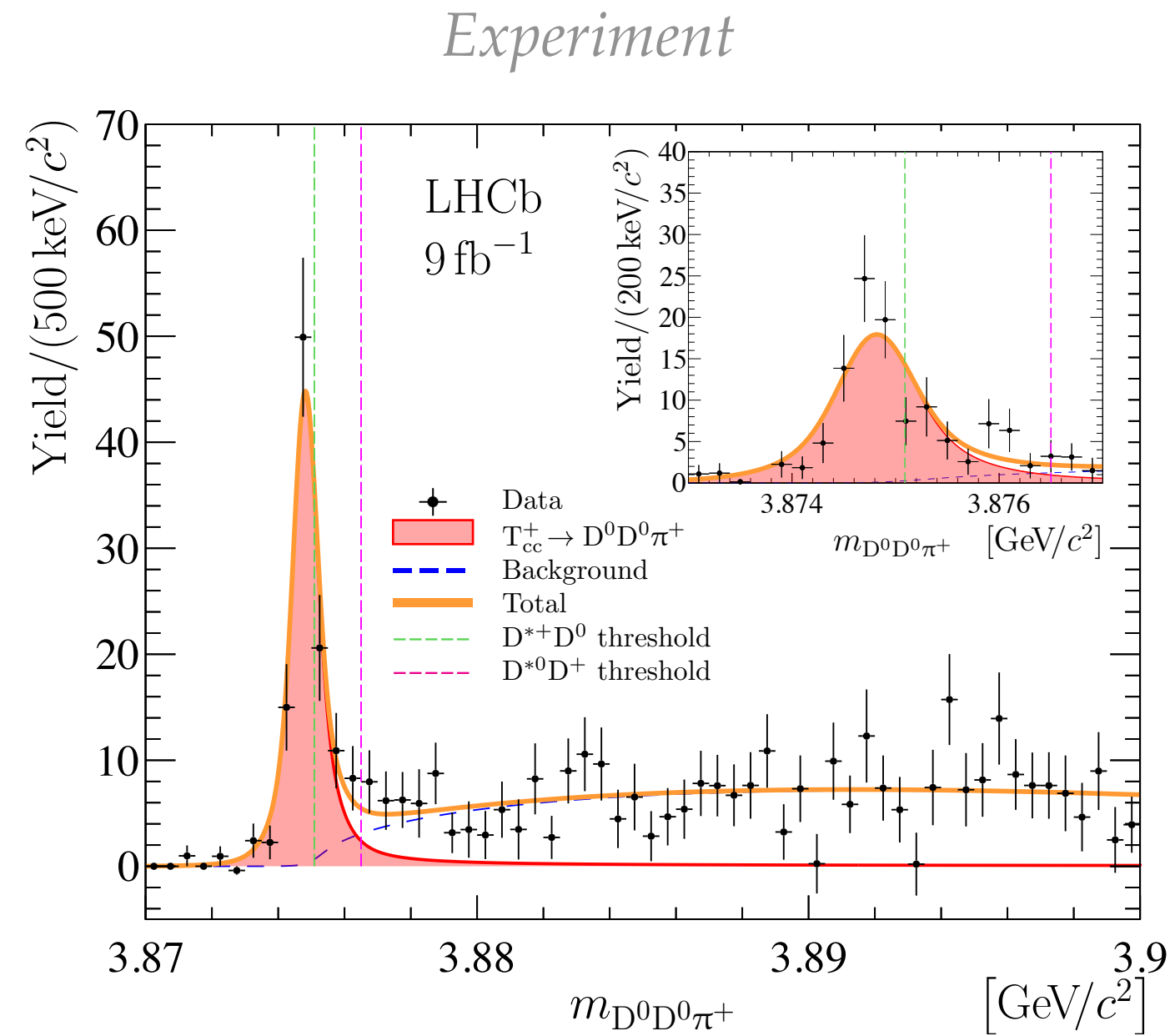
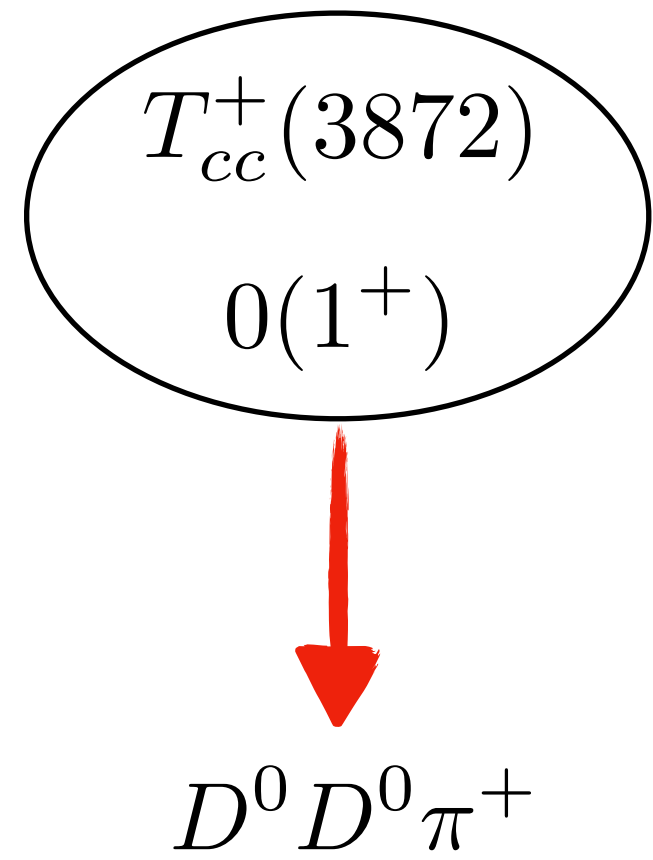
Outline

- Why **three body physics** is important?
- What toy model are we using?
- What is the left hand cut?
- How were the **orange points** derived from Lüscher formalism?
- How were the **solid lines** derived from solving Integral equations?

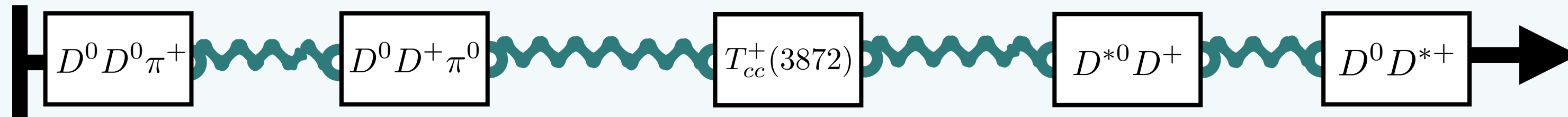


Why Three Body Physics?

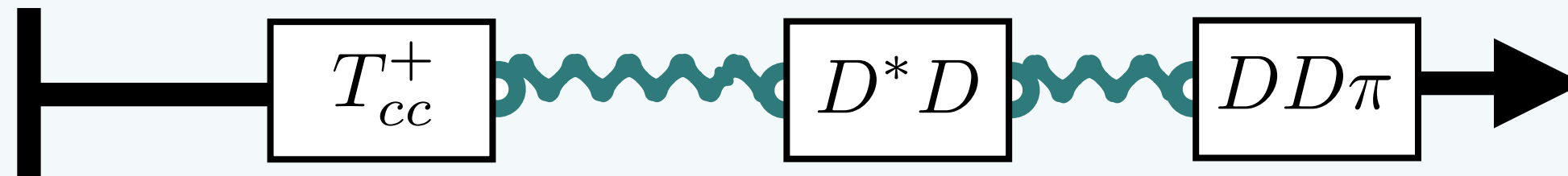
Interesting resonances decay to three particle final states under strong interactions



Physical Threshold

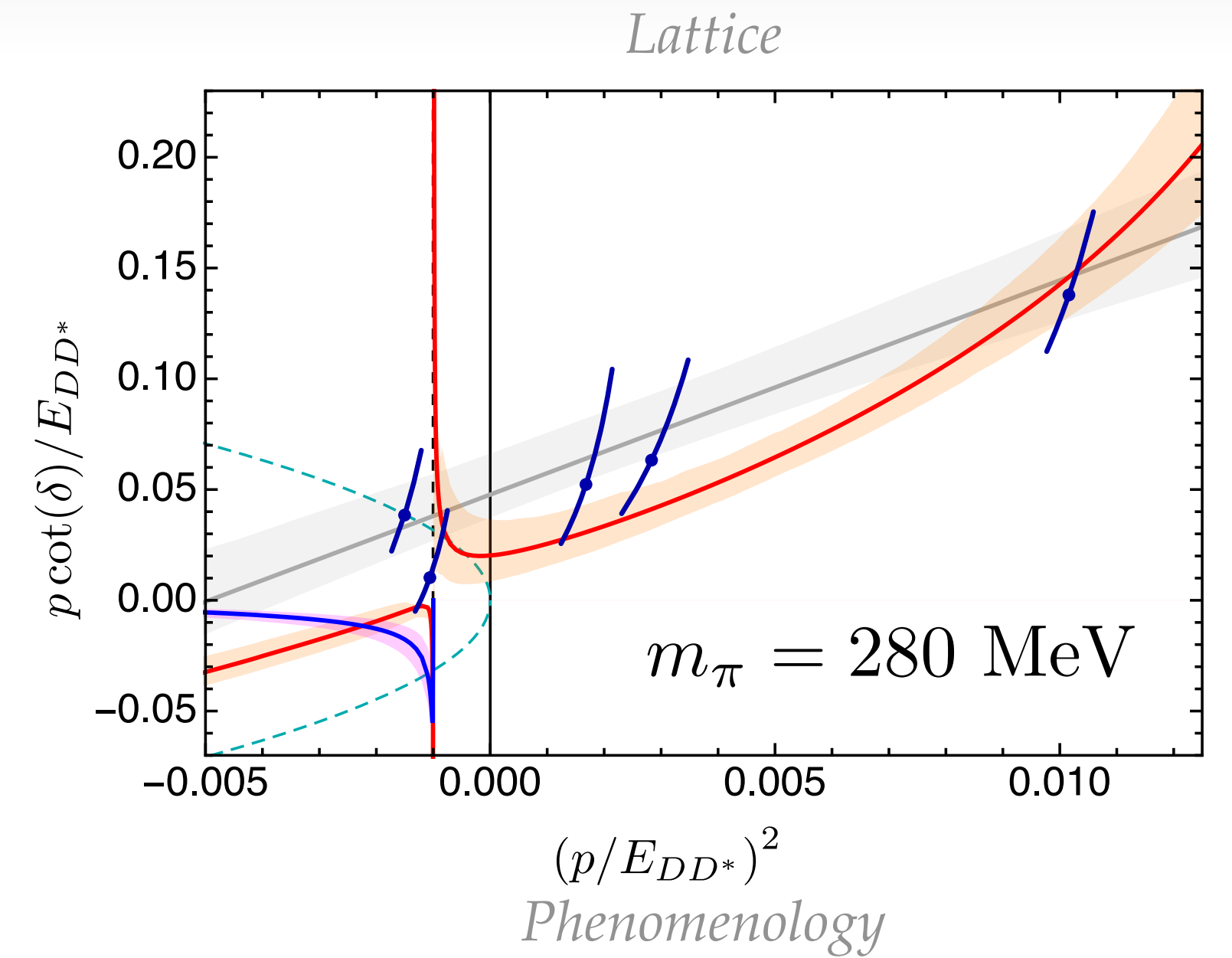
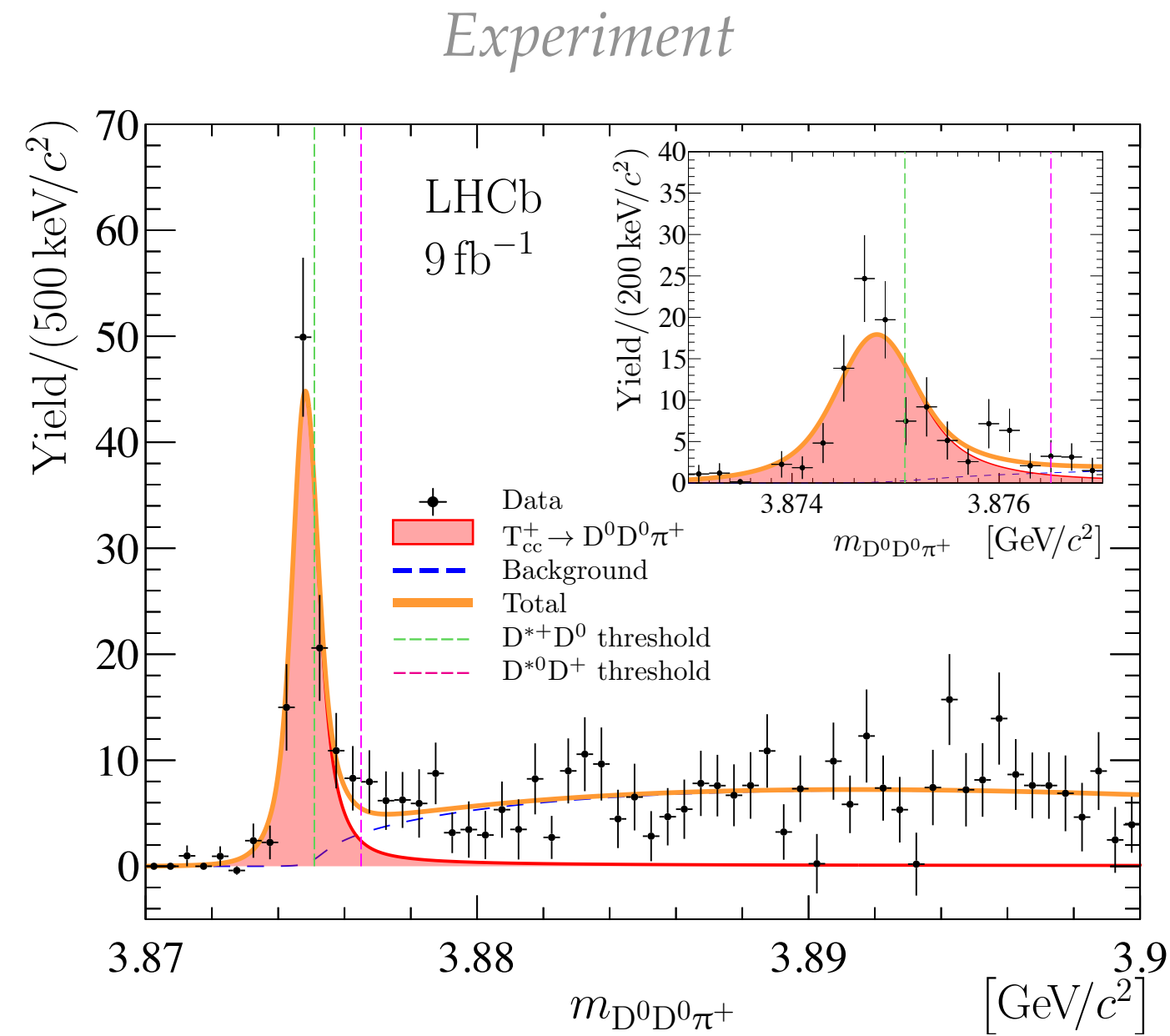
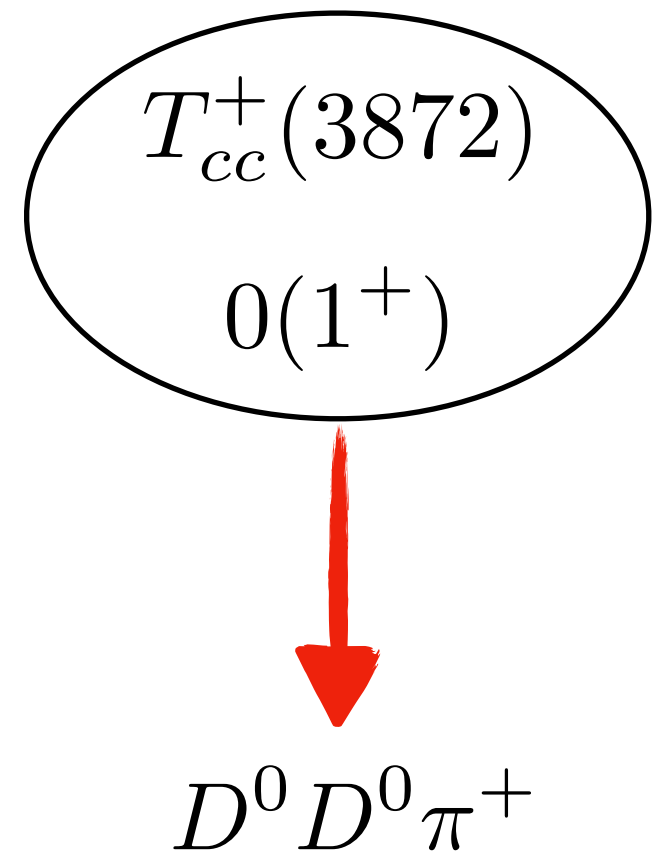


Threshold on the lattice

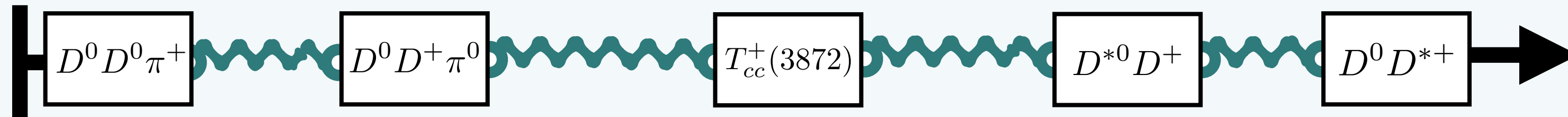


Why Three Body Physics?

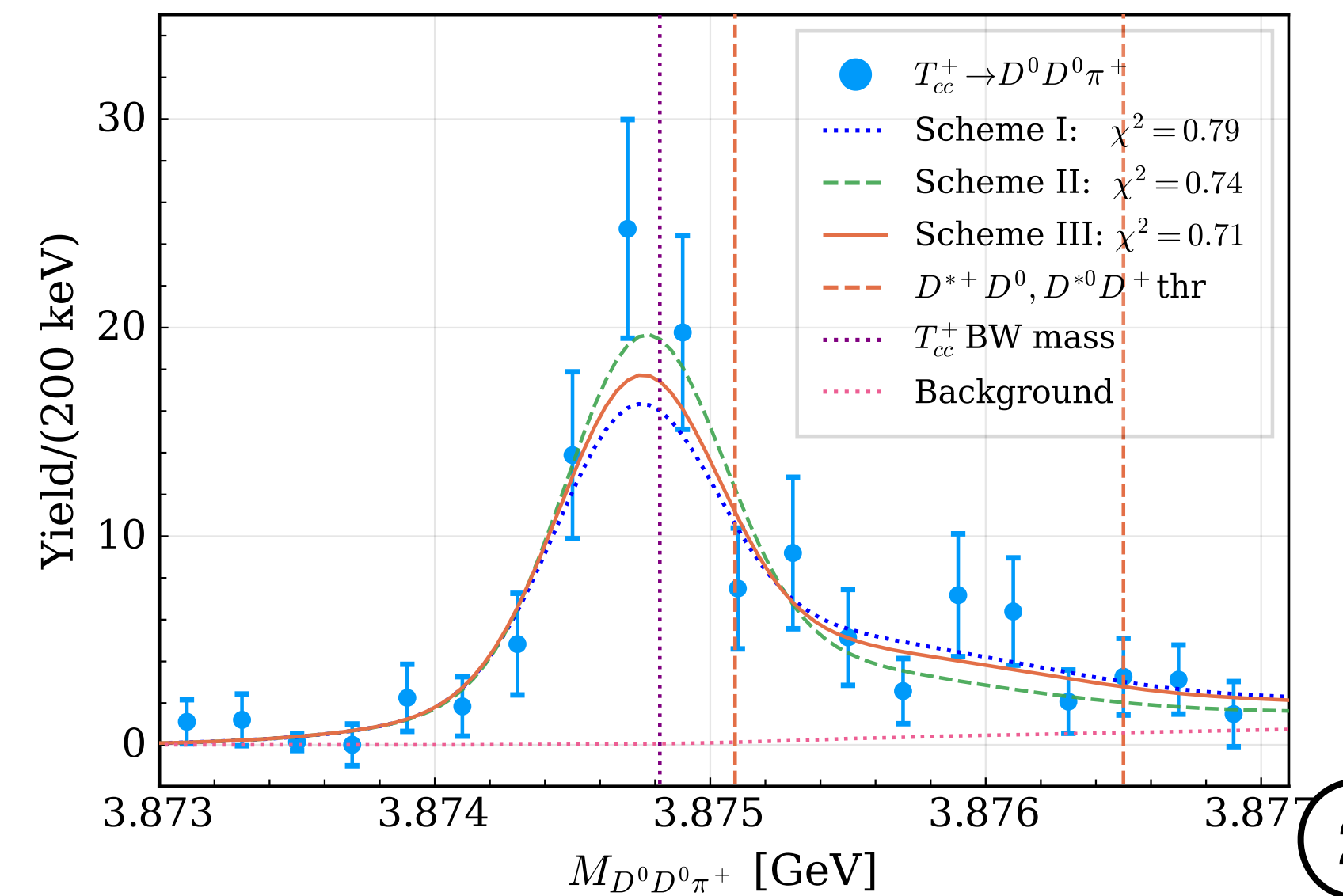
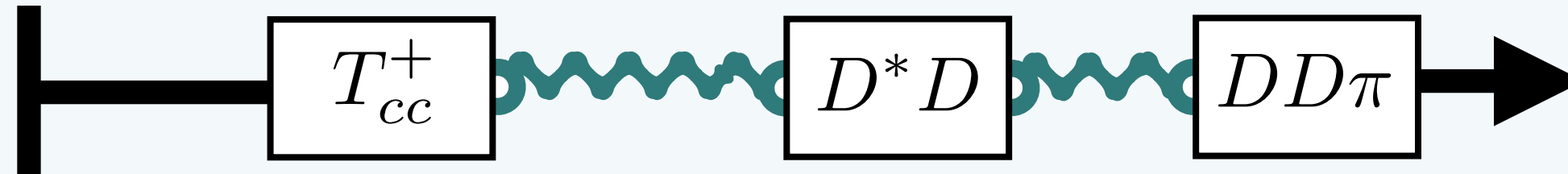
Interesting resonances decay to three particle final states under strong interactions



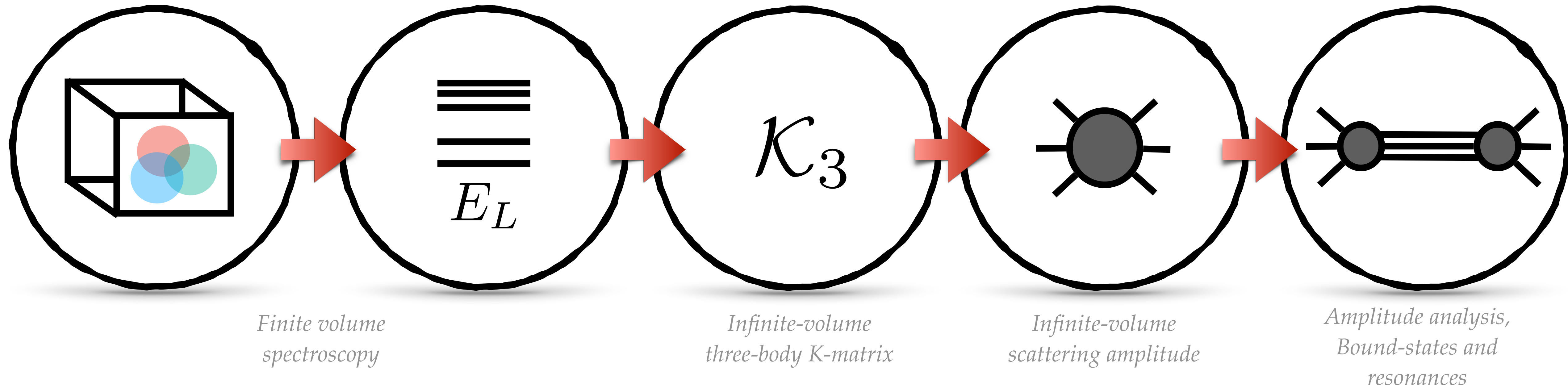
Physical Threshold



Threshold on the lattice



The Recipe



Hansen and Sharpe, 2014, 2015

Mai and Döring, 2017

Briceño, Hansen and Sharpe, 2018

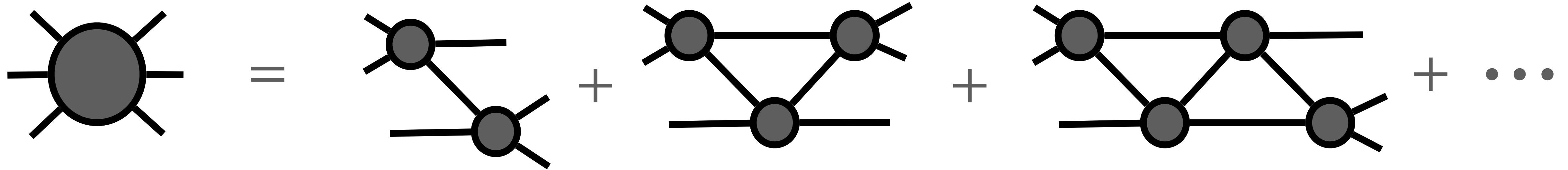
Hansen, Romero-López and Sharpe, 2020

Blanton and Sharpe, 2020

Jackura, 2022

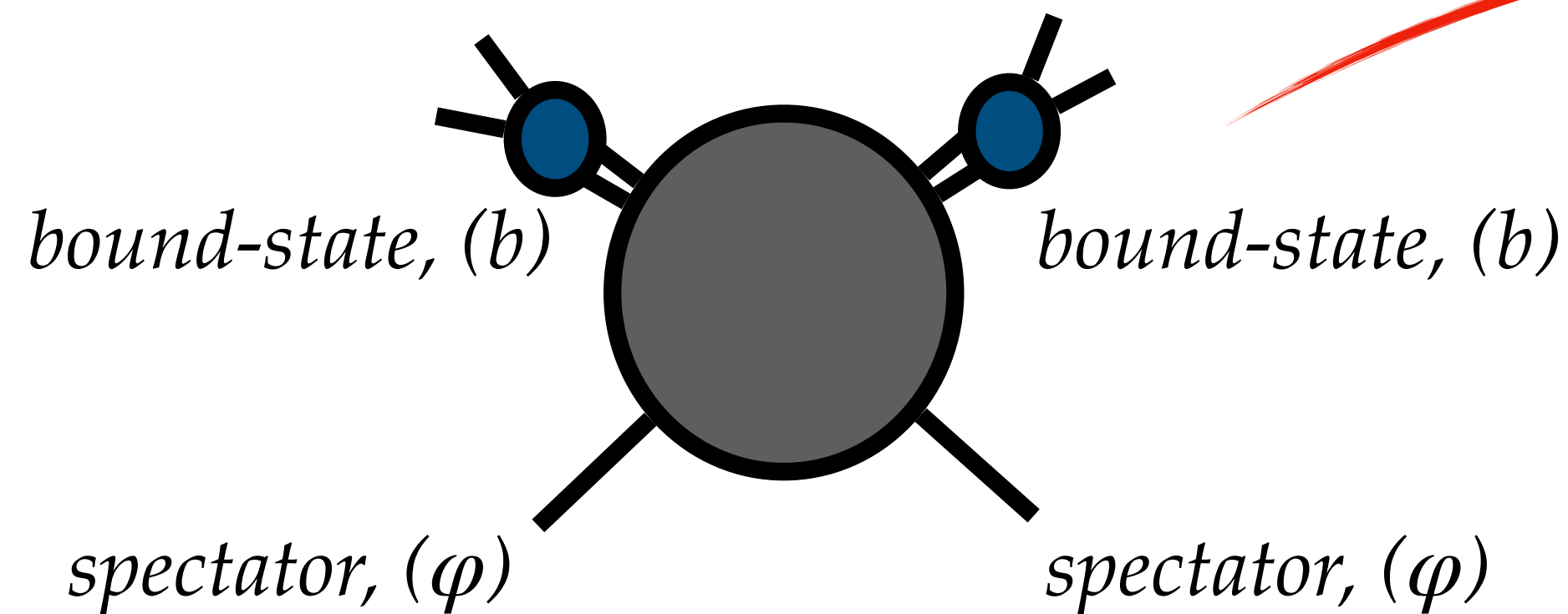
Toy Model

Assuming $\mathcal{K}_3 = 0$, and all identical scalar particles



Infinite volume integral equation
$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \mathcal{M}_2 \int_q G \mathcal{D}$$

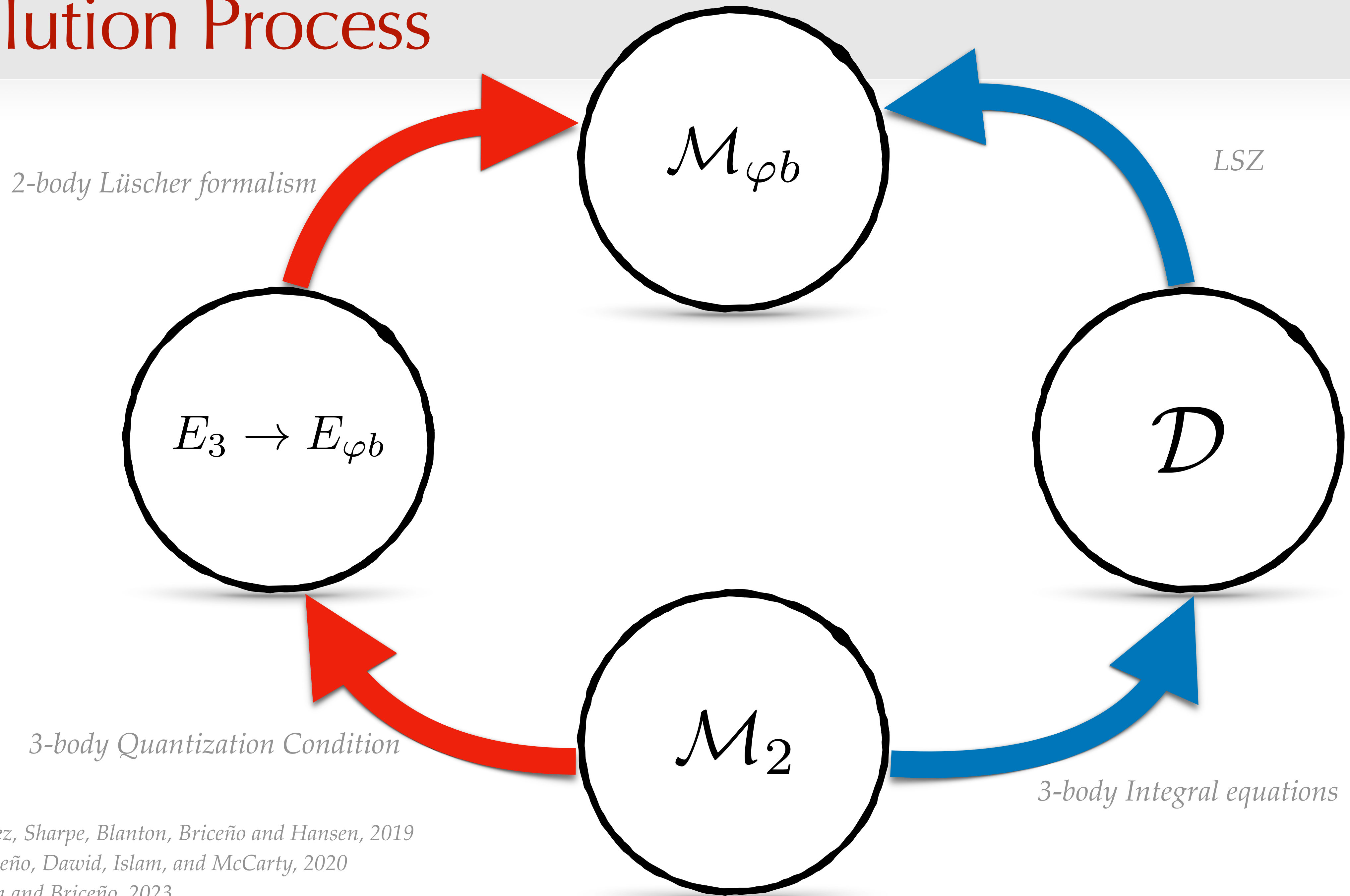
Finite volume Quantization condition
$$\det[\cancel{\mathcal{K}_3} + F_3^{-1}] = 0$$



$$\mathcal{D} = \frac{-g}{\sigma_p - m_b^2} \mathcal{M}_{\varphi b} \frac{-g}{\sigma_k - m_b^2}$$

Two-body bound-state residue g^2
Two-body bound-state mass m_b

Solution Process



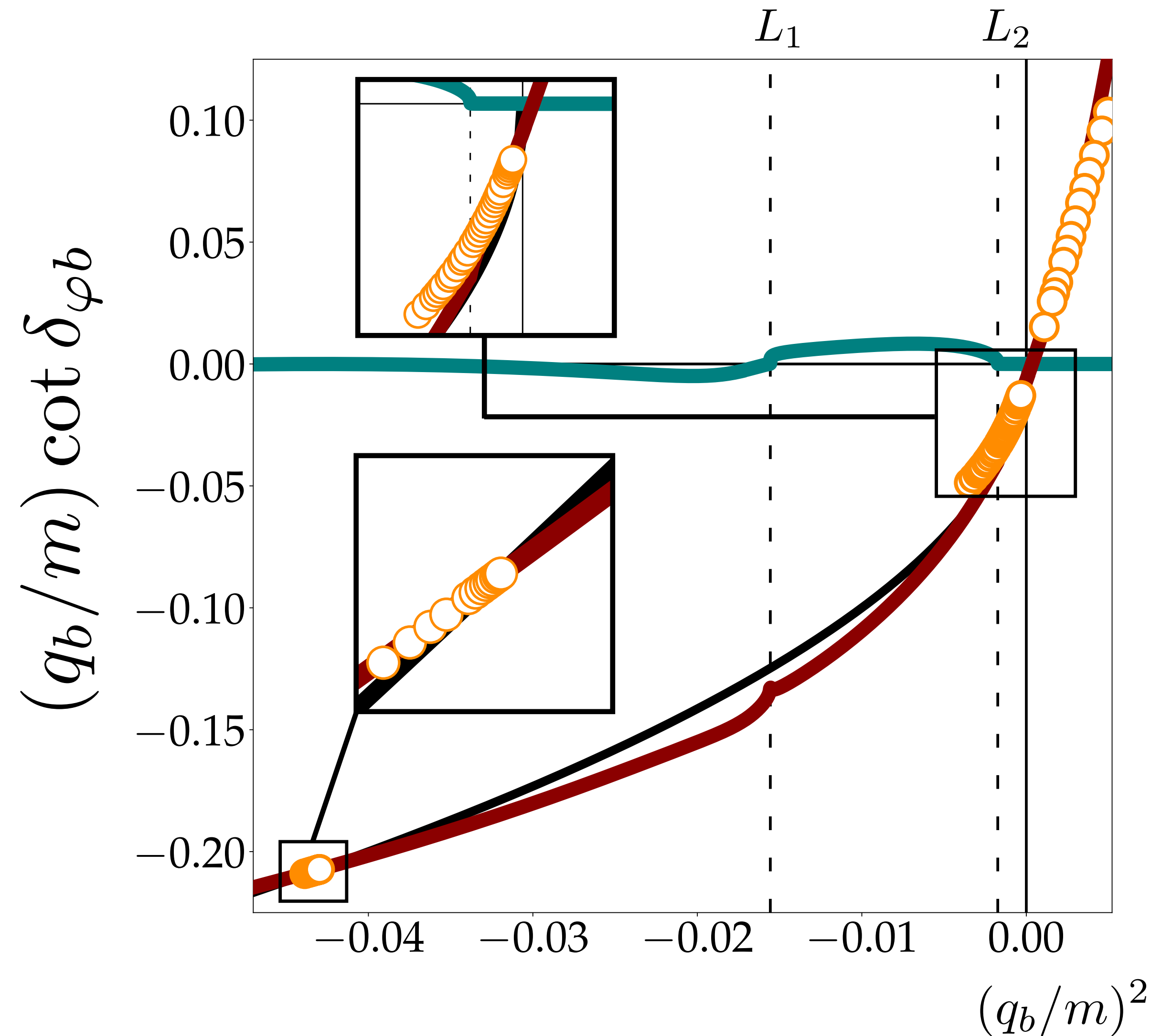
Romero-López, Sharpe, Blanton, Briceño and Hansen, 2019

Jackura, Briceño, Dawid, Islam, and McCarty, 2020

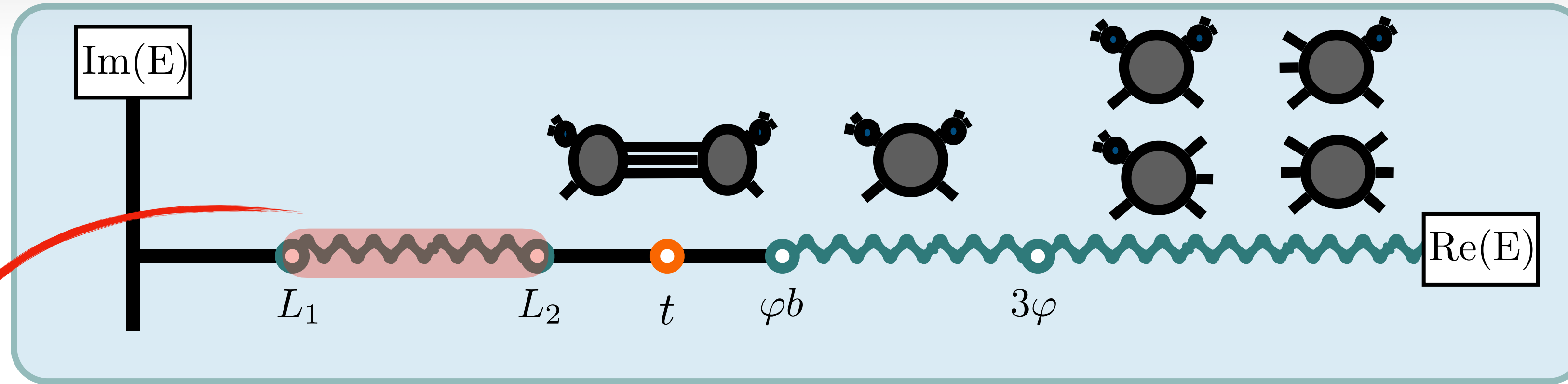
Dawid, Islam and Briceño, 2023

Outline

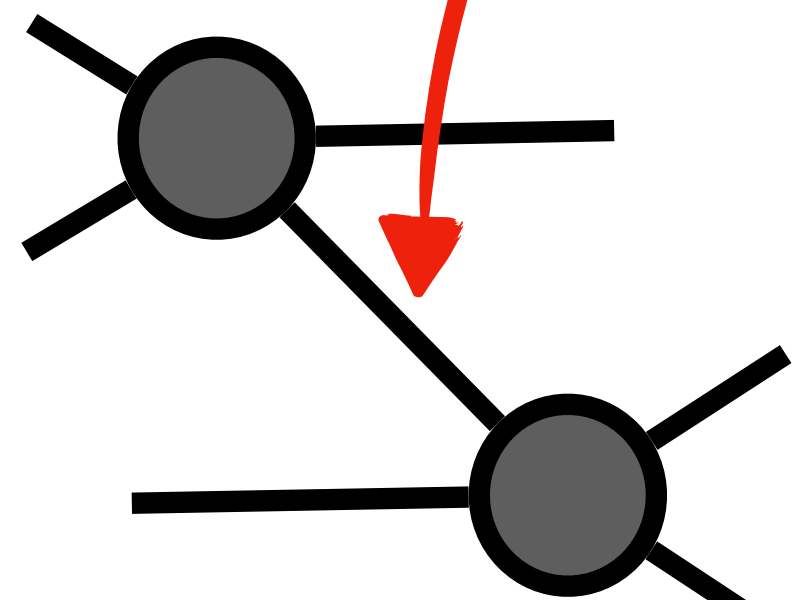
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- ✓ What toy model are we using?
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Left Hand Cut

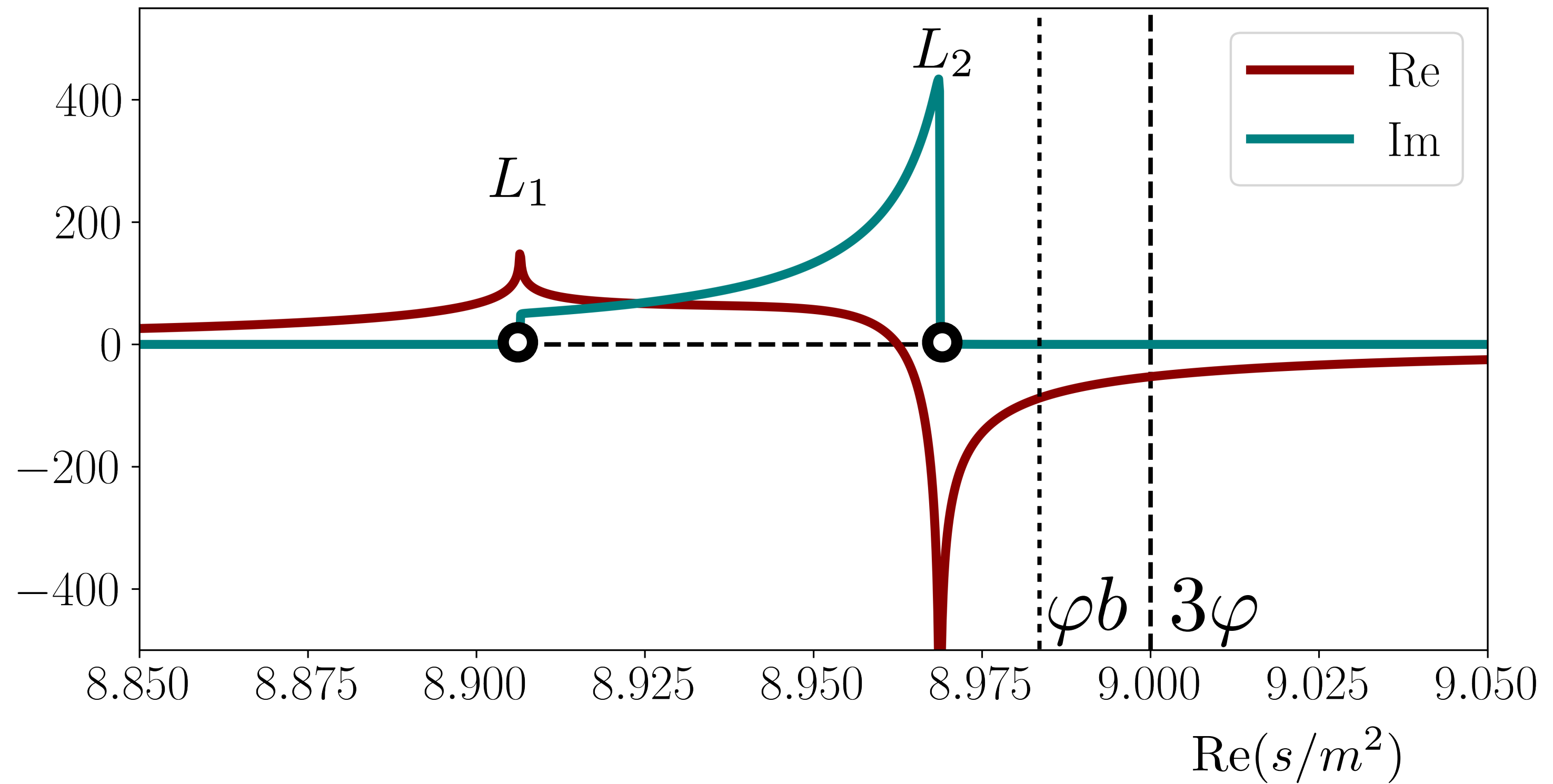


$$s = E^2$$



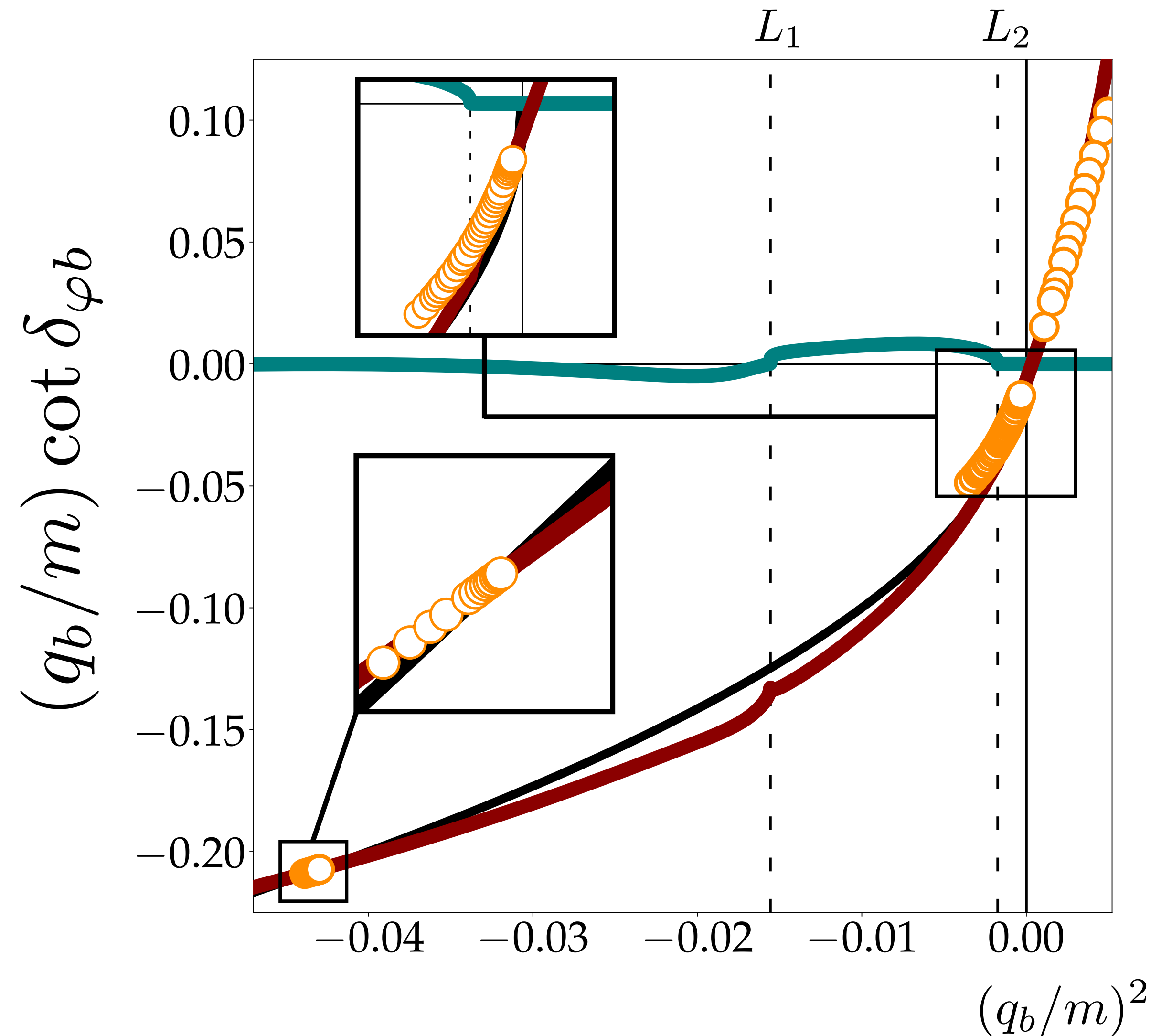
$$G \sim \log \left(\frac{Z - 1}{Z + 1} \right)$$

Branch points at $Z = \pm 1$



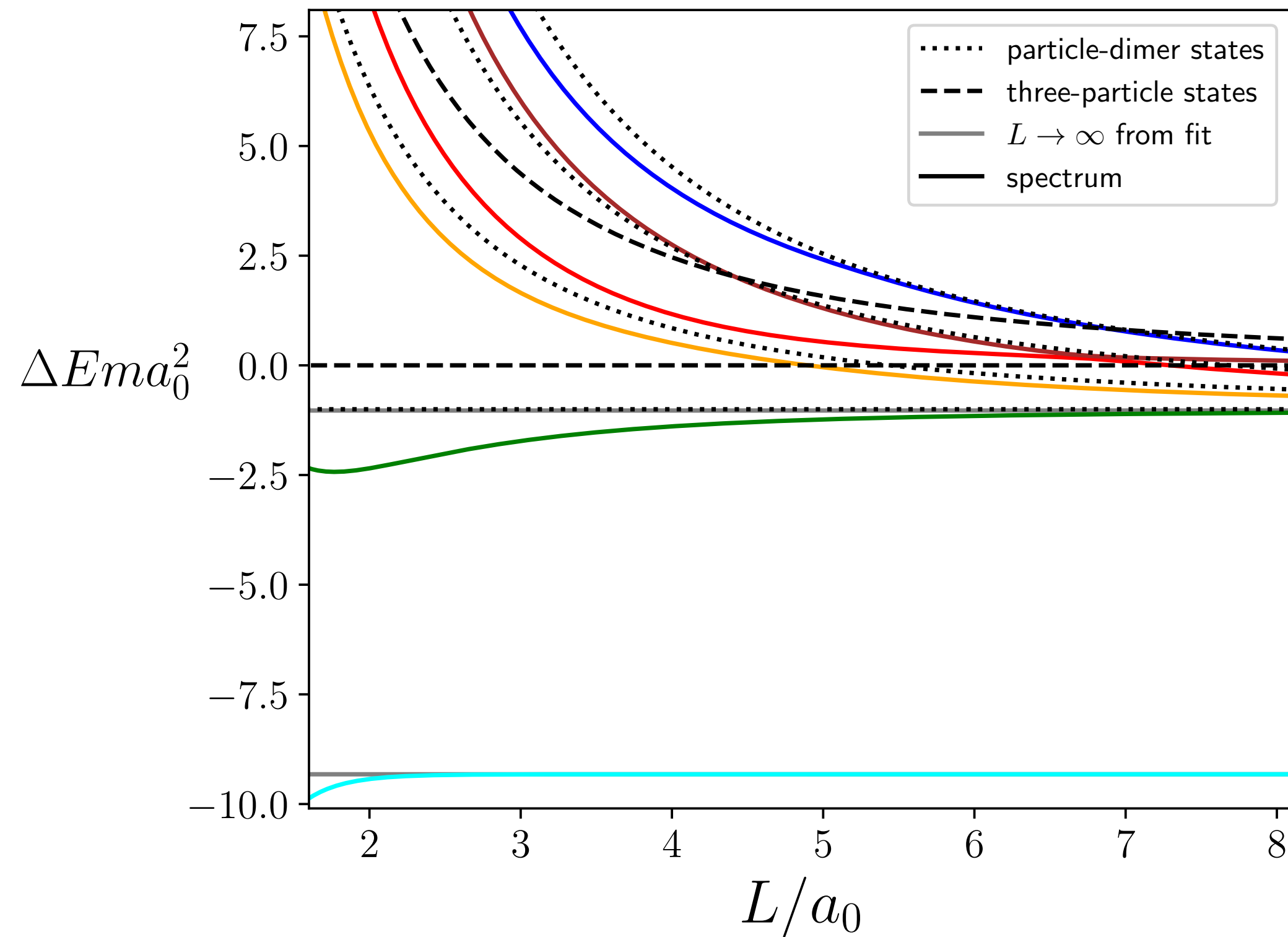
Outline

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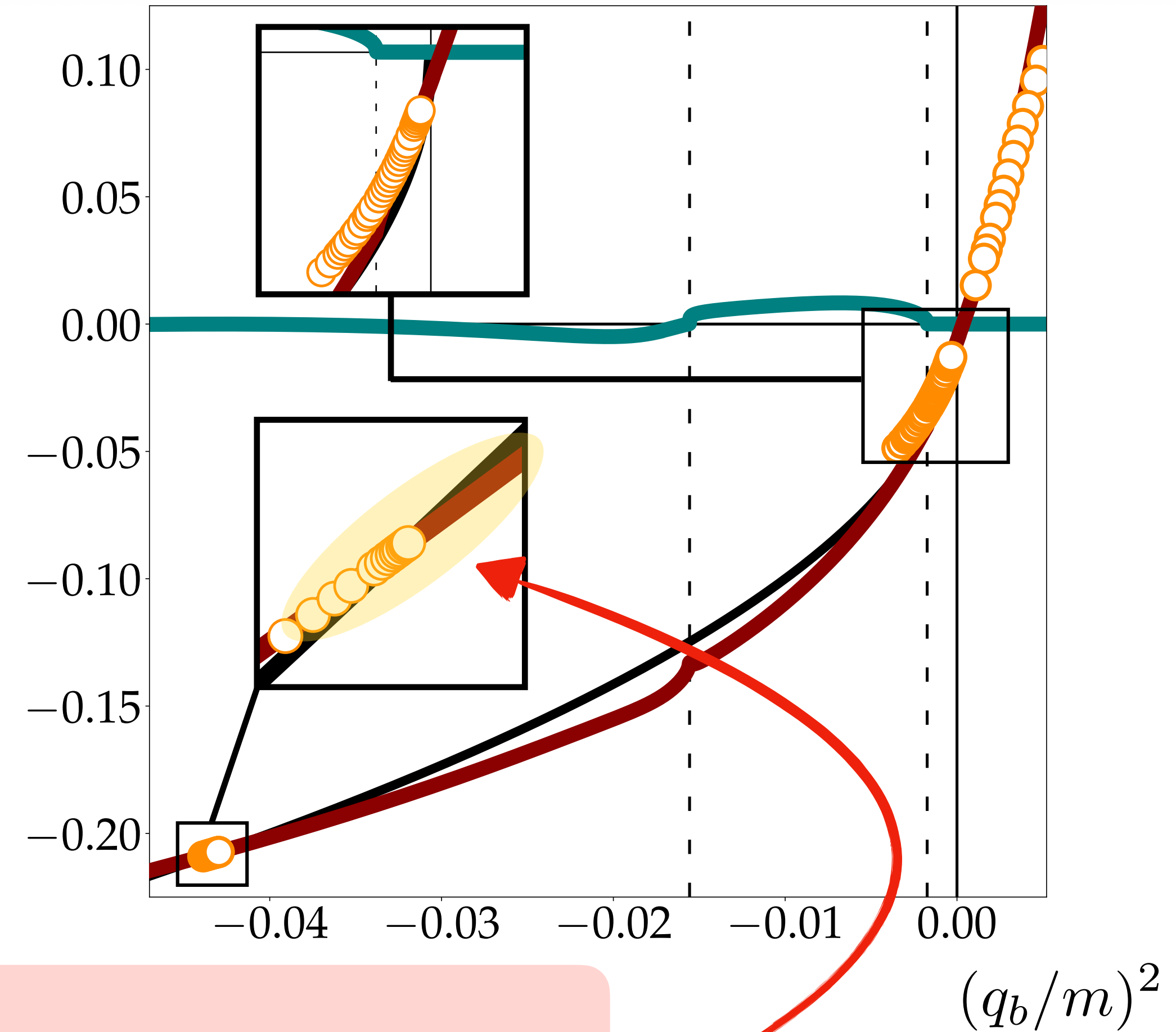


Phase Shifts from Lüscher formalism

$$\det[\mathcal{K}_3 + F_3^{-1}] = 0$$



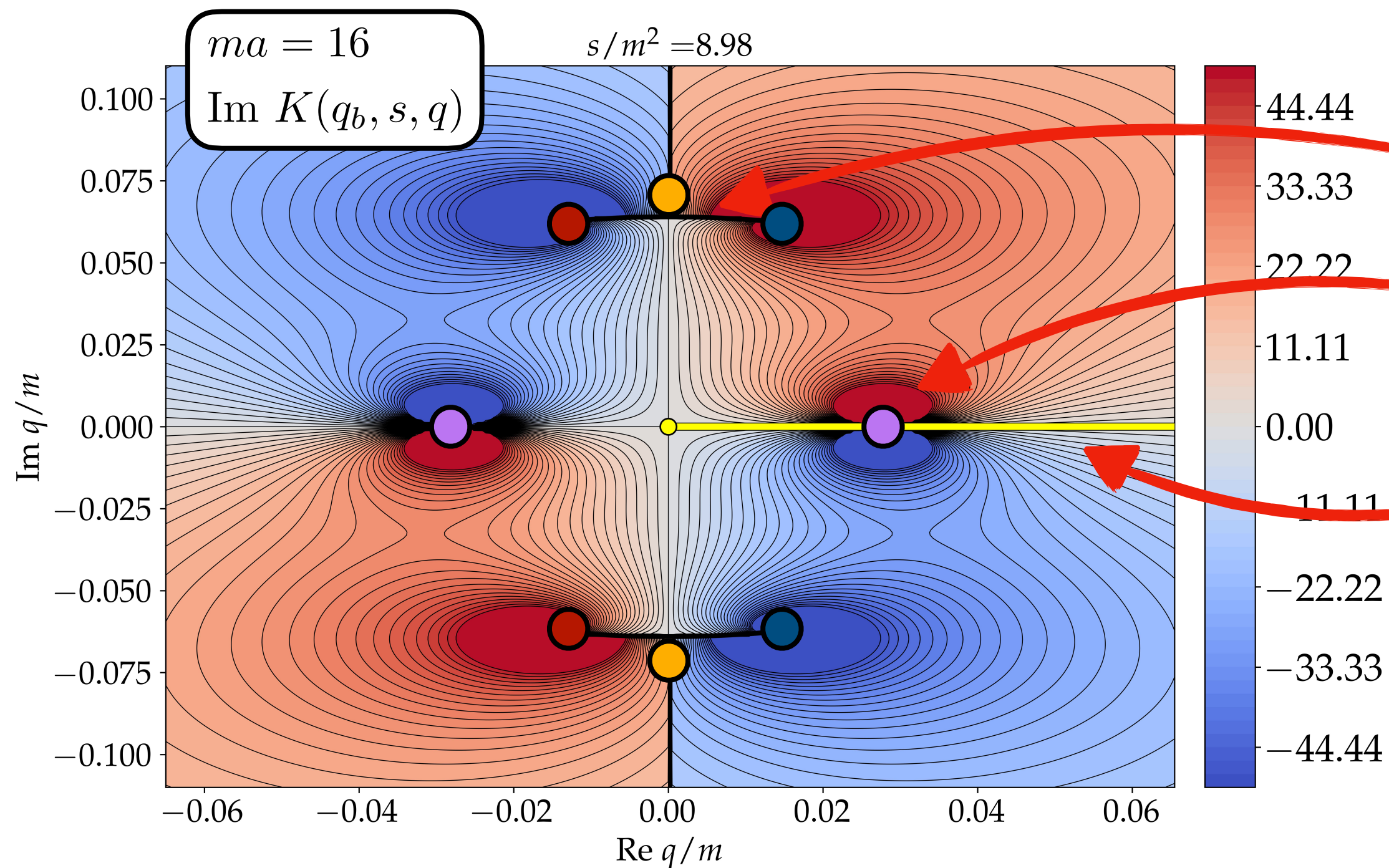
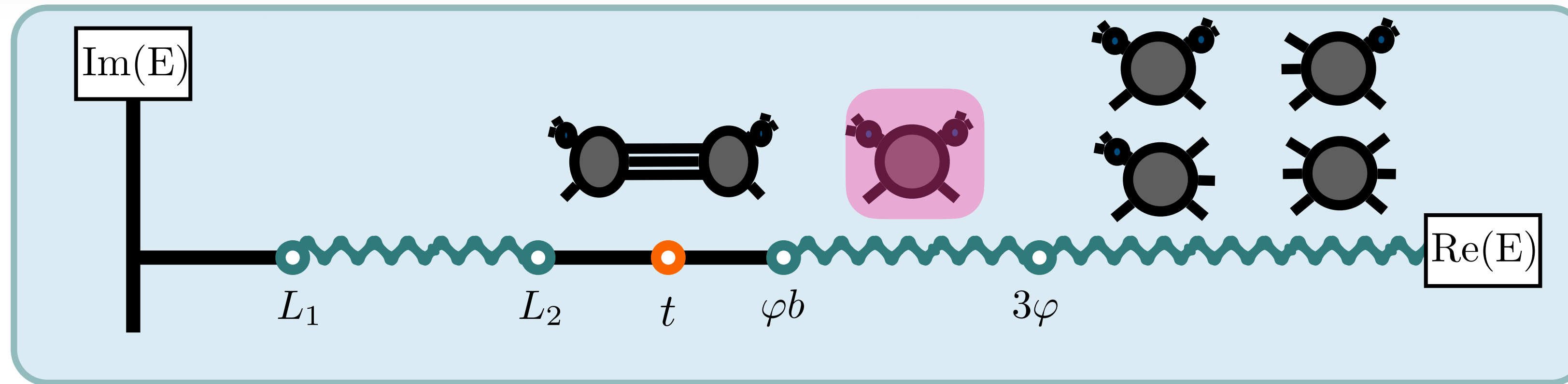
$$(q_b/m) \cot \delta_{\varphi b}$$



$$\det[\rho_{\varphi b} \cot \delta_{\varphi b} + F_2] = 0$$

Spectra obtained using 3-body Quantization condition
 Phase shift calculated using 2-body Lüscher formalism

Phase Shifts from Integral Equations



$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \mathcal{M}_2 \int_q G \mathcal{D}$$

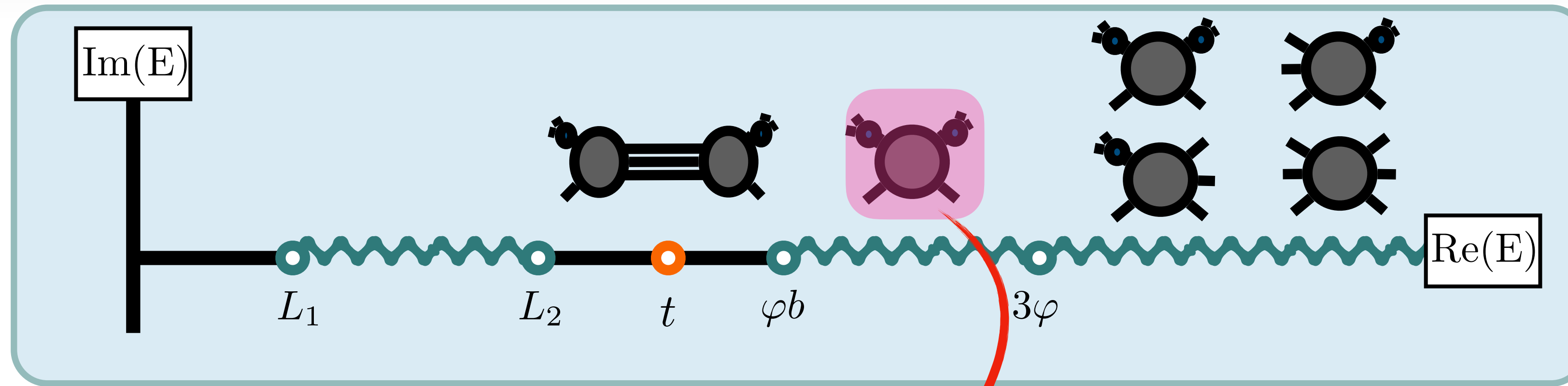
Singularity cut from the one particle exchange function

Two body amplitude has a pole in the integration region

Momentum integration interval

$$K(p, s, q) = \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s)$$

Phase Shifts from Integral Equations

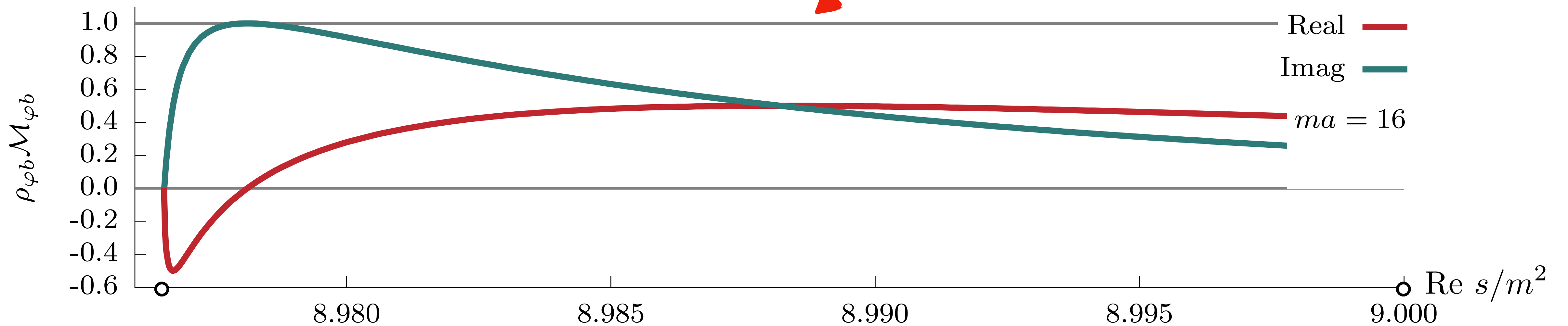


$$s = E^2$$

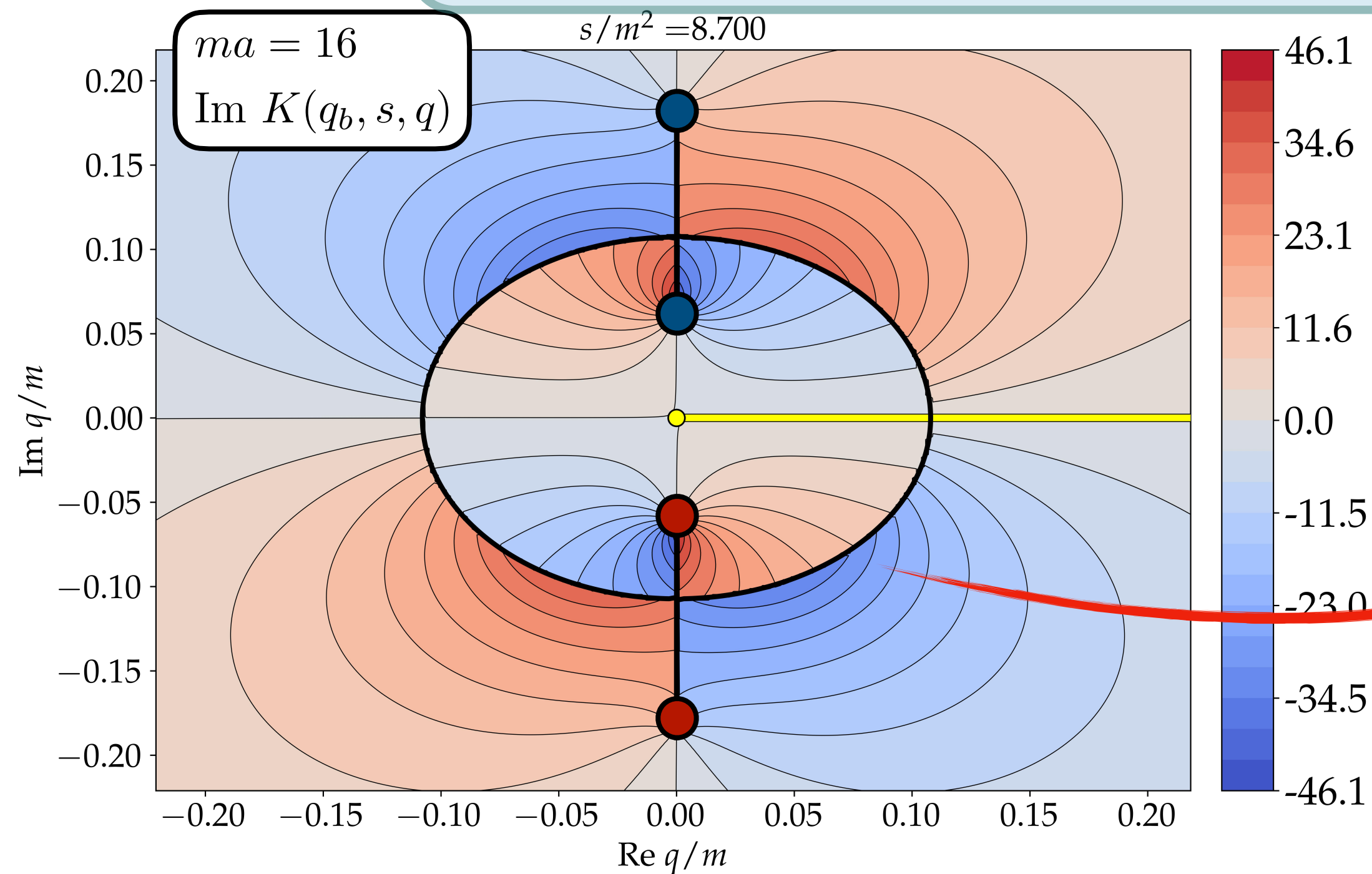
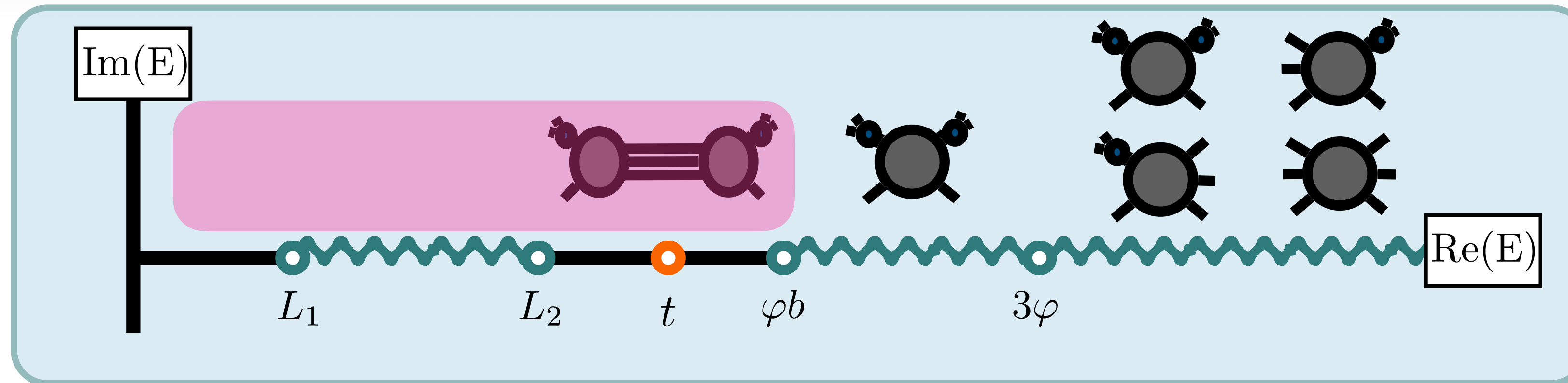
$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \mathcal{M}_2 \int_q G \mathcal{D}$$

Follows two-body unitarity condition:

$$|\rho_{\phi b} \mathcal{M}_{\phi b}| \leq 1$$



Phase Shifts from Integral Equations



$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \mathcal{M}_2 \int_q G \mathcal{D}$$

Logarithmic cut forms a circle around the end point of integration

$$K(p, s, q) = \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s)$$

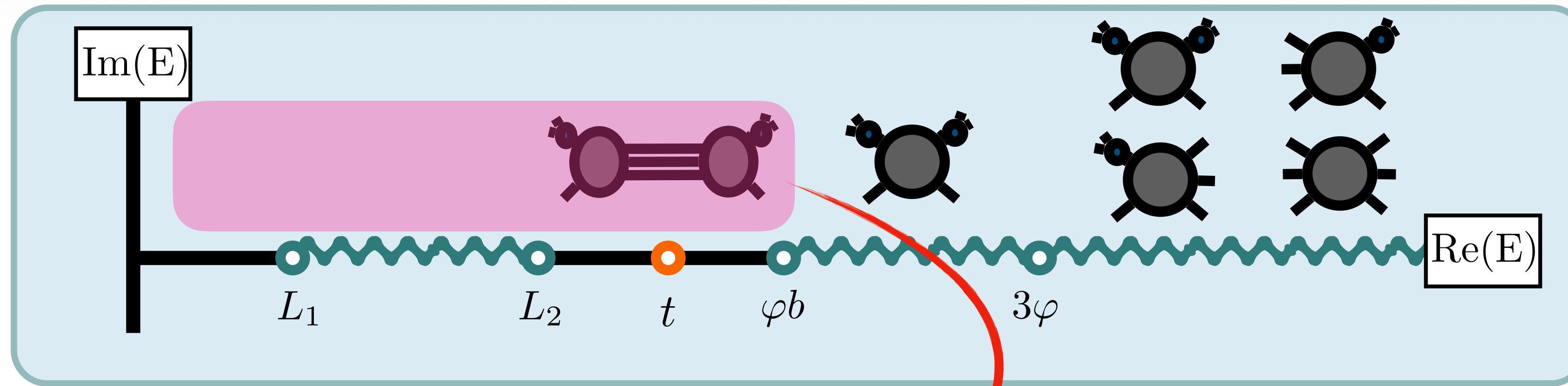
q_b is the spectator momenta corresponding to two-body bound-state energy, $\sigma_q = m_b^2$

Romero-López, Sharpe, Blanton, Briceño and Hansen, 2019

Jackura, Briceño, Dawid, Islam, and McCarty, 2020

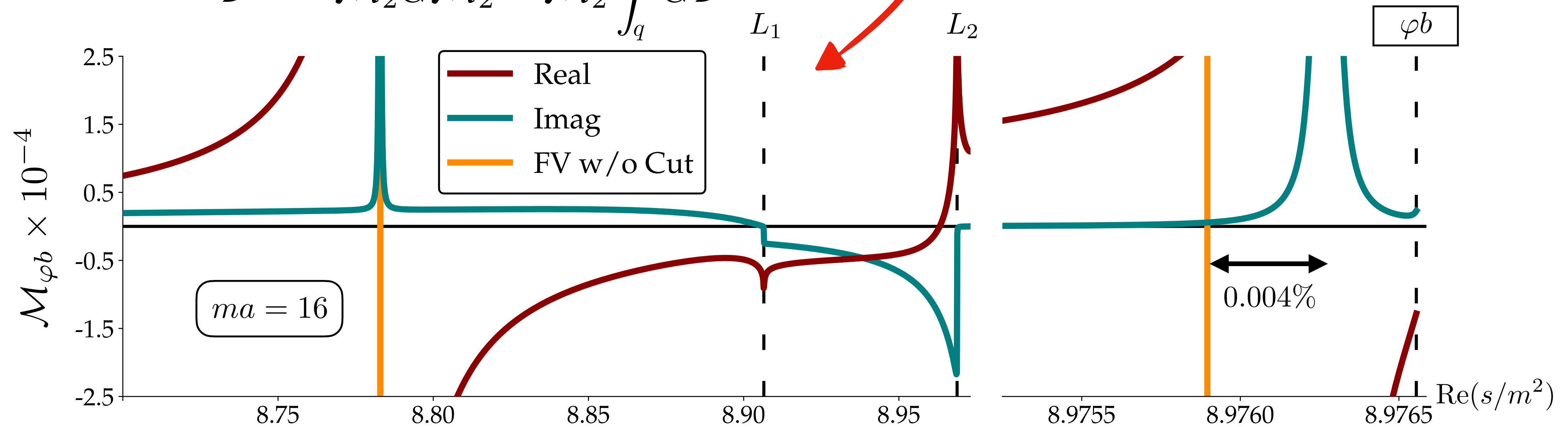
Dawid, Islam and Briceño, 2023

Phase Shifts from Integral Equations



$$s = E^2$$

$$\mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \mathcal{M}_2 \int_q G \mathcal{D}$$



Romero-López, Sharpe, Blanton, Briceño and Hansen, 2019

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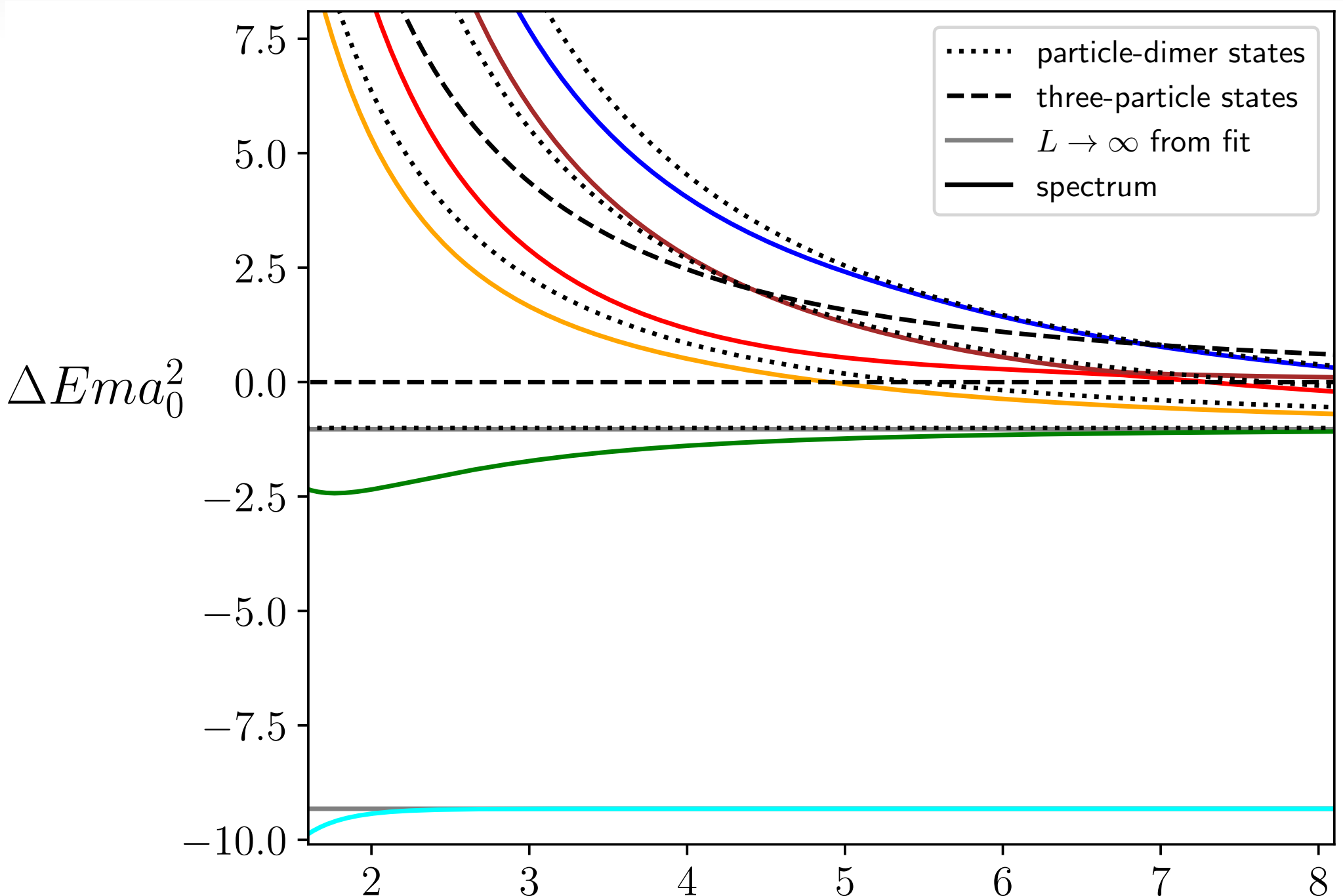
Dawid, Islam and Briceño, 2023

Breakdown near Left Hand Cut

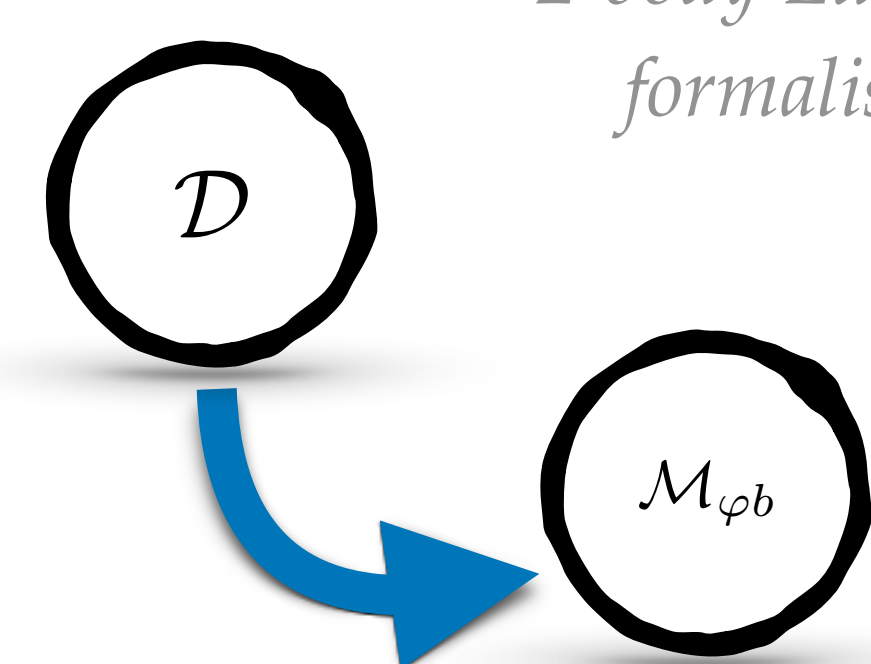
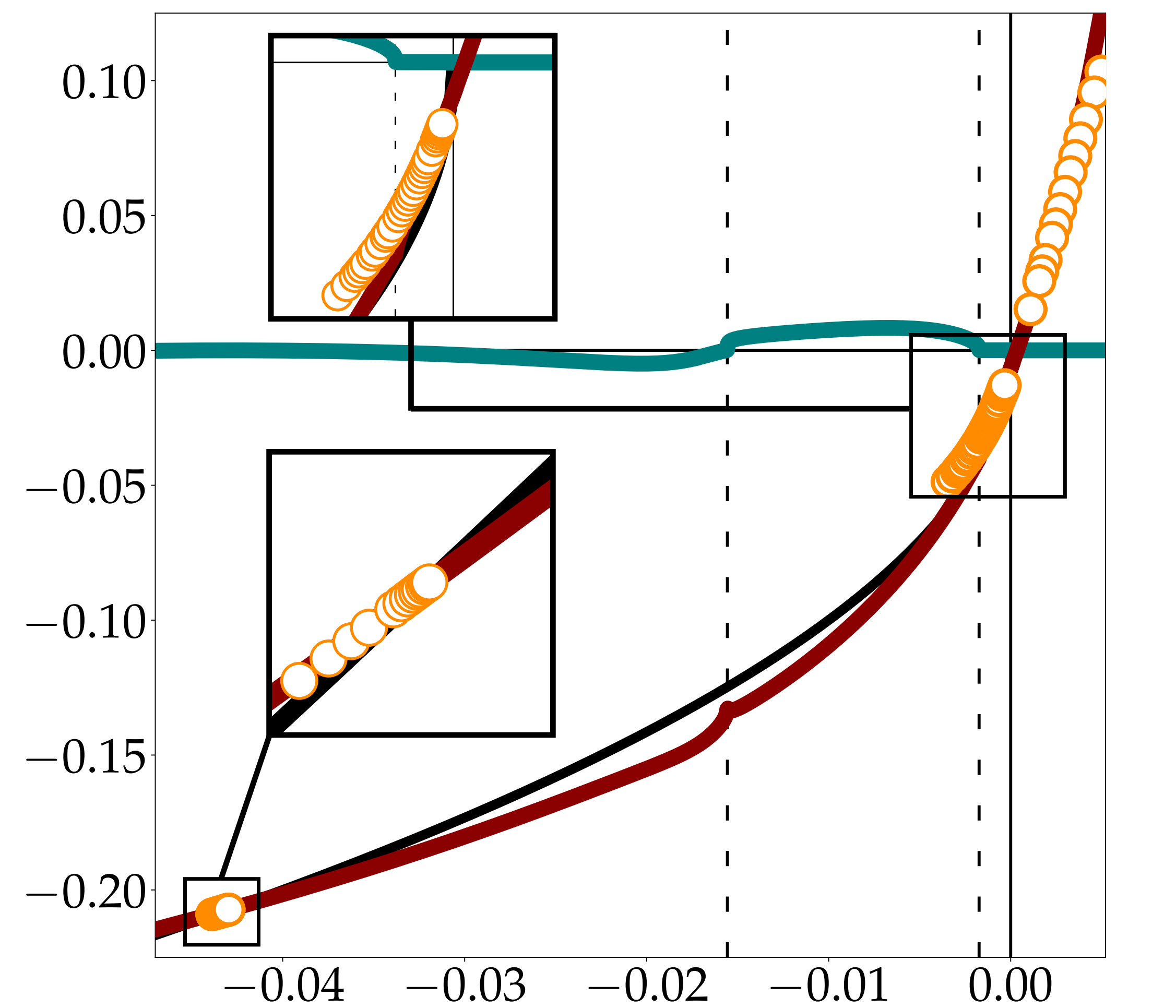
Romero-López, Sharpe, Blanton, Briceño and Hansen, 2019

Jackura, Briceño, Dawid, Islam, and McCarty, 2020

Dawid, Islam and Briceño, 2023



$(q_b/m) \cot \delta_{\varphi b}$



2-body Lüscher formalism

$$\det[\rho_{\varphi b} \cot \delta_{\varphi b} + F_2] = 0$$

Solving Integral equations:

$$q_b \cot \delta_{\varphi b} = 8\pi\sqrt{s}\mathcal{M}_{\varphi b}^{-1} + iq_b$$

$ma = 16$

$(q_b/m)^2$

Summary

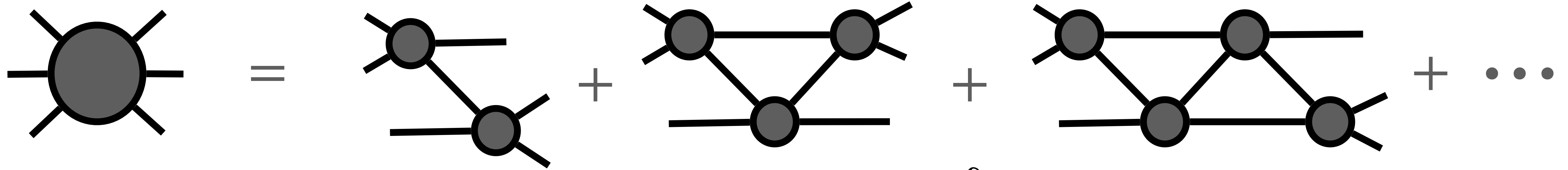
- We have shown breakdown of the Lüscher formalism in the vicinity of the left hand cuts
- We have addressed the complications in studying bound-state-spectator amplitude above and below threshold and shown possible solutions to them
- We have performed consistency checks with toy model that provides confidence for both finite-volume and infinite-volume formalisms

Thank you

Backup Slides

Three Body Scattering Amplitude

Assuming $\mathcal{K}_3 = 0$, and all identical scalar particles

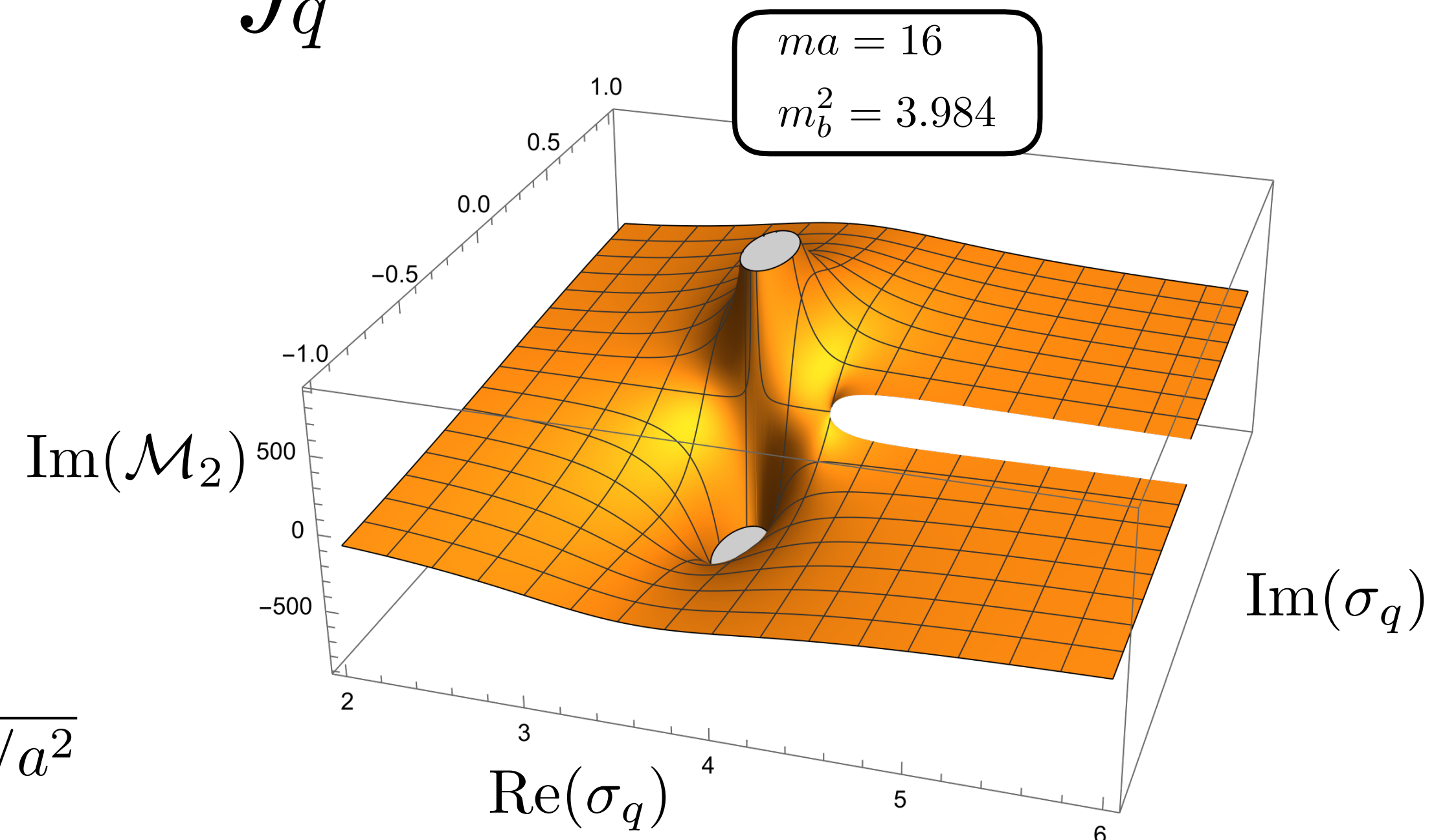


$$D = -\mathcal{M}_2 G \mathcal{M}_2 - \mathcal{M}_2 \int_q G D$$

Two-body scattering amplitude $\mathcal{M}_2 = \frac{1}{\mathcal{K}_2^{-1} - i\rho}$

Leading order effective range expansion $= \frac{1}{\rho \cot \delta_0 - i\rho}$

$$= \frac{16\pi \sqrt{\sigma_q}}{-\frac{1}{a} - i\sqrt{\frac{\sigma_q}{4} - m^2}}$$



If $a > 0$, we have a bound-state in the two body system with mass, $m_b = 2\sqrt{m^2 - 1/a^2}$

Solving Integral Equations:

1. Introduce $\mathcal{D} = \mathcal{M}_2 d\mathcal{M}_2$

2. Project onto partial wave, $\ell = 0$ sector

$$d_S(p, s, k) = -G_S(p, s, k) - \int_0^{q_{max}} dq \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s) d_S(q, s, k)$$

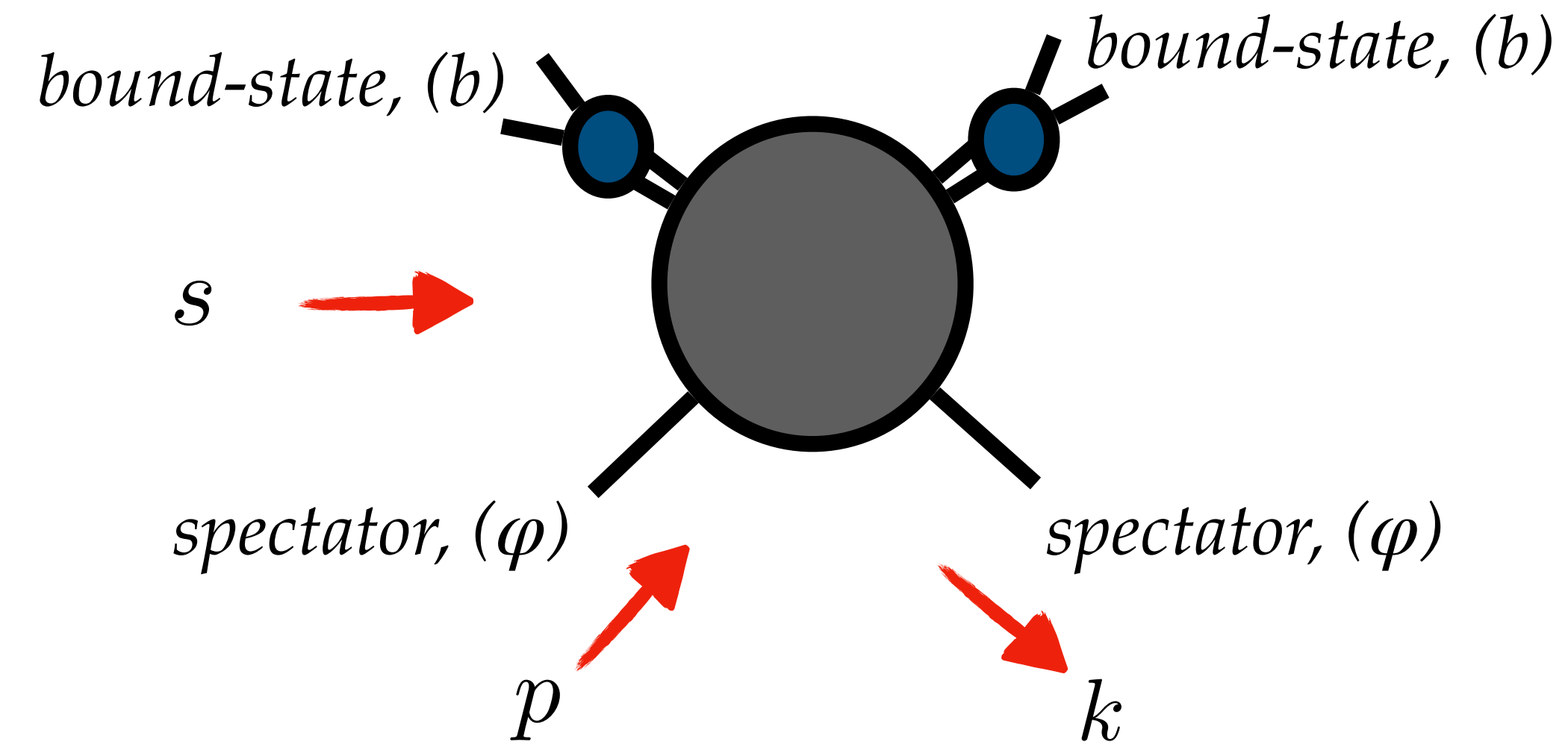
3. Soften singularities using non-zero epsilon, or isolate them

4. Write as matrix equation:

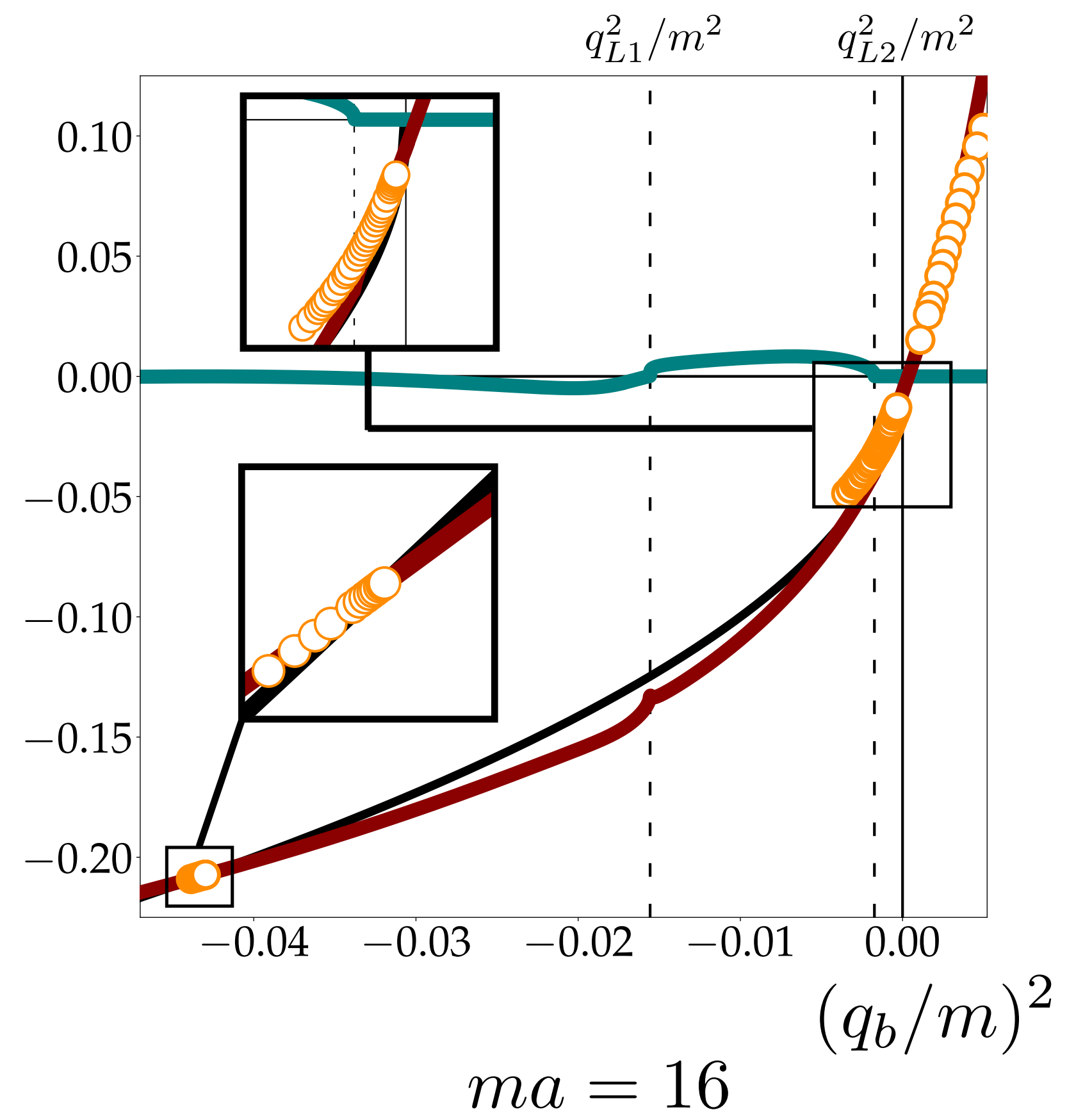
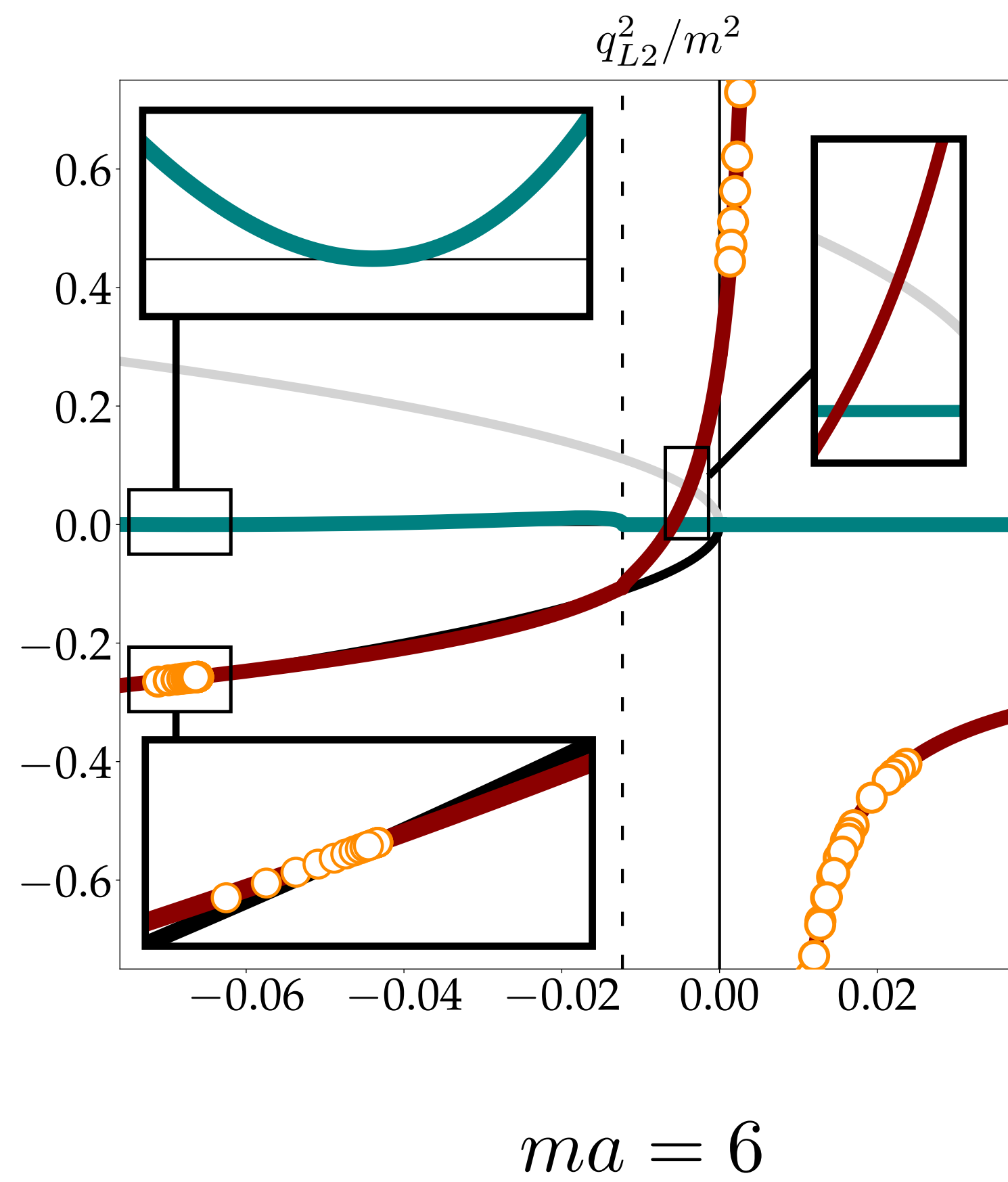
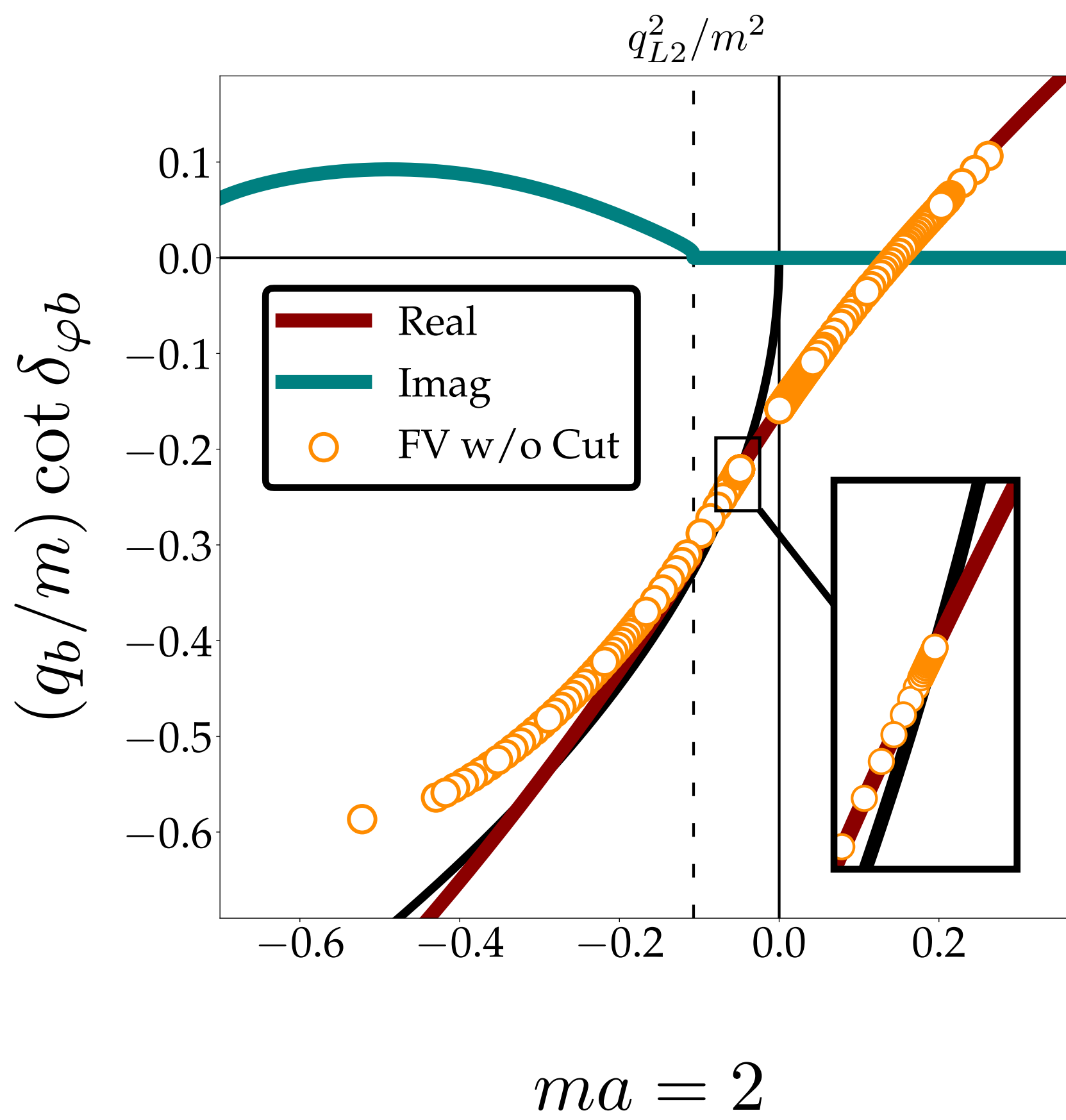
$$d(N, \epsilon) = -G(\epsilon) - P \cdot G(\epsilon) \cdot \mathcal{M}_2(\epsilon) \cdot d(N, \epsilon)$$

5. Recover the exact result by taking $N \rightarrow \infty, \epsilon \rightarrow 0$ limit

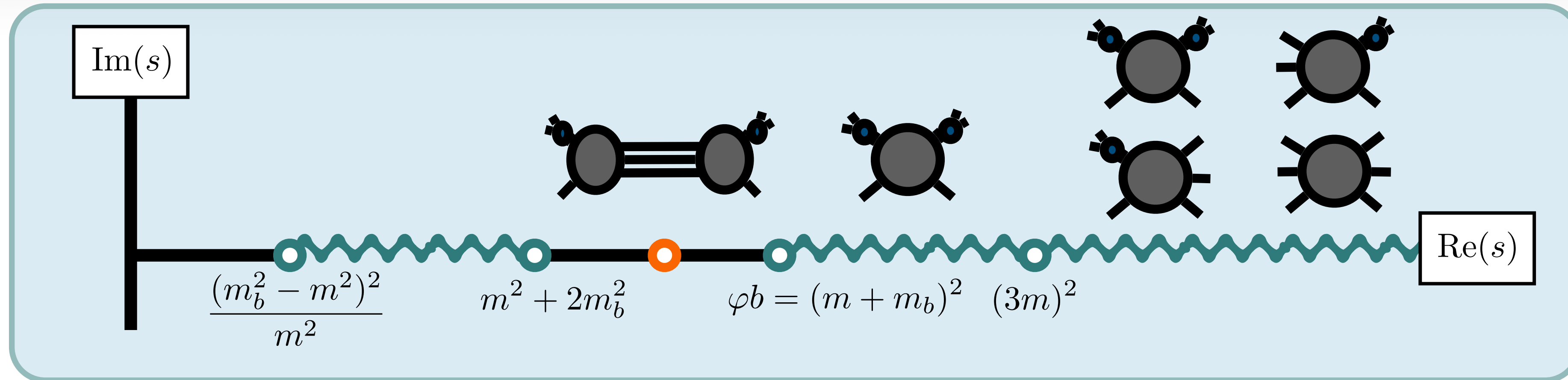
$$6. \mathcal{M}_{\varphi b} = g^2 \lim_{\sigma_p, \sigma_k \rightarrow m_b^2} d_S$$



Phase shifts:



Kinematic Region of Interest



$$d_S(p, s, k) = -G_S(p, s, k) - \int_0^{q_{max}} dq \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s) d_S(q, s, k)$$

φb Scattering Length, b_0

Romero-López, Sharpe, Blanton, Briceño and Hansen, 2019

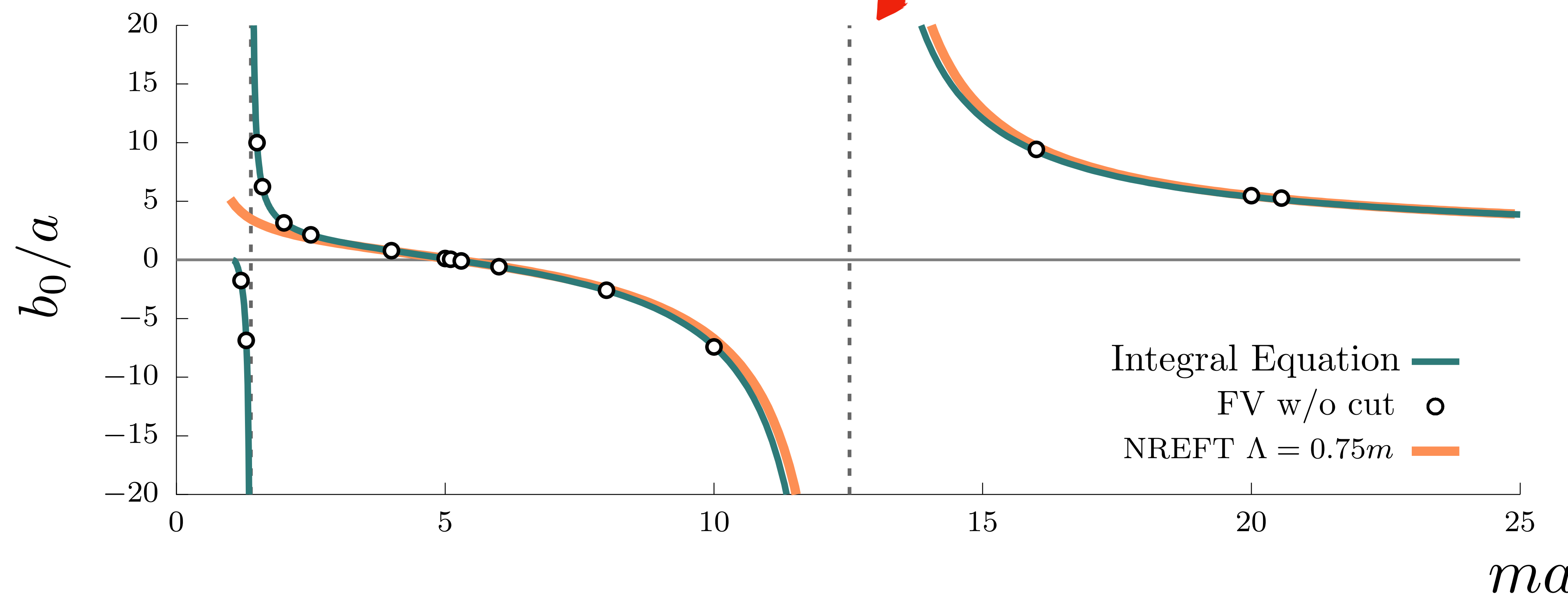
Jackura, Briceño, Dawid, Islam, and McCarty, 2020

Bedaque, Hammer, van Kolck, 1999

Fit to leading order effective range expansion to get φb scattering length, b_0

$$\lim_{q \rightarrow 0} q \cot \delta_{\varphi b} = -\frac{1}{b_0}$$

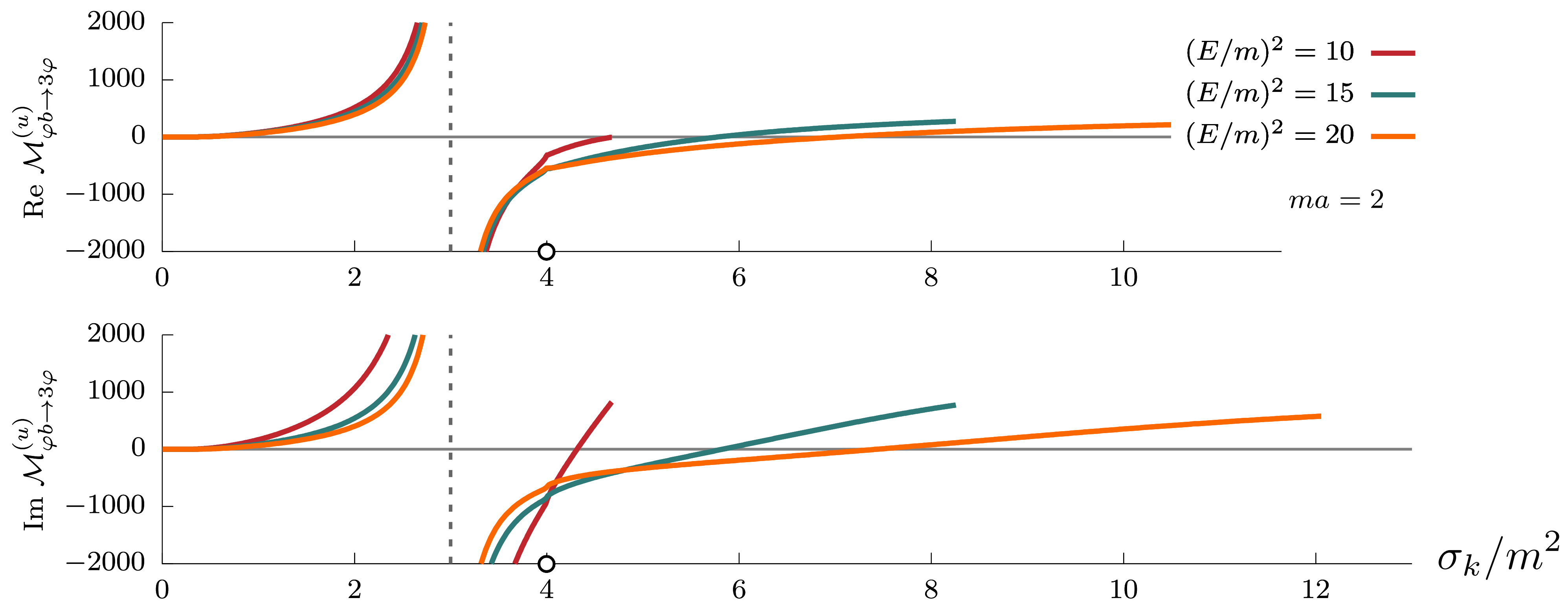
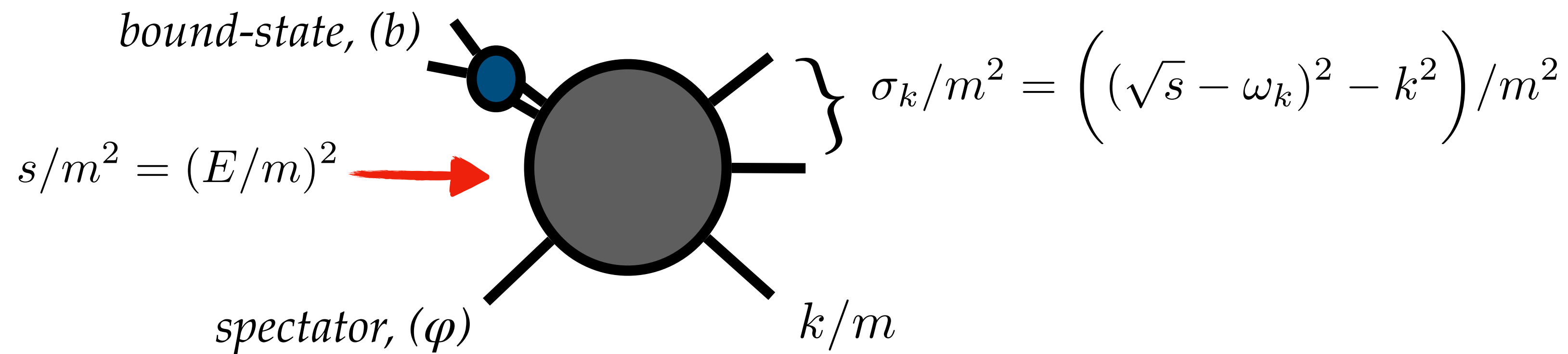
$$\mathcal{M}_{\varphi b} = \frac{8\pi\sqrt{s}}{q \cot \delta_{\varphi b} - iq}$$



Hints at possible trimers

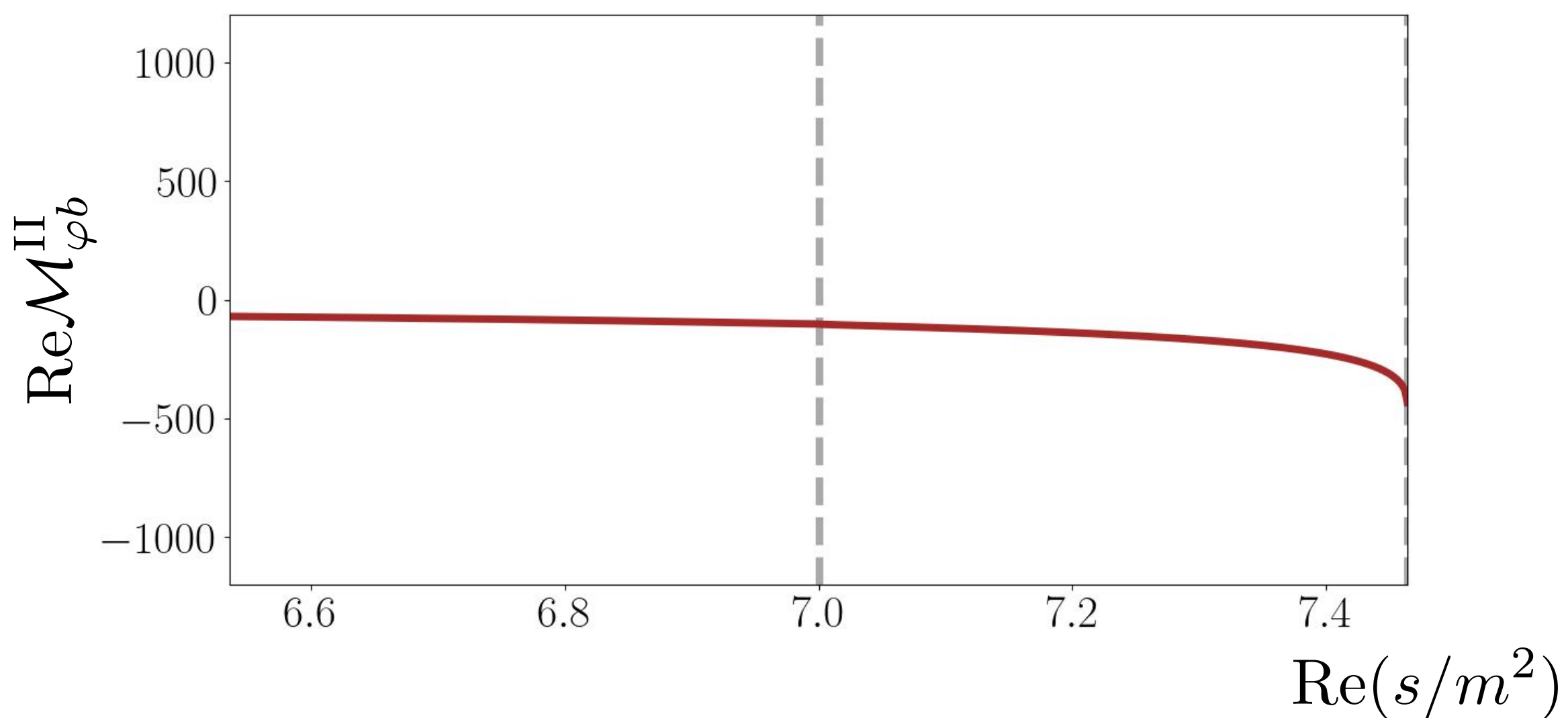
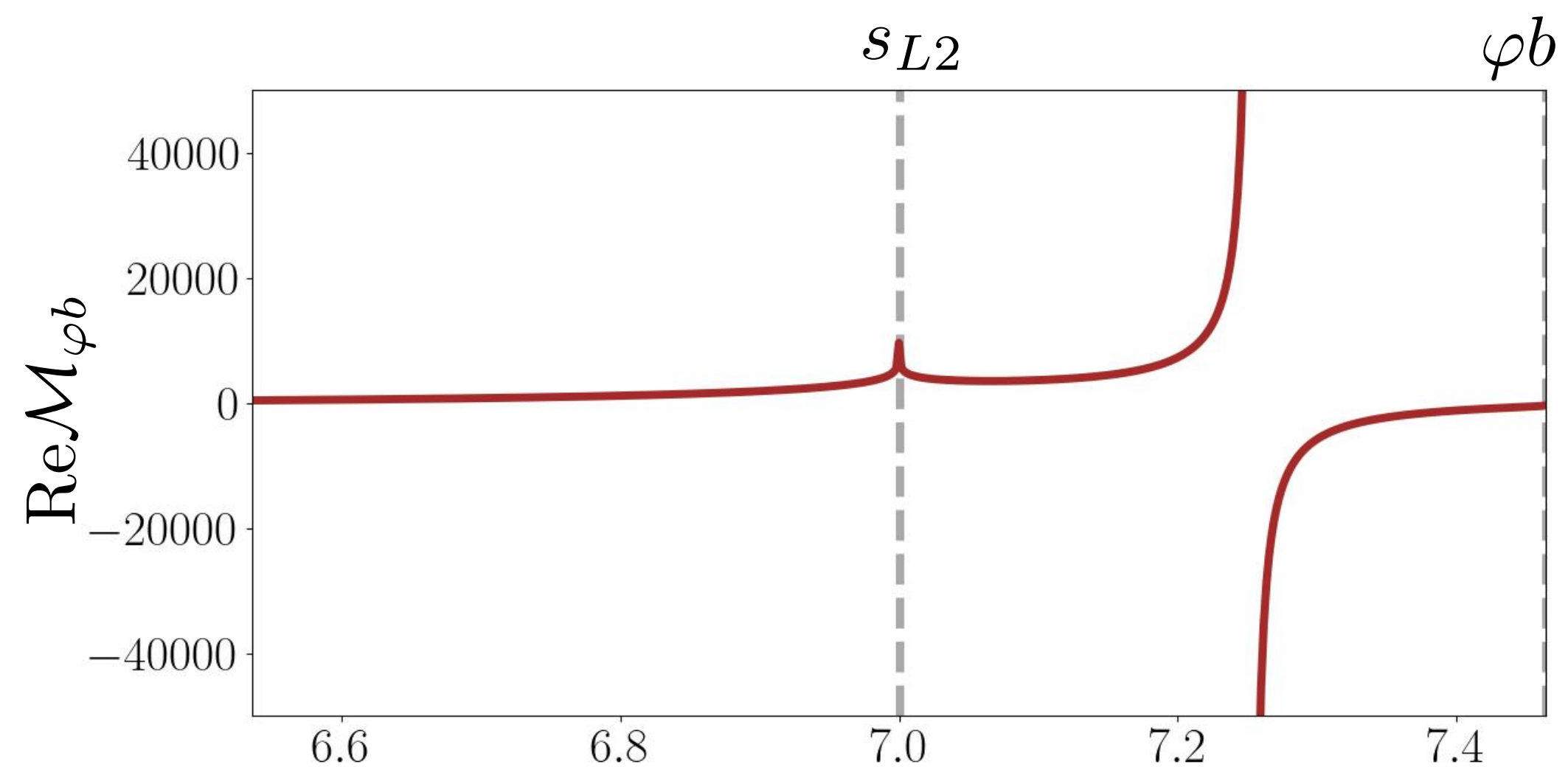
Integral Equation —
 FV w/o cut ○
 NREFT $\Lambda = 0.75m$ —

Three body breakups



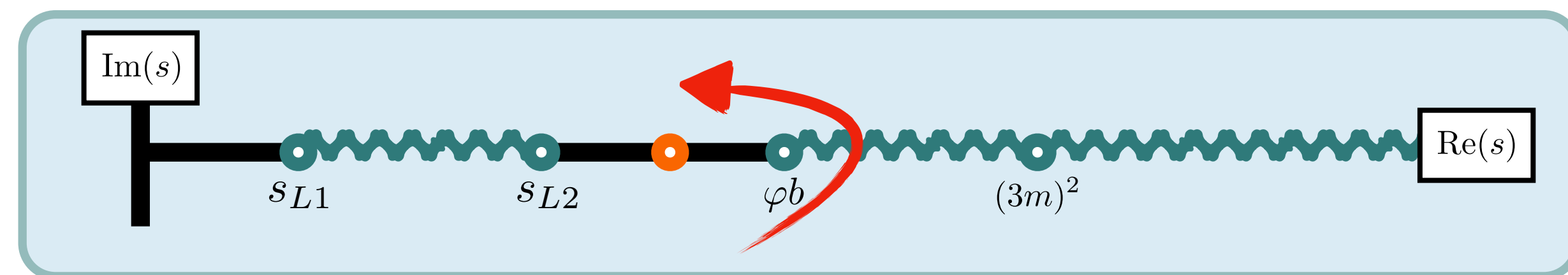
Three-Body Bound-states and Virtual States

$ma = 2.0000$

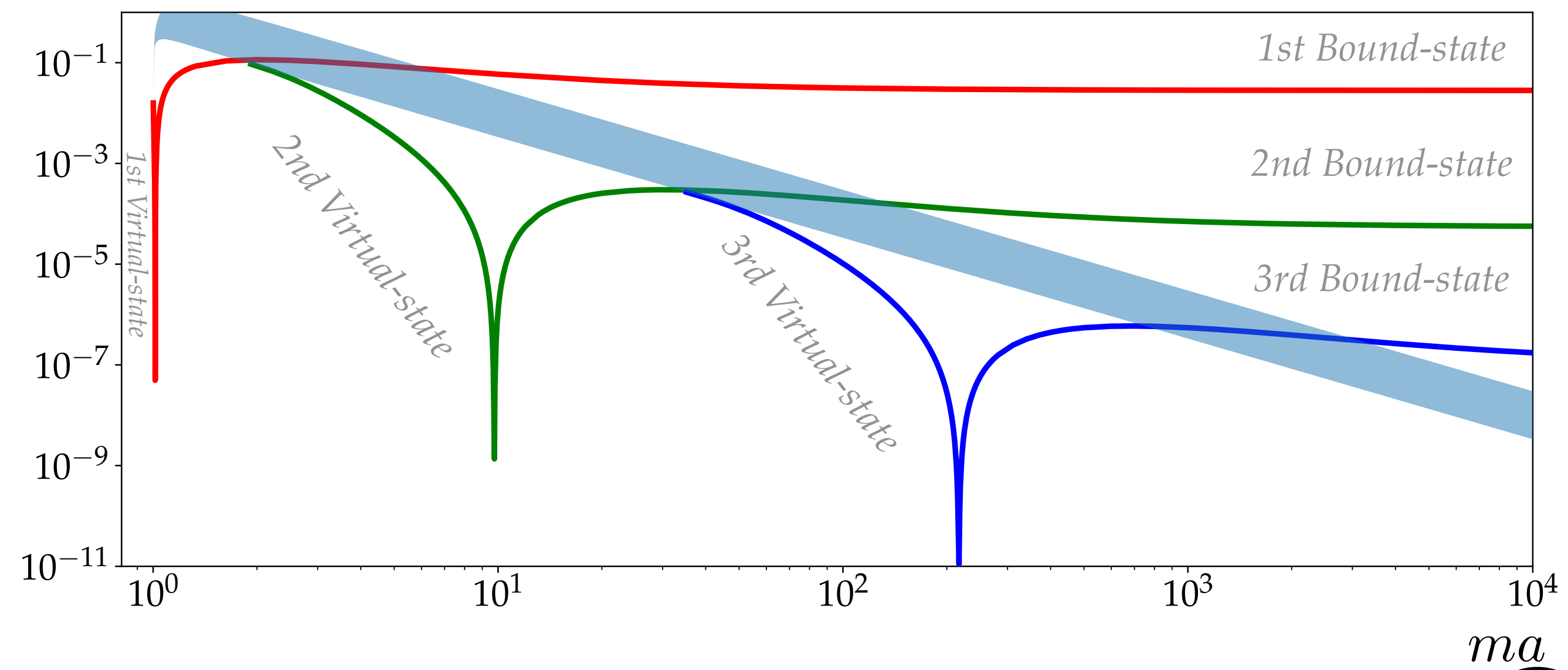


Amplitude on the Second Riemann Sheet

$$\mathcal{M}_{\varphi b}^{II}(s) = \frac{\mathcal{M}_{\varphi b}(s)}{1 + 2i\rho_{\varphi b}(s)\mathcal{M}_{\varphi b}(s)}$$

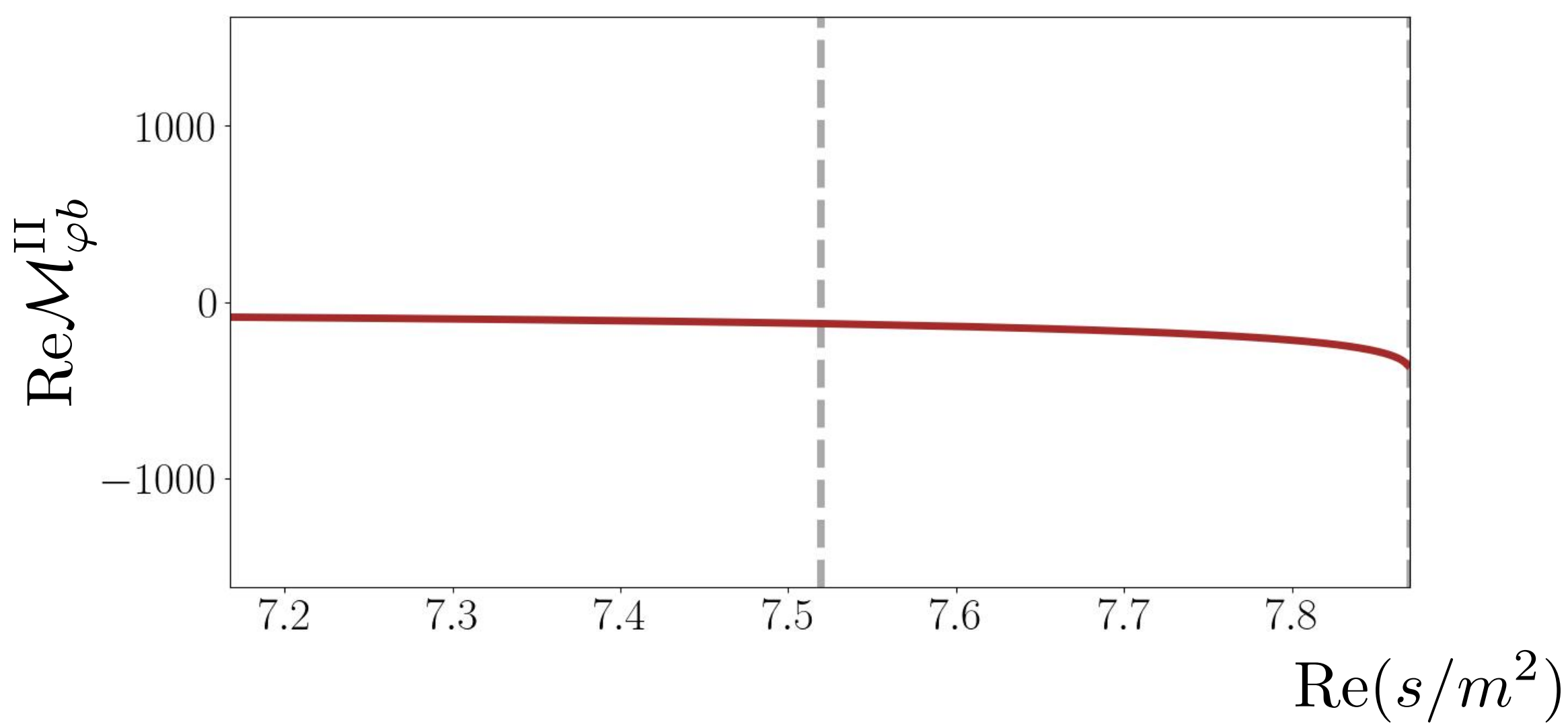
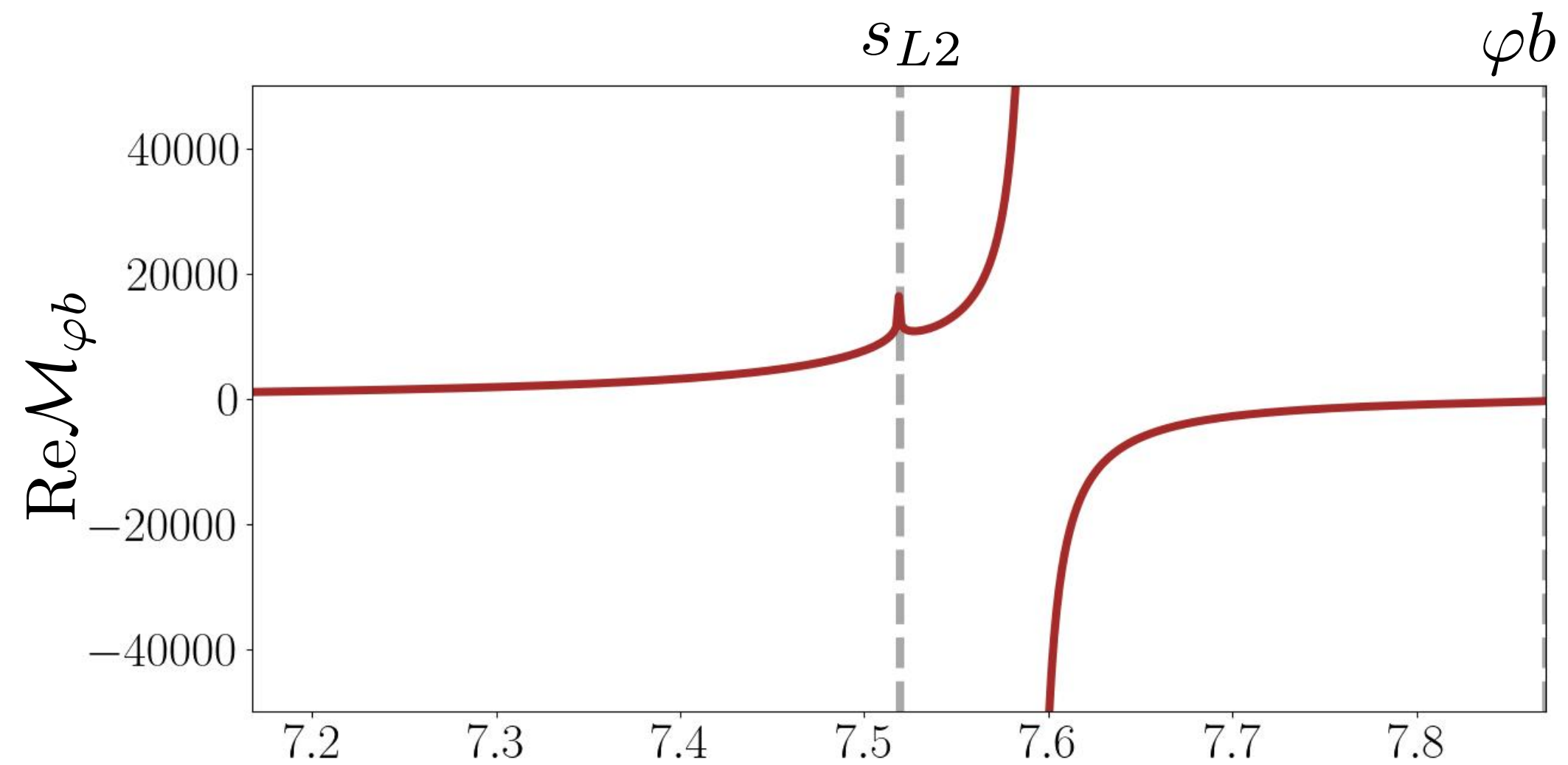


Binding energies E/m



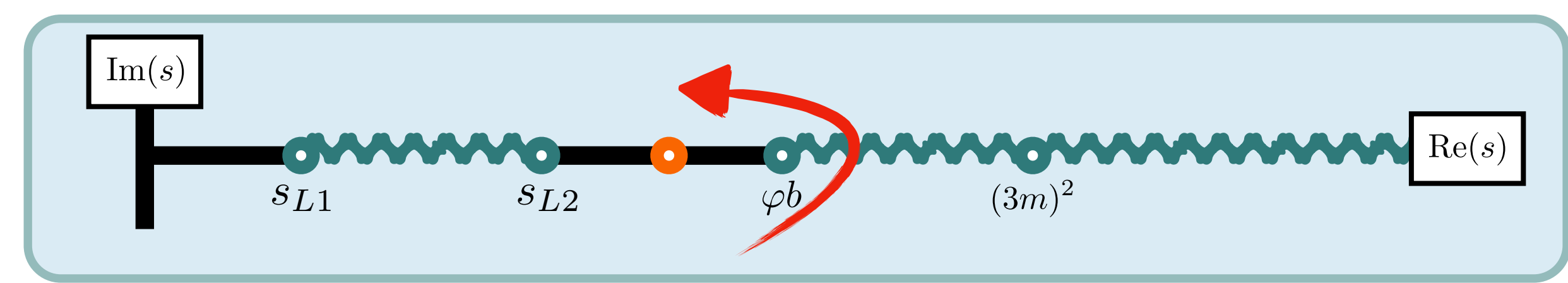
Three-Body Bound-states and Virtual States

$$ma = 2.3240$$

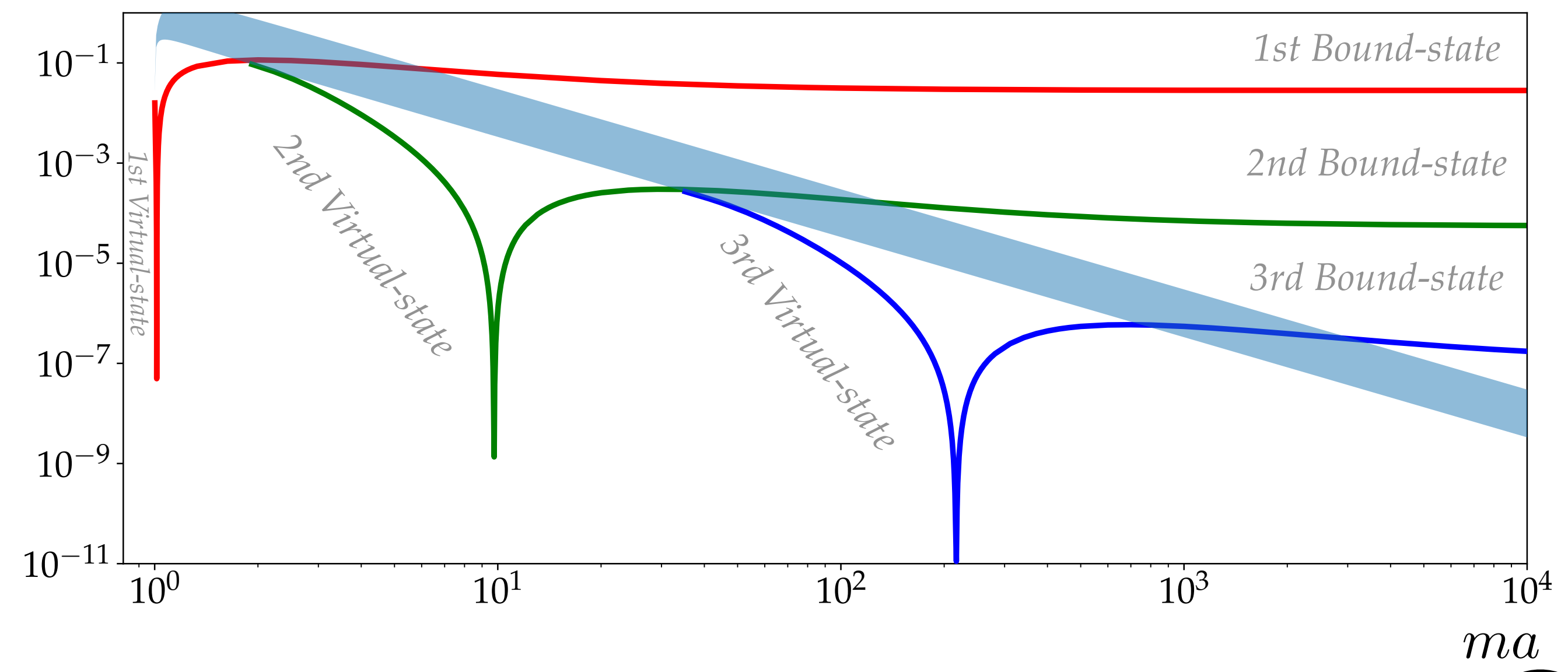


Amplitude on the Second Riemann Sheet

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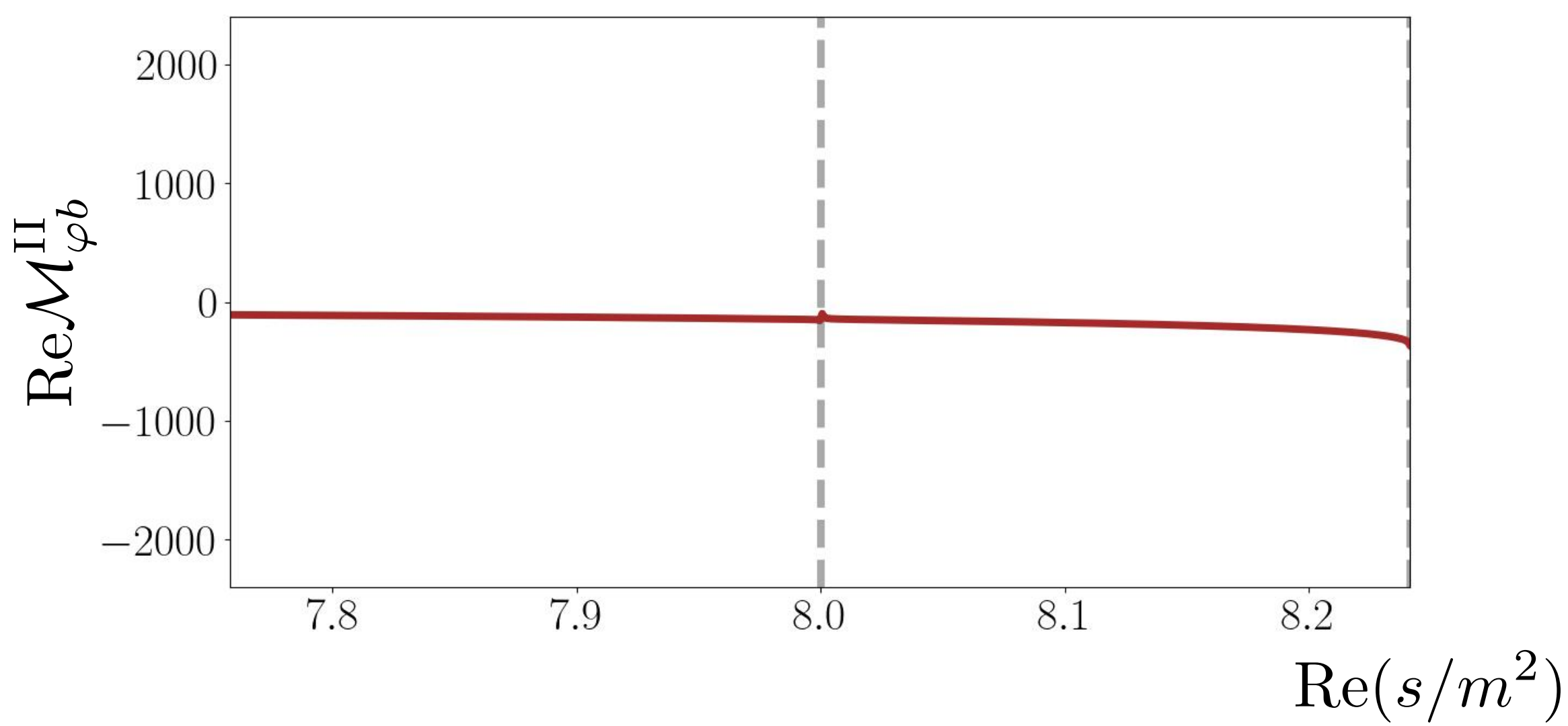
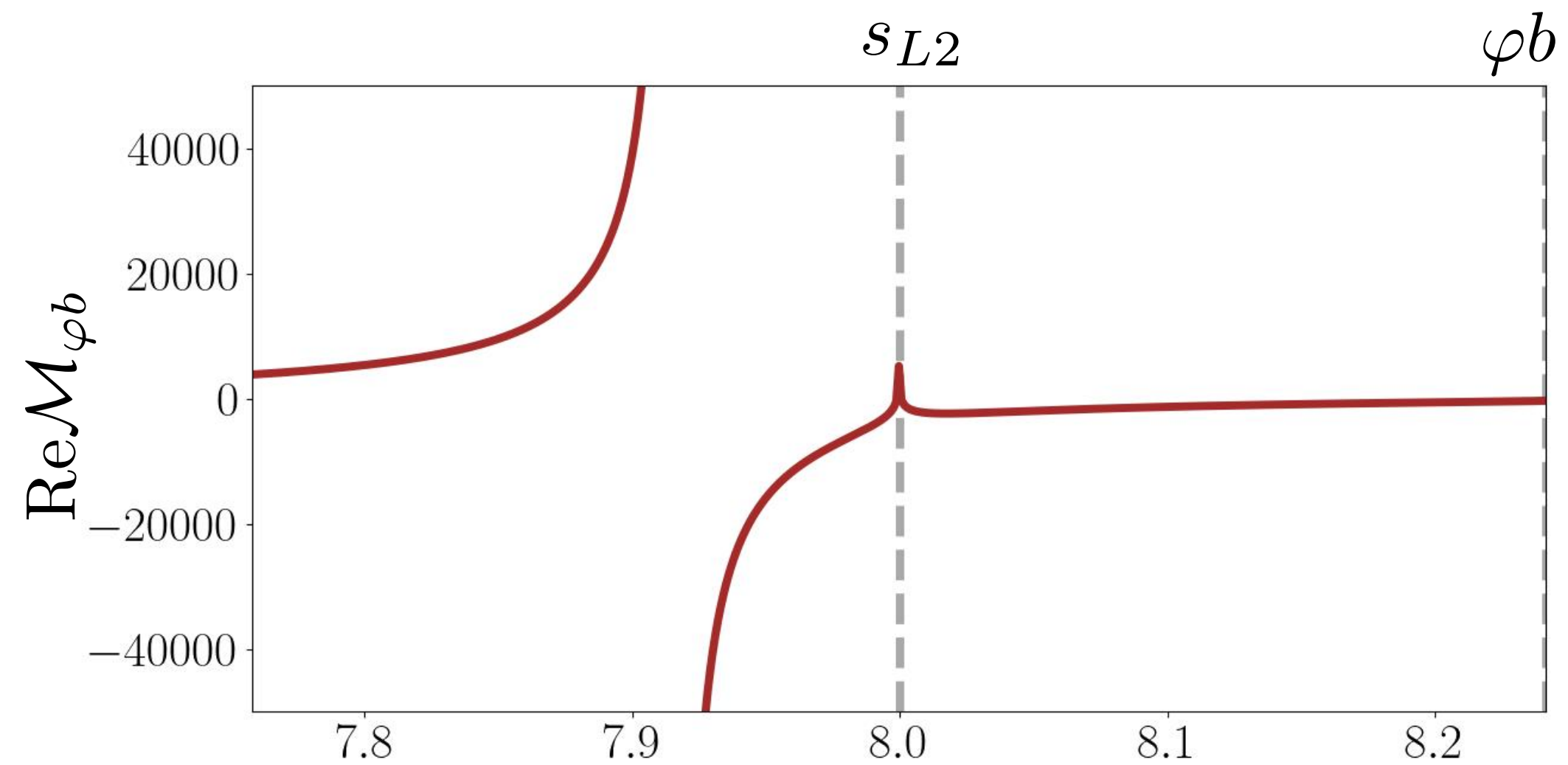


Binding energies E/m



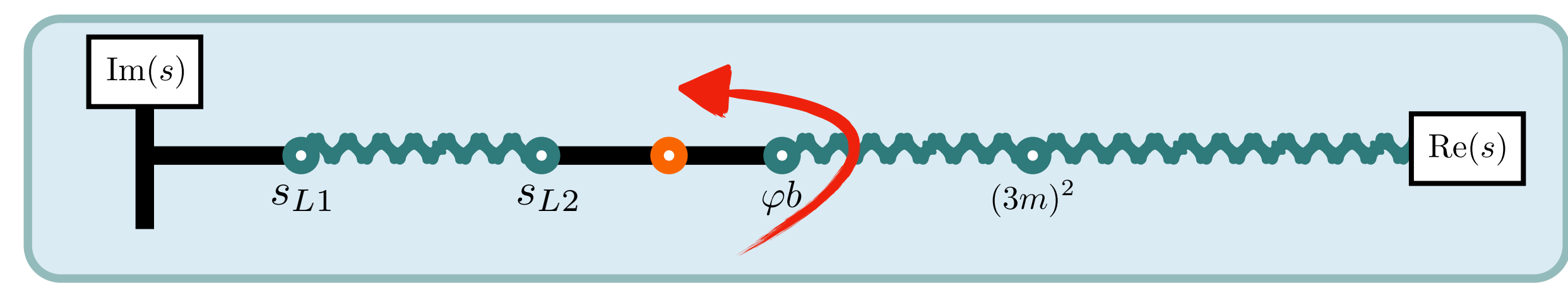
Three-Body Bound-states and Virtual States

$ma = 2.8280$

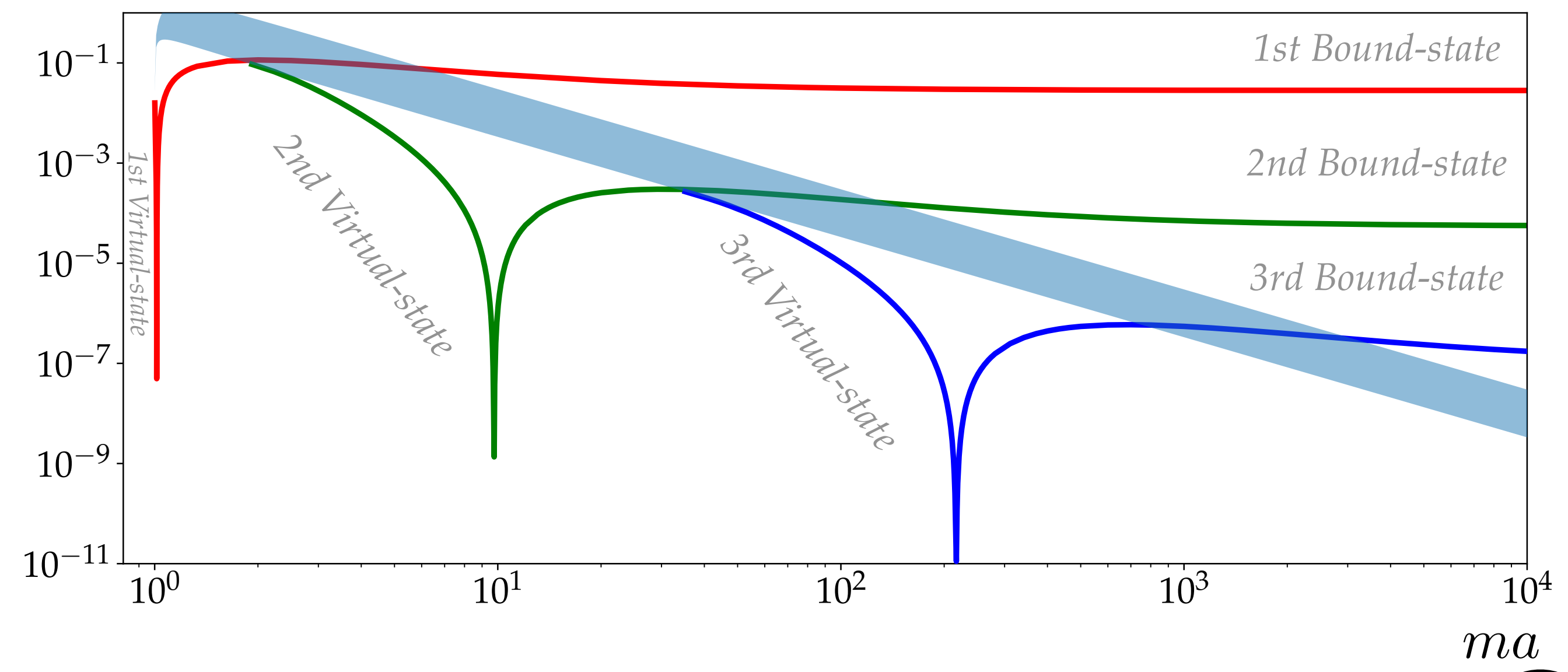


Amplitude on the Second Riemann Sheet

$$\mathcal{M}_{\varphi b}^{II}(s) = \frac{\mathcal{M}_{\varphi b}(s)}{1 + 2i\rho_{\varphi b}(s)\mathcal{M}_{\varphi b}(s)}$$

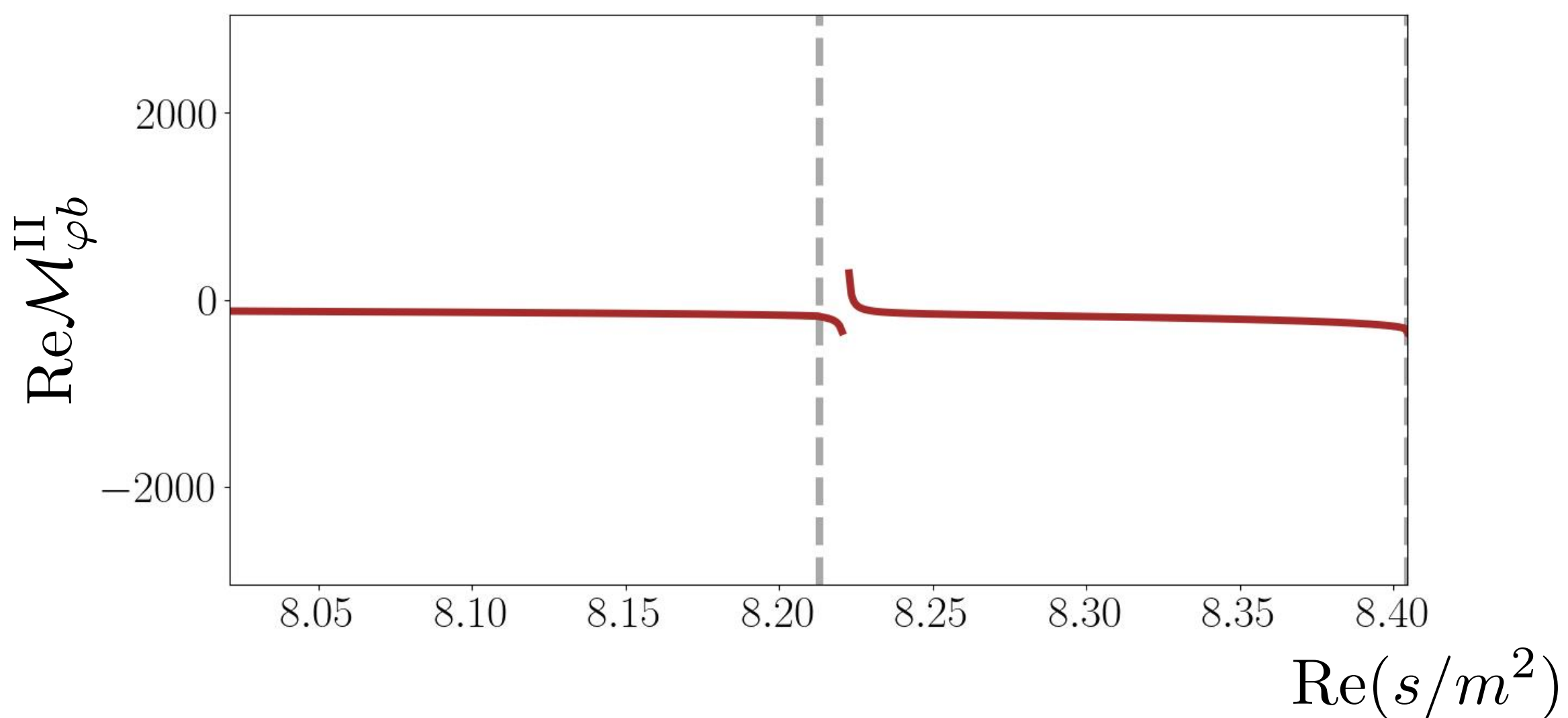
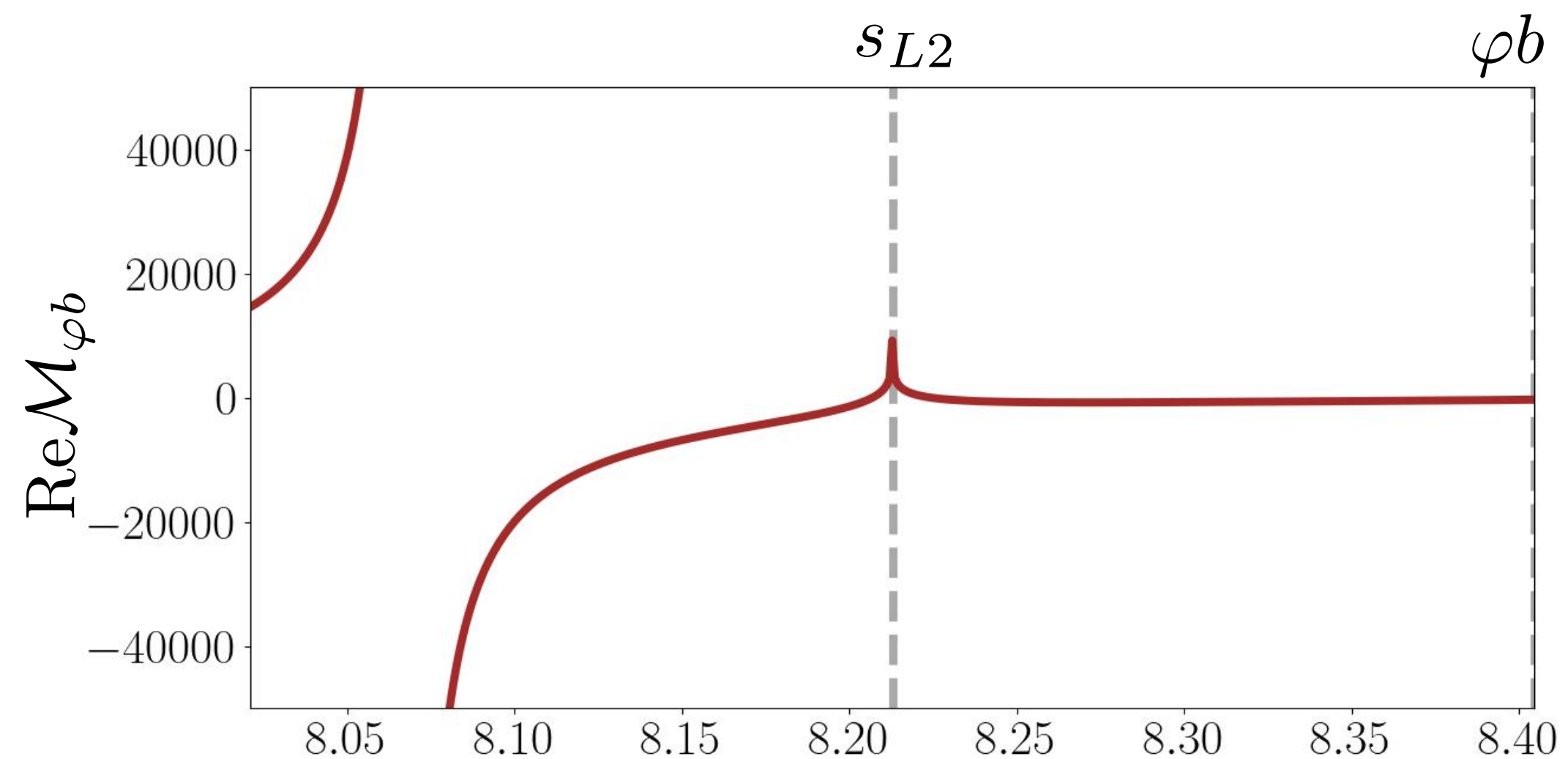


Binding energies E/m



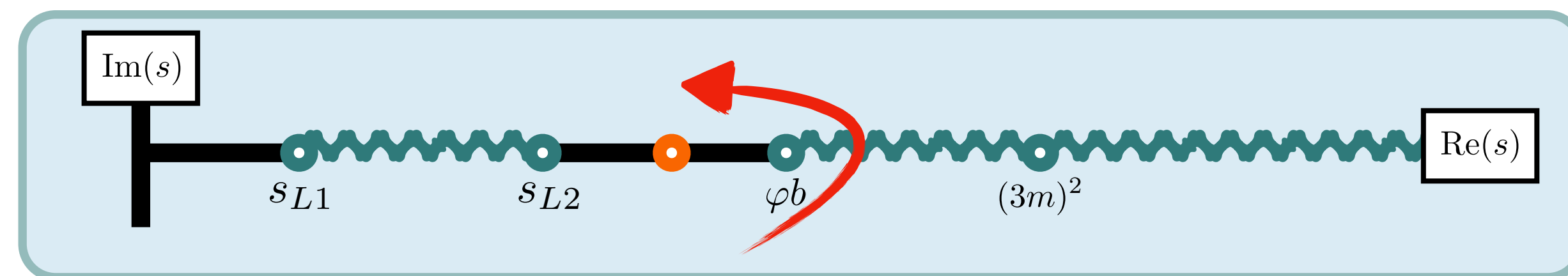
Three-Body Bound-states and Virtual States

$ma = 3.1880$

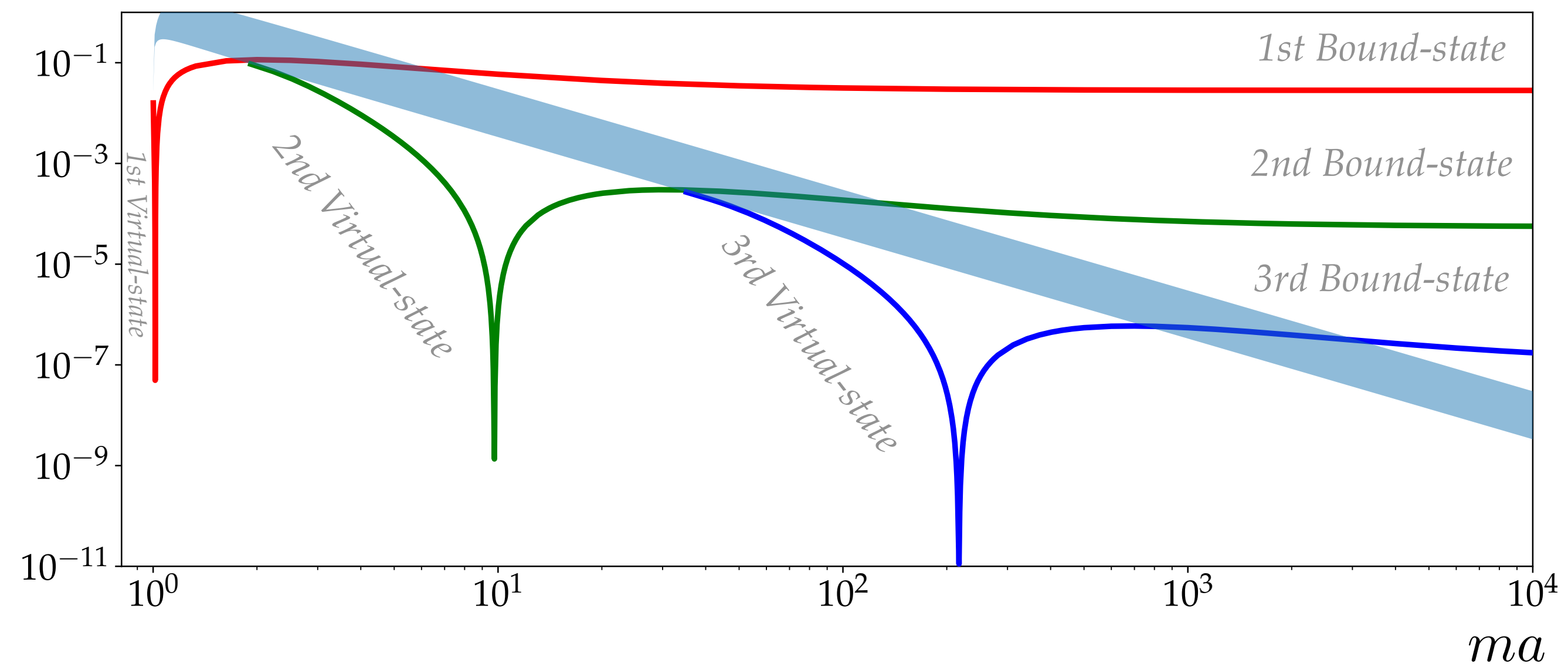


Amplitude on the Second Riemann Sheet

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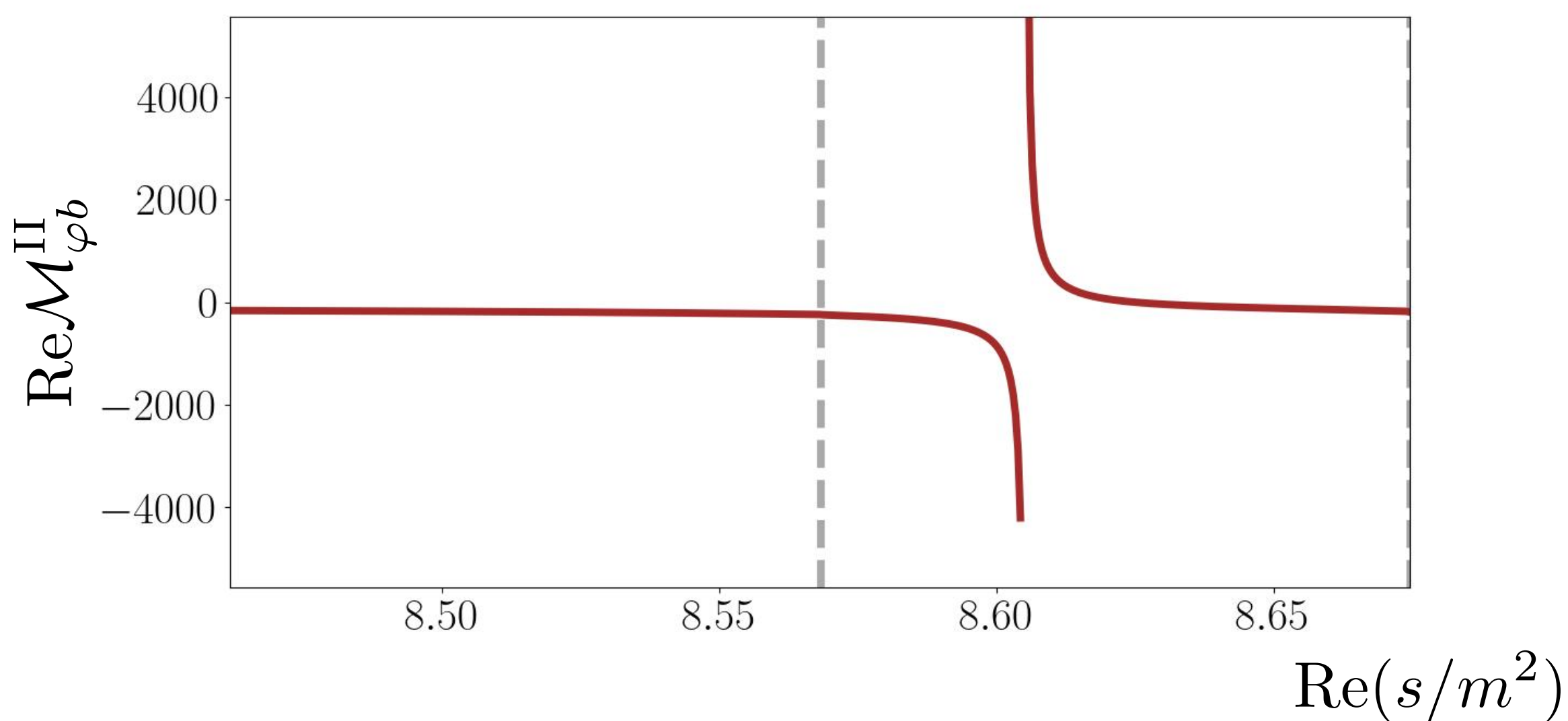
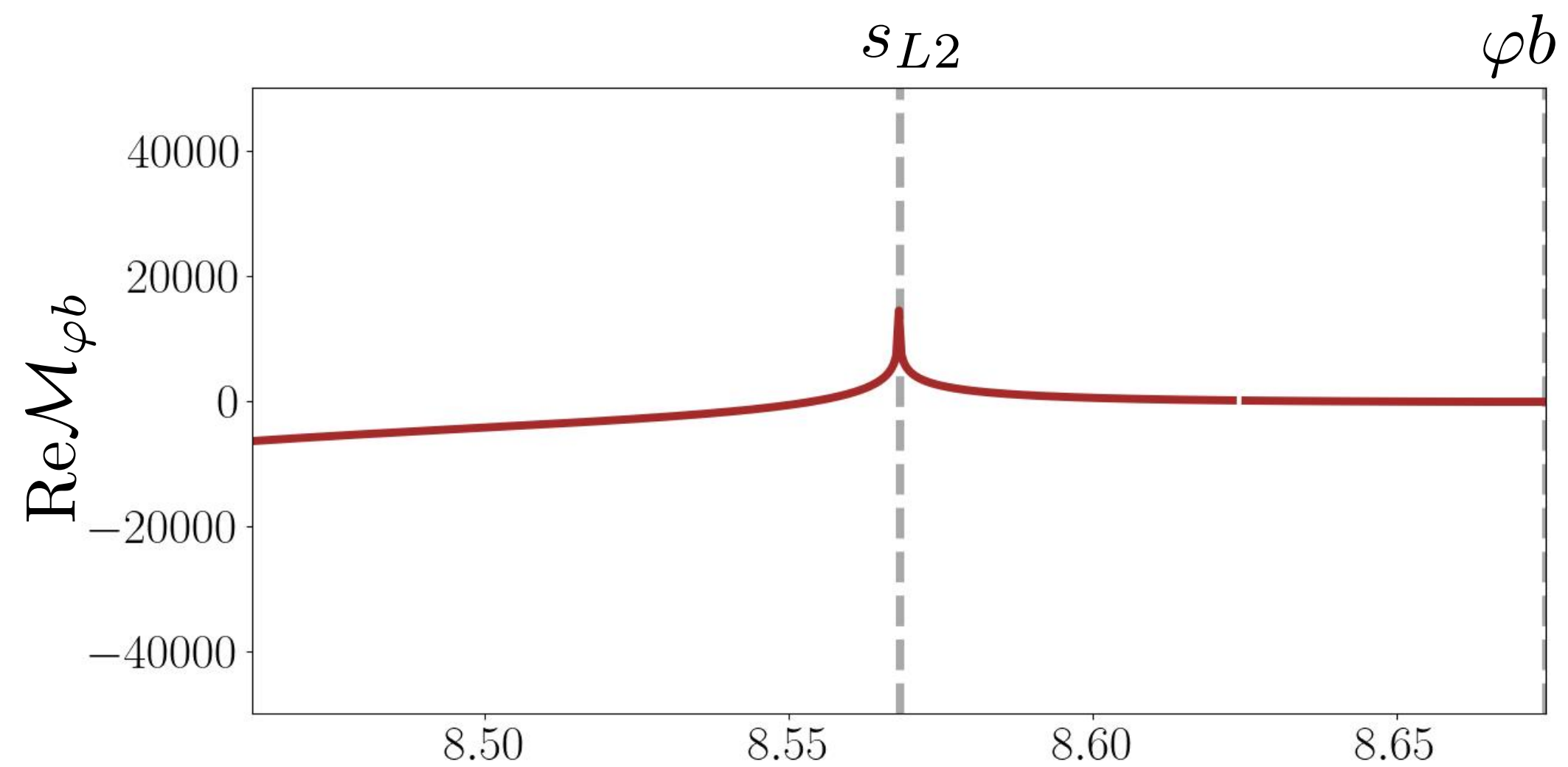


Binding energies E/m



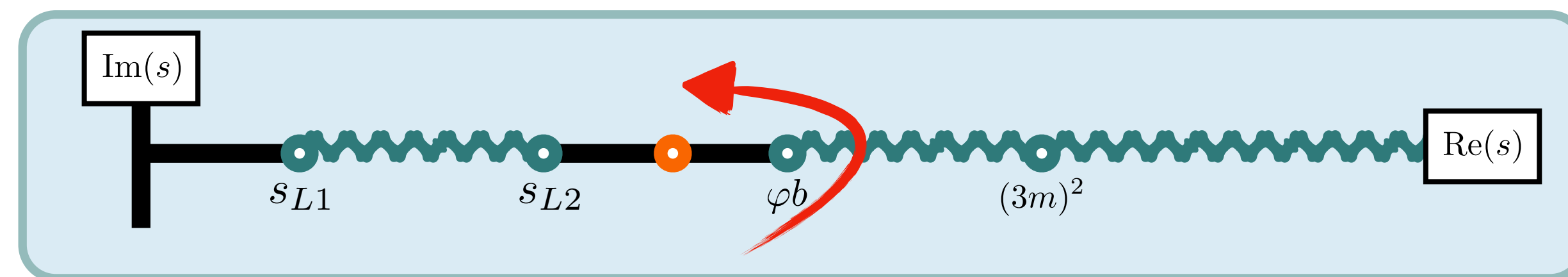
Three-Body Bound-states and Virtual States

$ma = 4.3040$

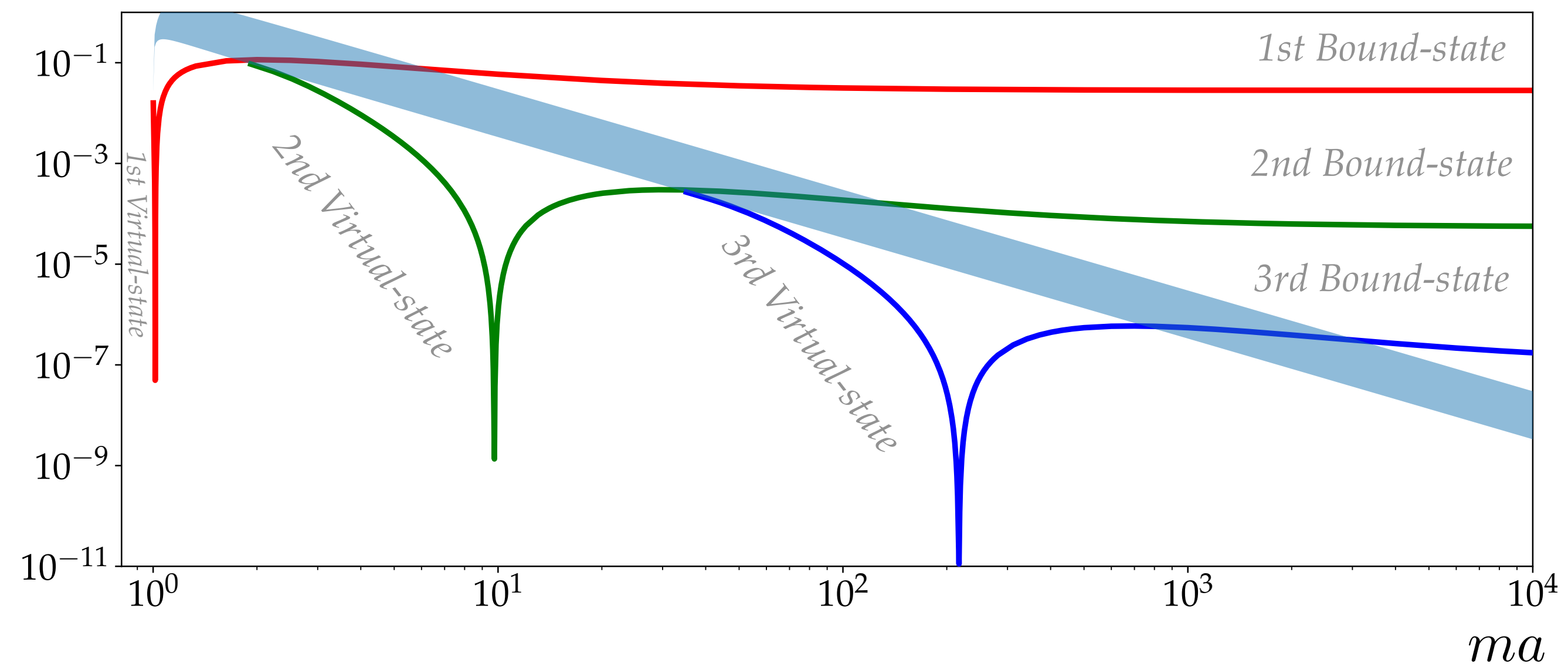


Amplitude on the Second Riemann Sheet

$$\mathcal{M}_{\varphi b}^{II}(s) = \frac{\mathcal{M}_{\varphi b}(s)}{1 + 2i\rho_{\varphi b}(s)\mathcal{M}_{\varphi b}(s)}$$

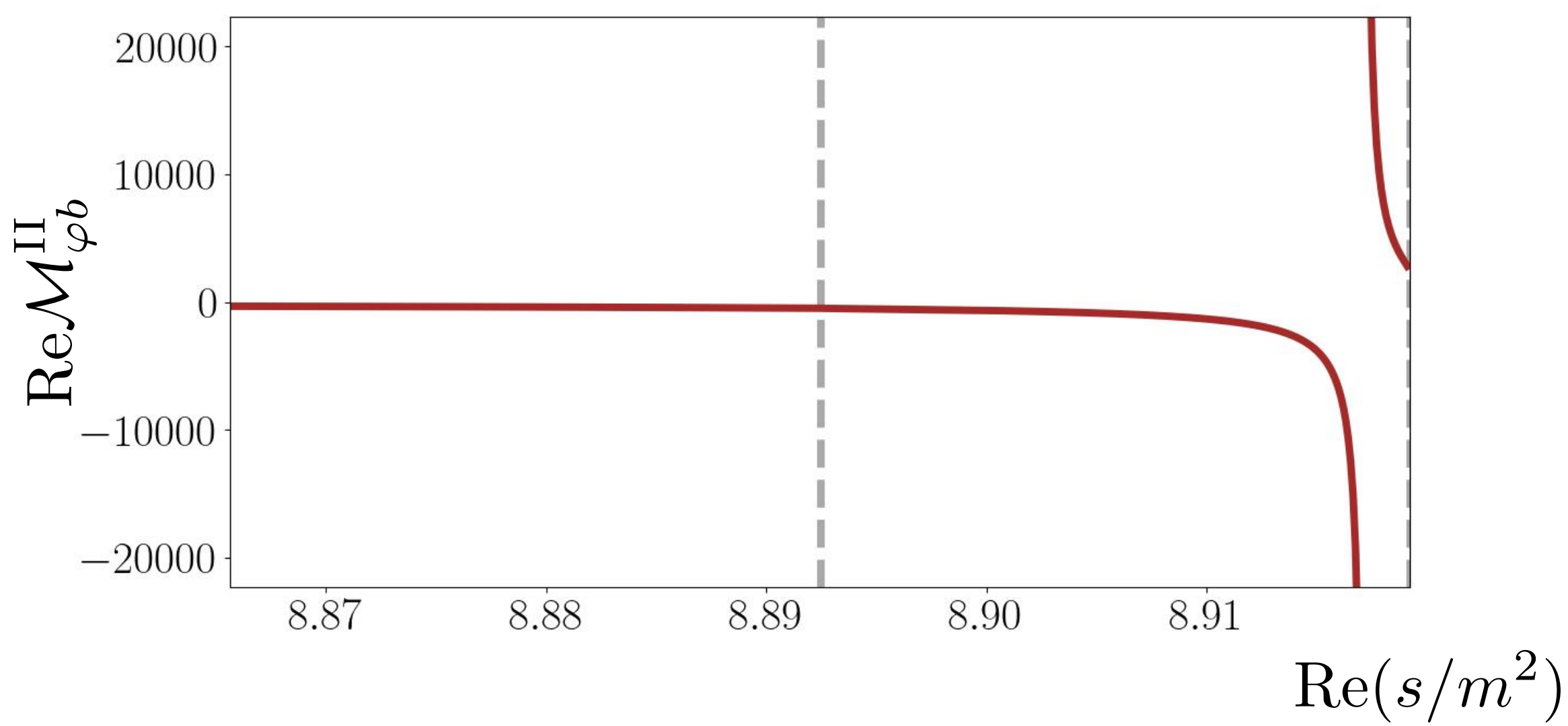
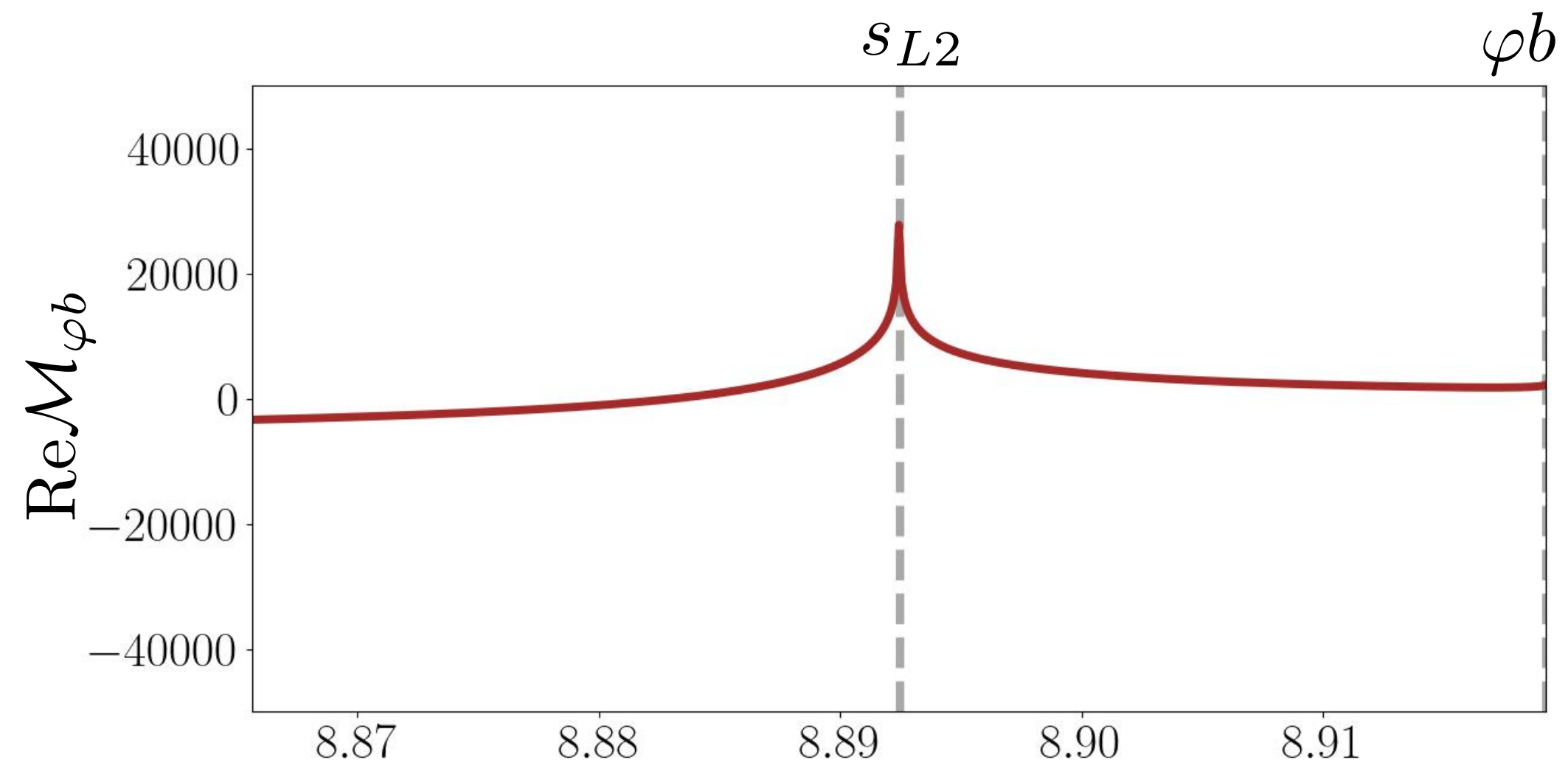


Binding energies E/m



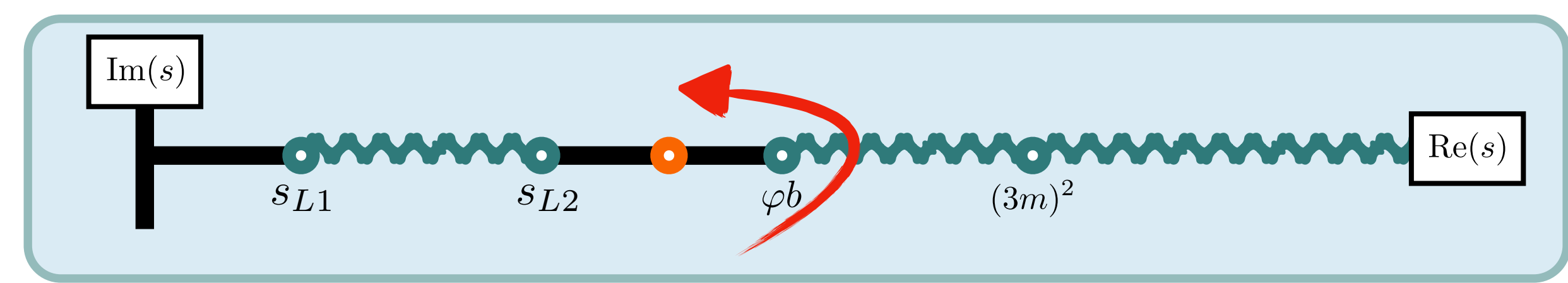
Three-Body Bound-states and Virtual States

$ma = 8.6240$

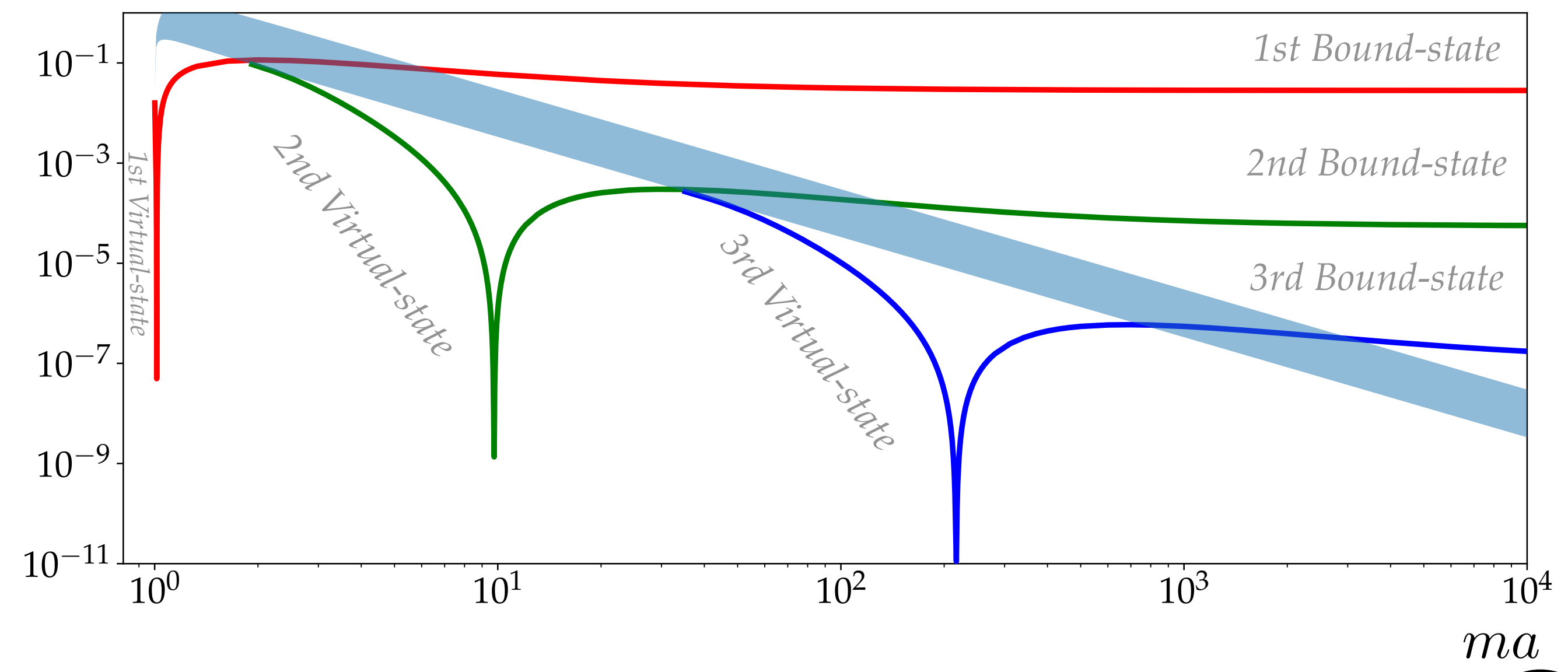


Amplitude on the Second Riemann Sheet

$$\mathcal{M}_{\varphi b}^{II}(s) = \frac{\mathcal{M}_{\varphi b}(s)}{1 + 2i\rho_{\varphi b}(s)\mathcal{M}_{\varphi b}(s)}$$

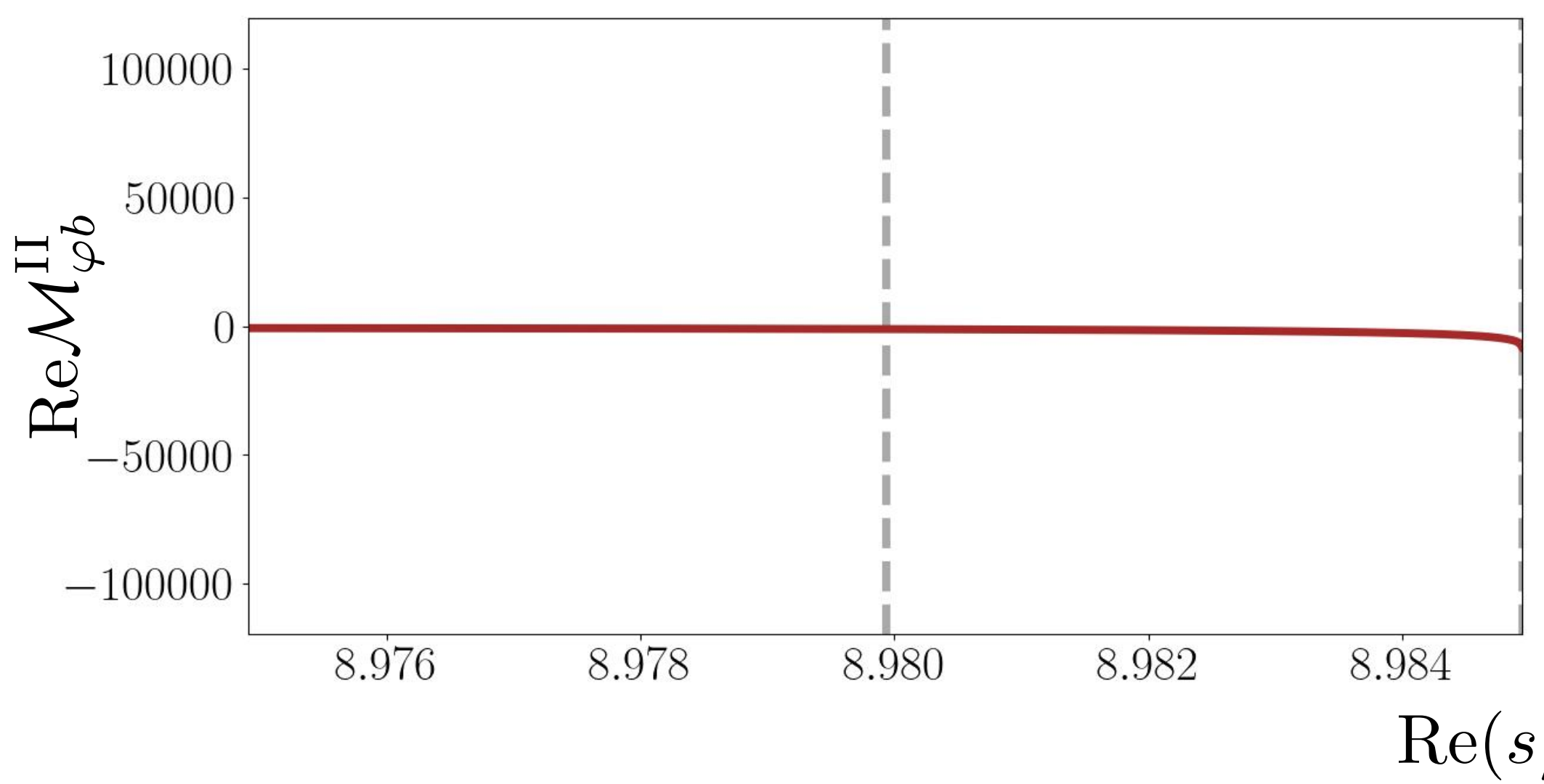
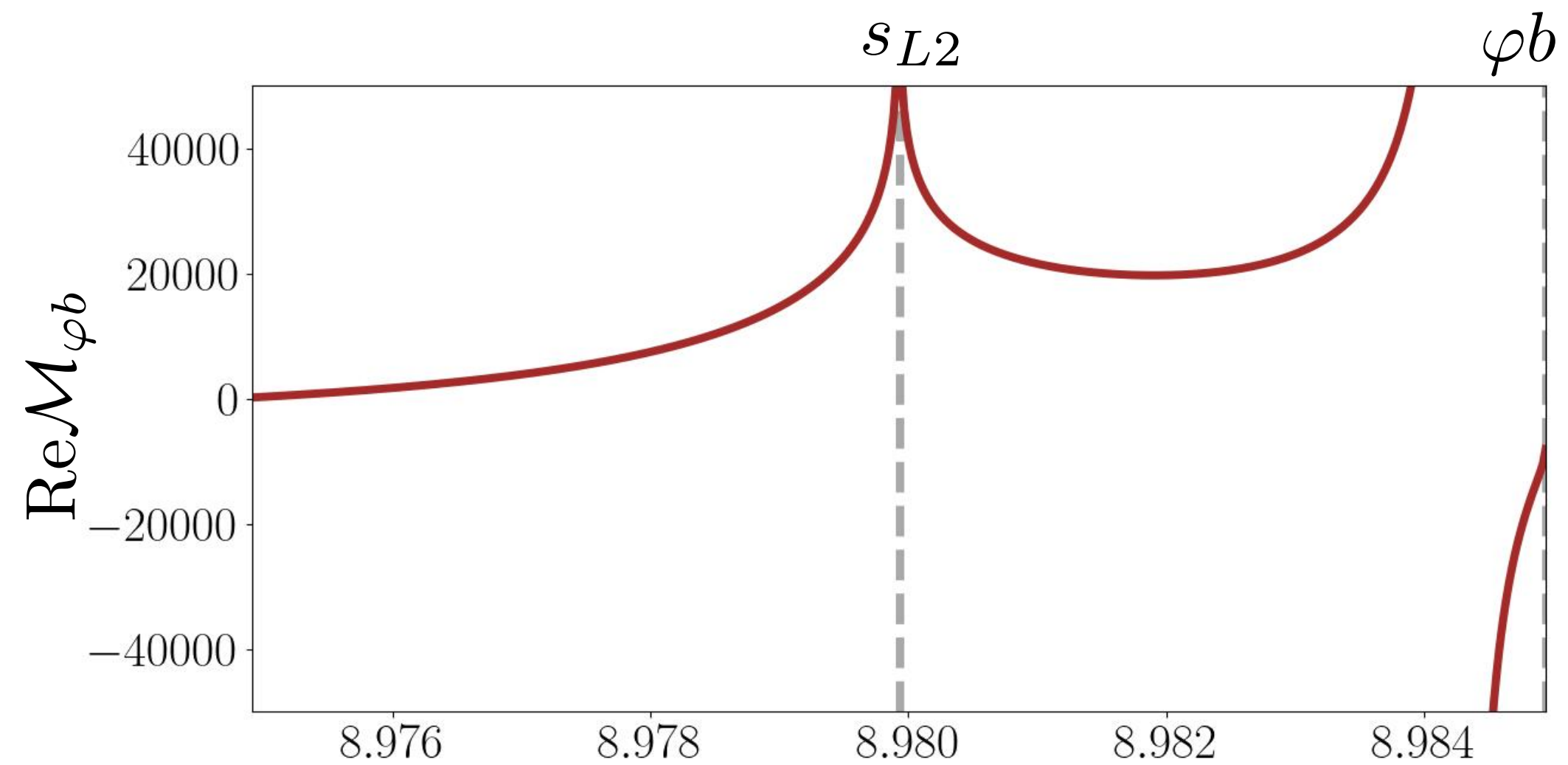


Binding energies E/m



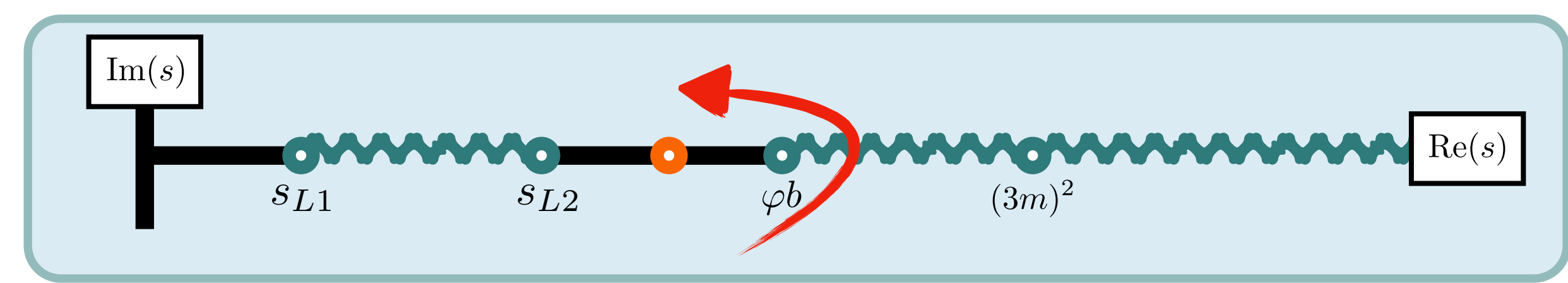
Three-Body Bound-states and Virtual States

$$ma = 19.9640$$

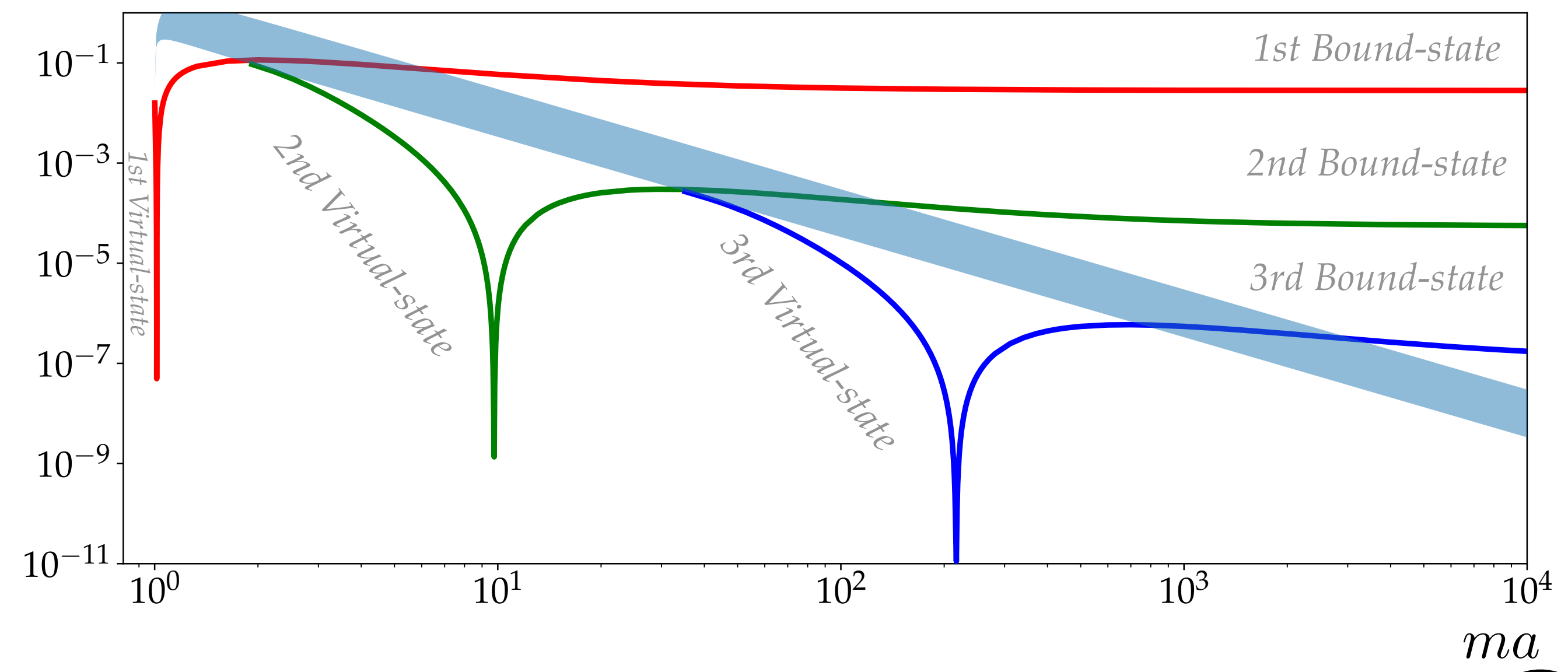


Amplitude on the Second Riemann Sheet

$$\mathcal{M}_{\varphi b}^{\text{II}}(s) = \frac{\mathcal{M}_{\varphi b}(s)}{1 + 2i\rho_{\varphi b}(s)\mathcal{M}_{\varphi b}(s)}$$



Binding energies E/m



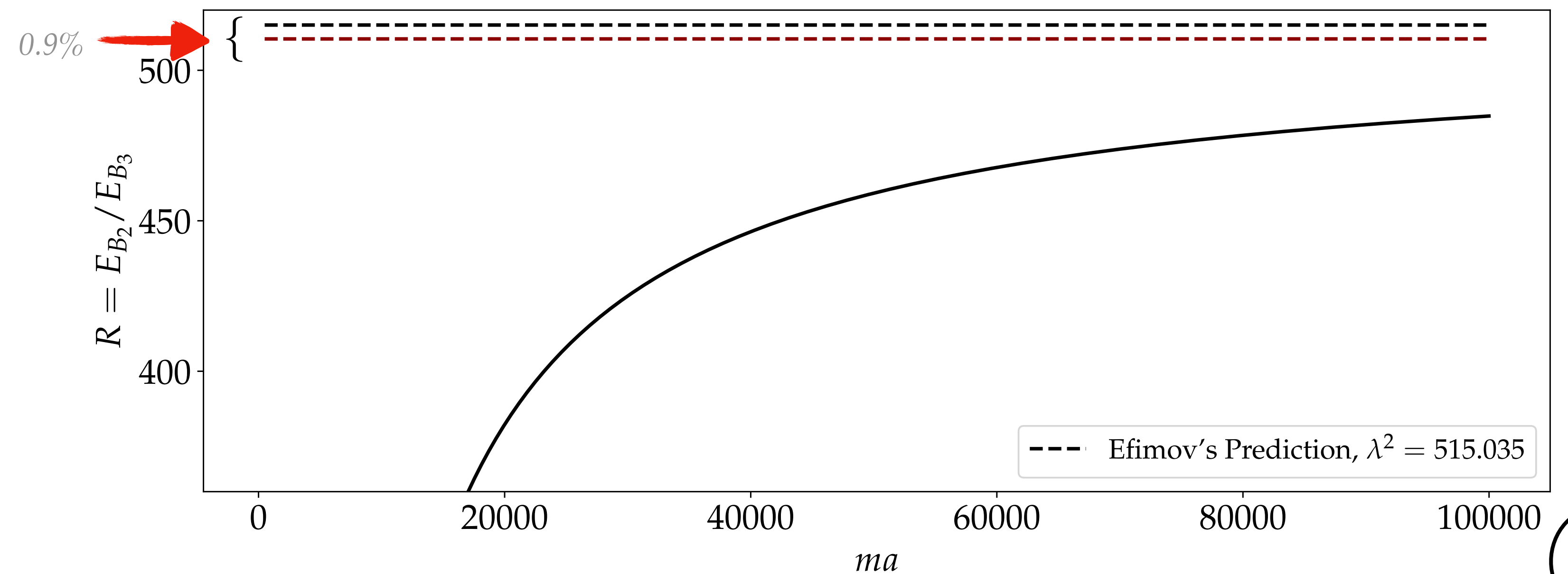
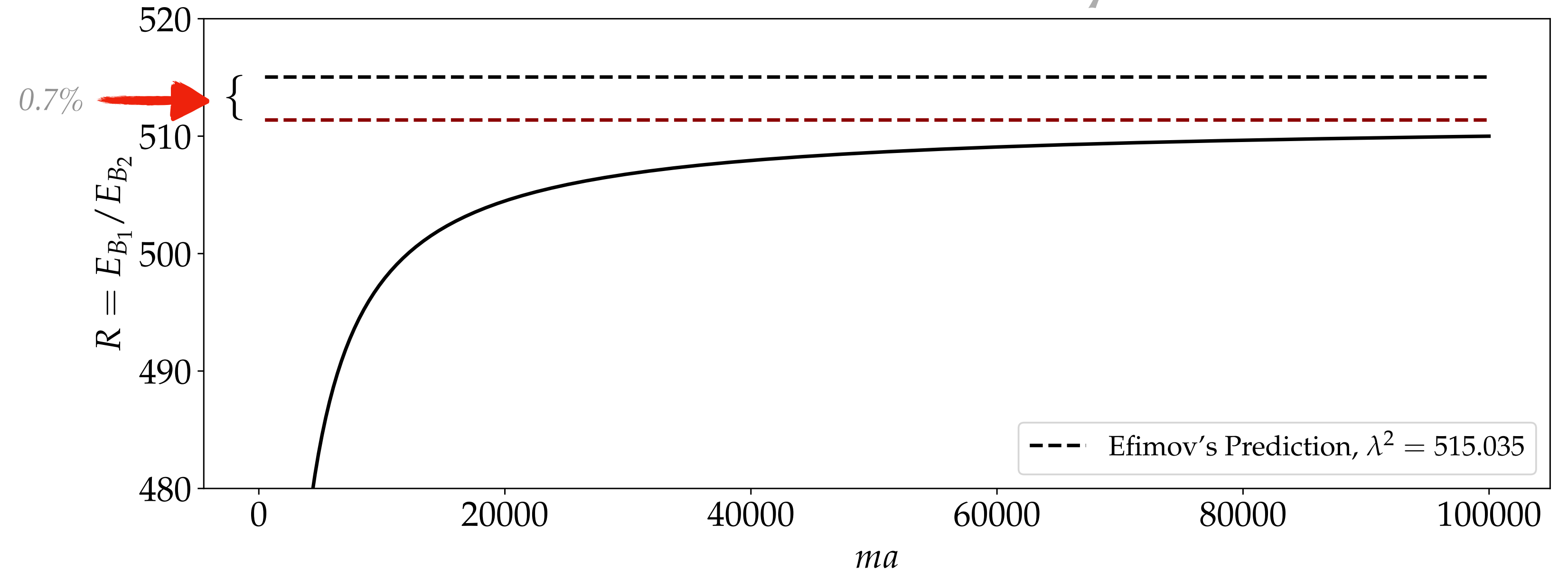
Recovering Efimov Physics

Preliminary Results

For finite two body scattering length, ma , we have a finite number of three body bound-states

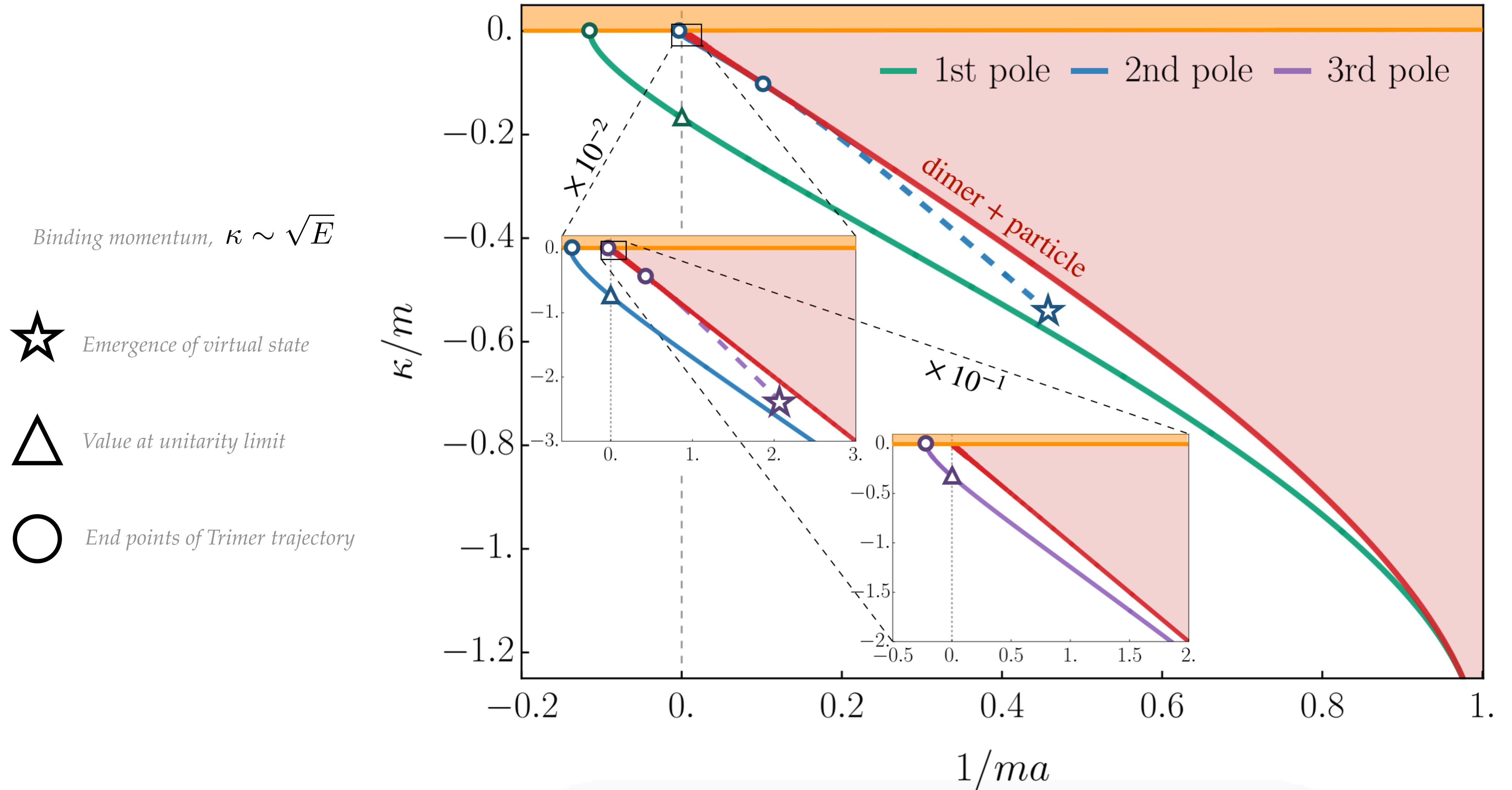
For a specific scattering length, we take the ratio of the binding energies of two consecutive three-body bound-states

For increasing scattering length, the ratio becomes close to Efimov's prediction



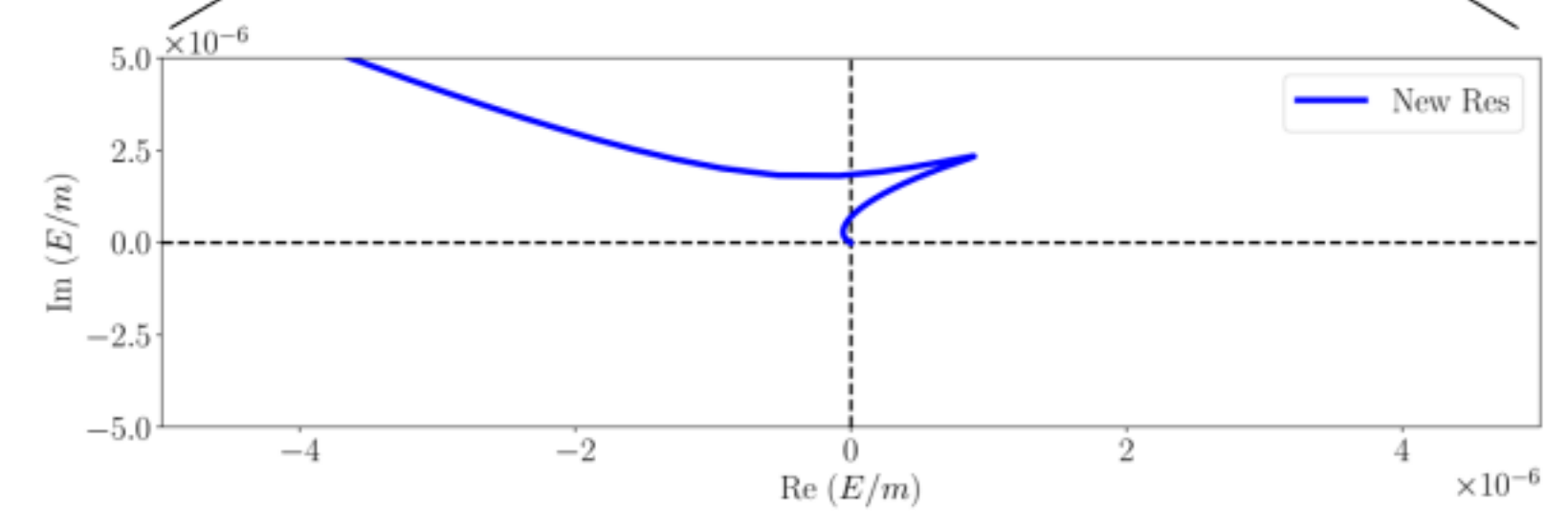
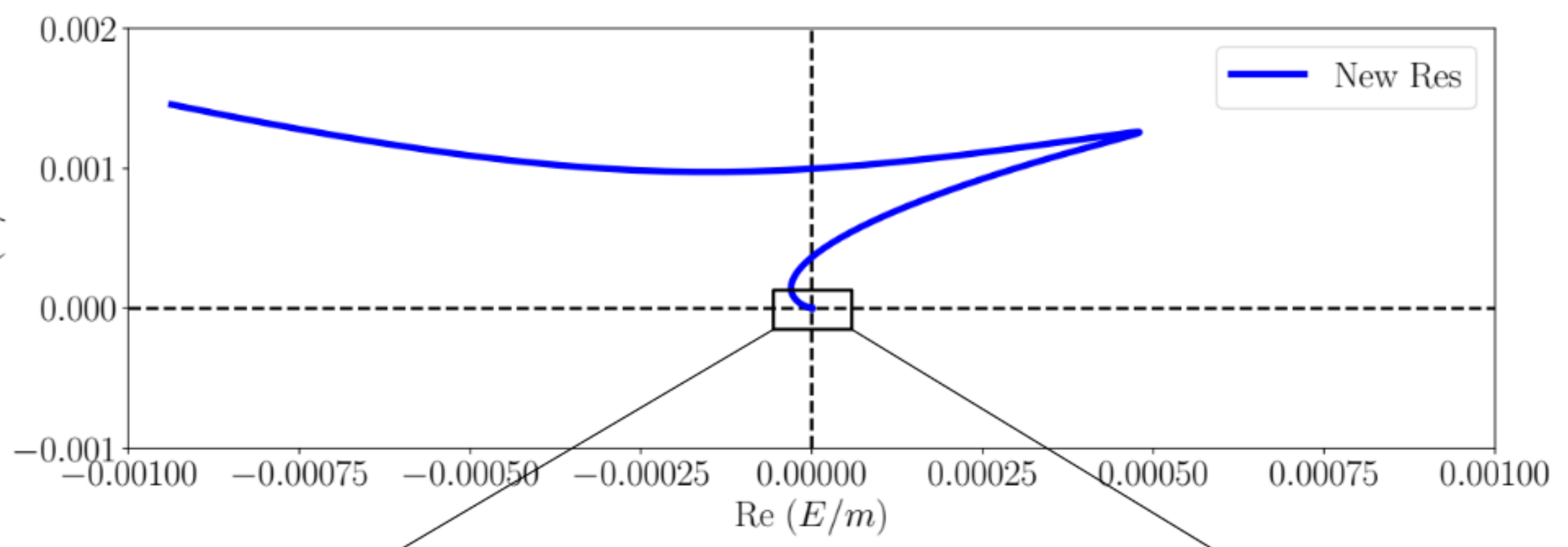
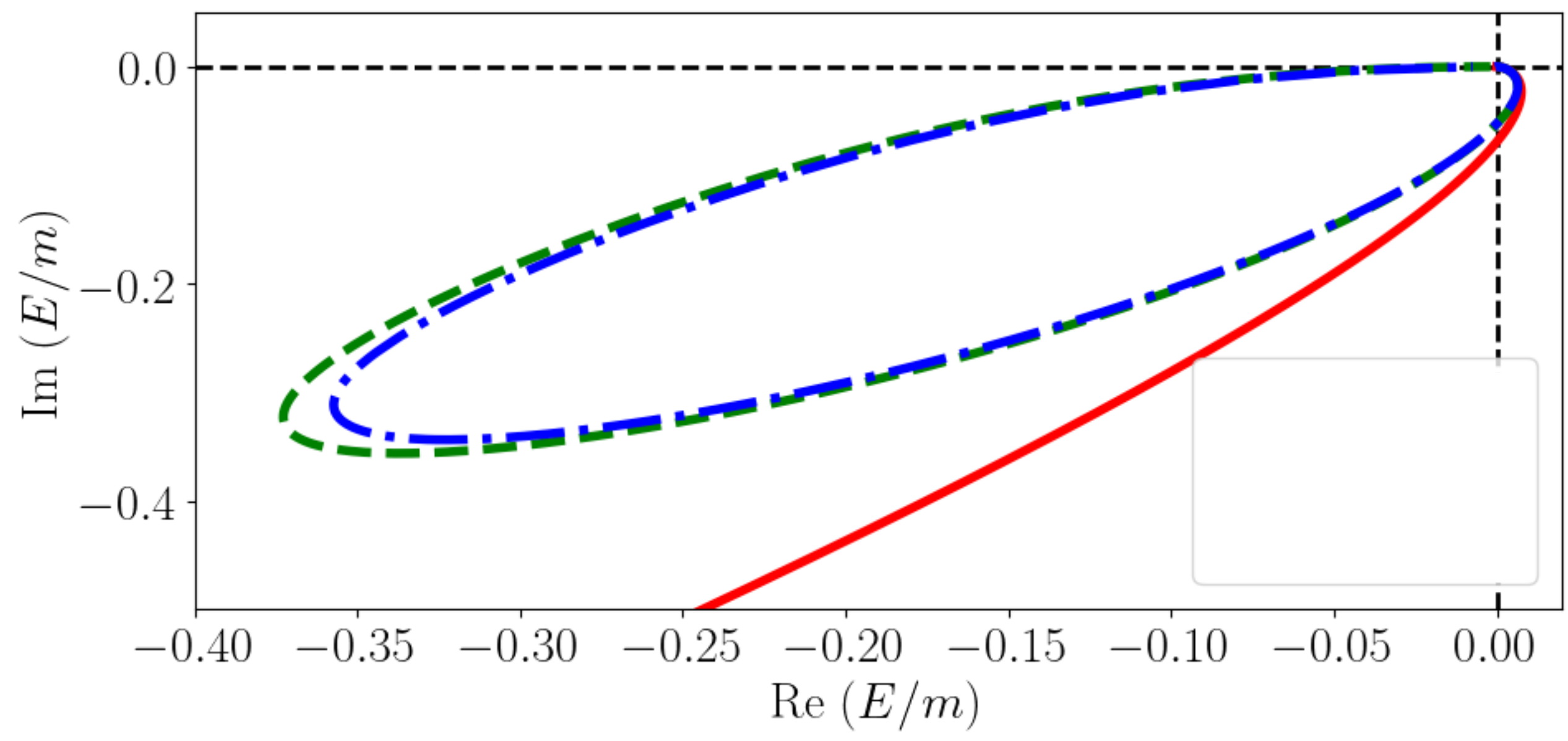
Trimer Trajectories

Preliminary Results



Resonances

Preliminary Results



Mesons

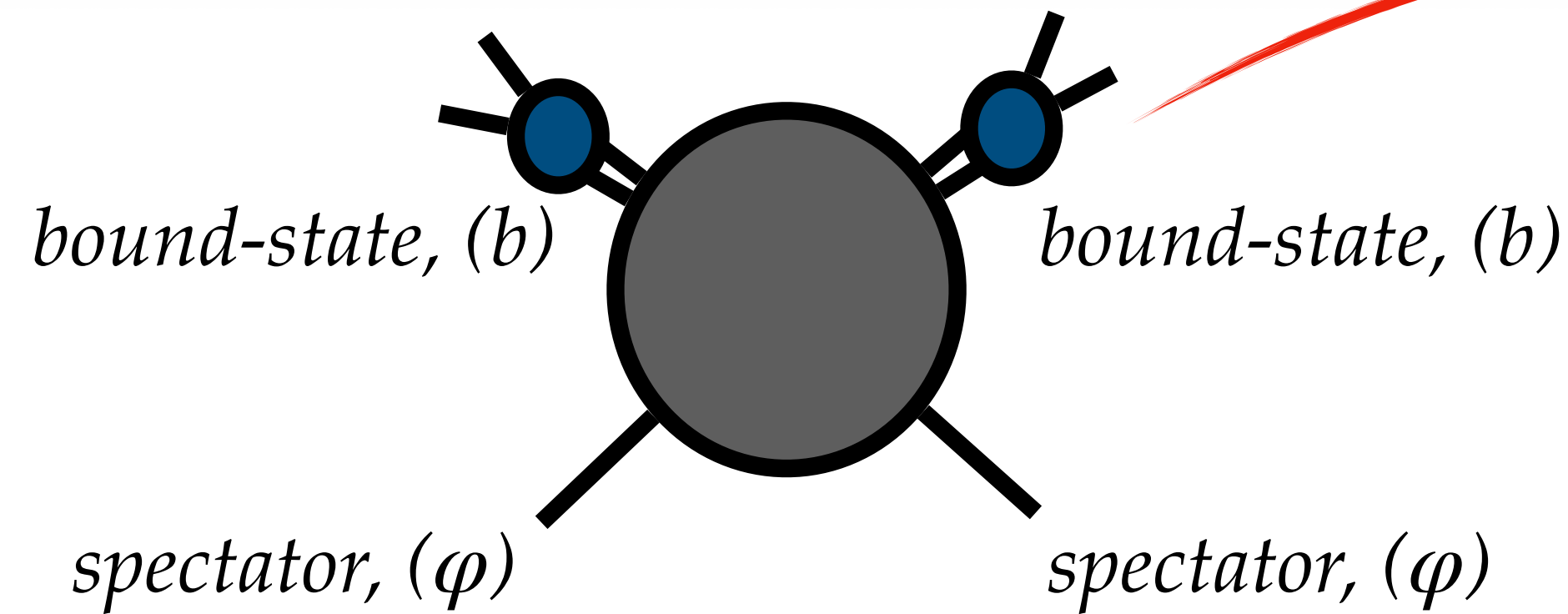
$n^{2s+1}\ell_J$	J^{PC}	$l = 1$ $u\bar{d}, \bar{u}d,$ $\frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$	$l = \frac{1}{2}$ $u\bar{s}, d\bar{s};$ $\bar{d}s, \bar{u}s$	$l = 0$ f'	$l = 0$ f
1^1S_0	0^{-+}	π	K	η	$\eta'(958)$
1^3S_1	1^{--}	$\rho(770)$	$K^*(892)$	$\phi(1020)$	$\omega(782)$
1^1P_1	1^{+-}	$b_1(1235)$	K_{1B}^a	$h_1(1415)$	$h_1(1170)$
1^3P_0	0^{++}	$a_0(1450)$	$K_0^*(1430)$	$f_0(1710)$	$f_0(1370)$
1^3P_1	1^{++}	$a_1(1260)$	K_{1A}^a	$f_1(1420)$	$f_1(1285)$
1^3P_2	2^{++}	$a_2(1320)$	$K_2^*(1430)$	$f_2'(1525)$	$f_2(1270)$
1^1D_2	2^{-+}	$\pi_2(1670)$	$K_2(1770)^a$	$\eta_2(1870)$	$\eta_2(1645)$
1^3D_1	1^{--}	$\rho(1700)$	$K^*(1680)^b$	$\phi(2170)^d$	$\omega(1650)$
1^3D_2	2^{--}		$K_2(1820)^a$		
1^3D_3	3^{--}	$\rho_3(1690)$	$K_3^*(1780)$	$\phi_3(1850)$	$\omega_3(1670)$
1^3F_4	4^{++}	$a_4(1970)$	$K_4^*(2045)$	$f_4(2300)$	$f_4(2050)$
1^3G_5	5^{--}	$\rho_5(2350)$	$K_5^*(2380)$		
2^1S_0	0^{-+}	$\pi(1300)$	$K(1460)$	$\eta(1475)^c$	$\eta(1295)$
2^3S_1	1^{--}	$\rho(1450)$	$K^*(1410)^b$	$\phi(1680)$	$\omega(1420)$
2^3P_1	1^{++}	$a_1(1640)$			
2^3P_2	2^{++}	$a_2(1700)$	$K_2^*(1980)$	$f_2(1950)$	$f_2(1640)$

$n^{2s+1}\ell_J$	J^{PC}	$l = 0$ $c\bar{c}$	$l = \frac{1}{2}$ $c\bar{u}, c\bar{d};$ $\bar{c}u, \bar{c}d$	$l = 0$ $c\bar{s};$ $\bar{c}s$
1^1S_0	0^{-+}	$\eta_c(1S)$	D	D_s^\pm
1^3S_1	1^{--}	$J/\psi(1S)$	D^*	$D_s^{*\pm}$
1^3P_0	0^{++}	$\chi_{c0}(1P)$	$D_0^*(2300)^a$	$D_{s0}^*(2317)^{\pm b}$
1^3P_1	1^{++}	$\chi_{c1}(1P)$	$D_1(2430)$	$D_{s1}(2460)^{\pm b}$
1^1P_1	1^{+-}	$h_c(1P)$	$D_1(2420)$	$D_{s1}(2536)^\pm$
1^3P_2	2^{++}	$\chi_{c2}(1P)$	$D_2^*(2460)$	$D_{s2}^*(2573)^\pm$
2^1S_0	0^{-+}	$\eta_c(2S)$	$D_0(2500)^0$	$D_{s0}(2590)^+$
2^3S_1	1^{--}	$\psi(2S)$	$D_1^*(2760)^0$	$D_{s1}^*(2700)^{\pm c}$
1^3D_1	1^{--}	$\psi(3770)$		$D_{s1}^*(2860)^{\pm c}$
1^3D_2	2^{--}	$\psi_2(3823)$	$D_2(2740)^0$	
2^3P_J	$0^{++}, 1^{++}$	$\chi_{c0}(3860)$		
	2^{++}	$\chi_{c2}(3930)$		
3^3S_1	1^{--}	$\psi(4040)$		
2^3D_1	1^{--}	$\psi(4160)$		
4^3S_1	1^{--}	$\psi(4415)$		
1^3D_3	3^{--}	$\psi_3(3842)$	$D_3^*(2750)$	$D_{s3}^*(2860)^\pm$

Scalar and tensor resonances

	Mass (MeV)	Width (MeV)	$\mathcal{B}(f \rightarrow \pi\pi)$ (%)	$\mathcal{B}(f \rightarrow K\bar{K})$ (%)	$\mathcal{B}(f \rightarrow 4\pi)$ (%)
$f_0(1370)$	1200–1500	300–500	< 10 [11]	35 ± 13 [79]	> 72 [33]
$f_0(1500)$	1506 ± 6	112 ± 9	34.5 ± 2.2	8.5 ± 1.0	48.9 ± 3.3
$f_0(1710)$	1704 ± 12	123 ± 18	$3.9^{+3.0}_{-2.4}$ [80]	36 ± 12 [26]	—
[$f_0(2020)$]	1992 ± 16	442 ± 60	—	—	—
[$f_0(2200)$]	2187 ± 14	207 ± 40	—	—	—
[$f_0(2330)$]	2324 ± 35	195 ± 71	—	—	—
$f_2(1270)$	1275.5 ± 0.8	$186.7^{+2.2}_{-2.5}$	$84.2^{+2.9}_{-0.9}$	$4.6^{+0.5}_{-0.4}$	$10.4^{+1.6}_{-3.7}$ ^(a)
[$f_2(1430)$]	≈ 1430	—	—	—	—
$f'_2(1525)$	1517.4 ± 2.5	86 ± 5	0.83 ± 0.16	87.6 ± 2.2	—
[$f_2(1565)$]	1542 ± 19	122 ± 13	—	—	—
[$f_2(1640)$]	1639 ± 6	99^{+60}_{-40}	—	—	—
[$f_2(1810)$]	1815 ± 12	197 ± 22	21^{+2}_{-3} [80]	$2 \times 0.3^{+1.9}_{-0.2}$ [80]	—
[$f_2(1910)$]	1900 ± 9 ^(b)	167 ± 21 ^(b)	—	—	—
$f_2(1950)$	1936 ± 12	464 ± 24	—	—	—
$f_2(2010)$	2011^{+62}_{-76}	202^{+67}_{-62}	—	—	—
[$f_2(2150)$]	2157 ± 12	152 ± 30	—	—	—
[$f_2(2300)$]	2297 ± 28	149 ± 41	—	—	—

Consistency checks using Toy Model:



$$\mathcal{D} = \frac{-g}{\sigma_p - m_b^2} \mathcal{M}_{\varphi b} \frac{-g}{\sigma_k - m_b^2}$$

1. System dynamics fixed by two-body scattering length only.
2. Bound state pole in the integration interval, numerically most demanding scenario.
3. Finite volume results are available for comparison (Romero-López, Sharpe, Blanton, Briceño and Hansen, 2019)
4. Hints at **Efimov Physics**:

Unitarity Limit:

$$q \cot \delta_0 = -1/a = 0$$

Two body scattering amplitude has a pole at the threshold:

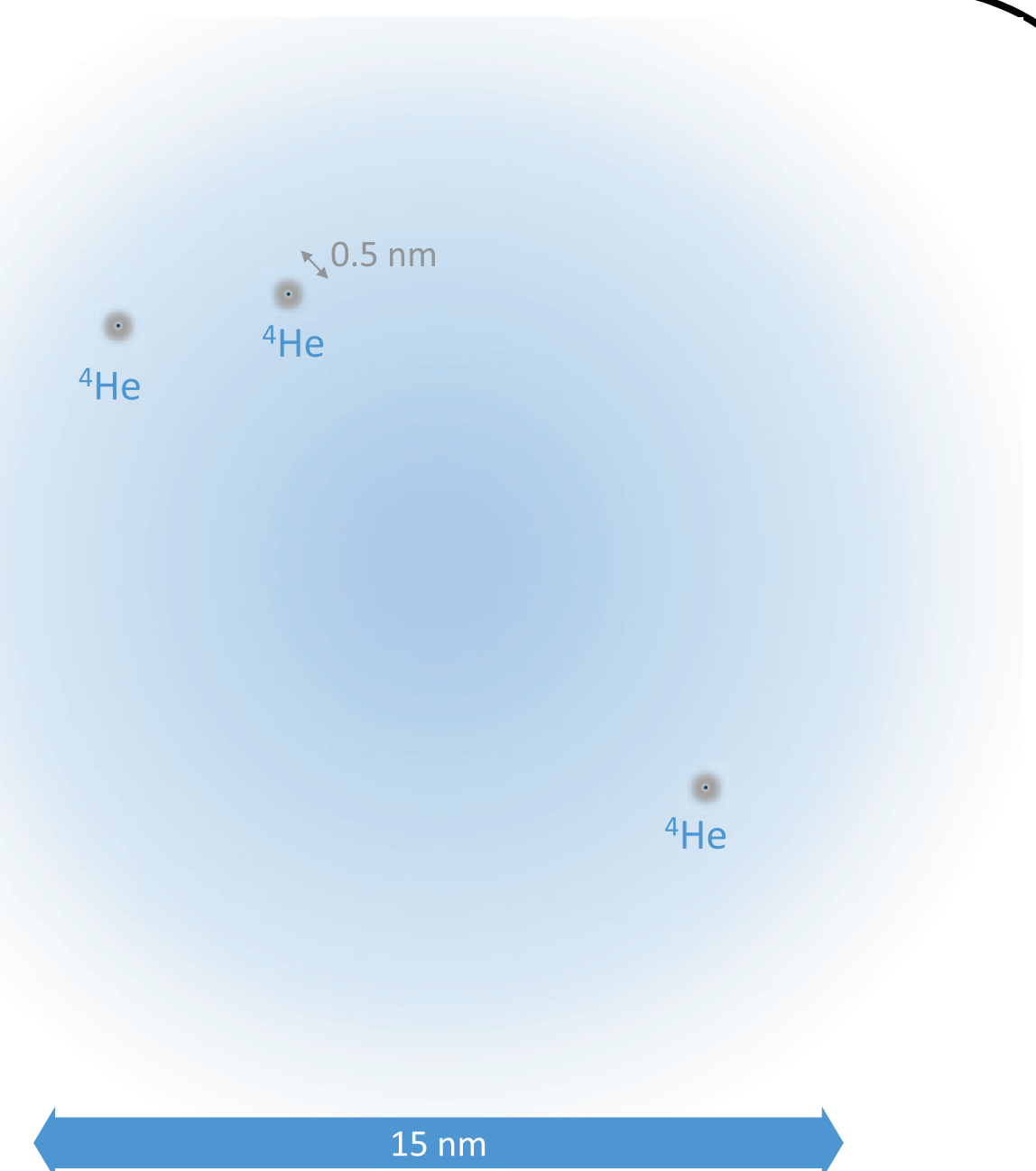
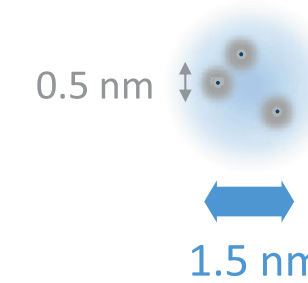
$$\mathcal{M}_2 = \frac{16\pi \sqrt{\sigma_q}}{-i\sqrt{\frac{\sigma_q}{4} - m^2}}$$

Geometrically separated, infinite tower of three-body bound-states:

$$E_{N+1} = E_N / \lambda^2$$

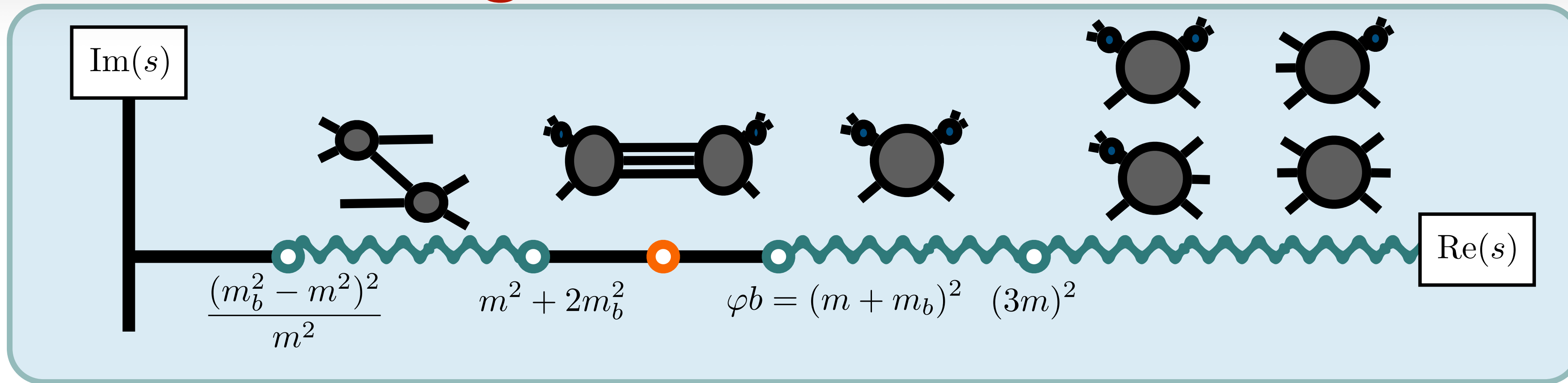
$$\lambda = 22.69438$$

Helium-4 trimer (He_3)
(ground state)

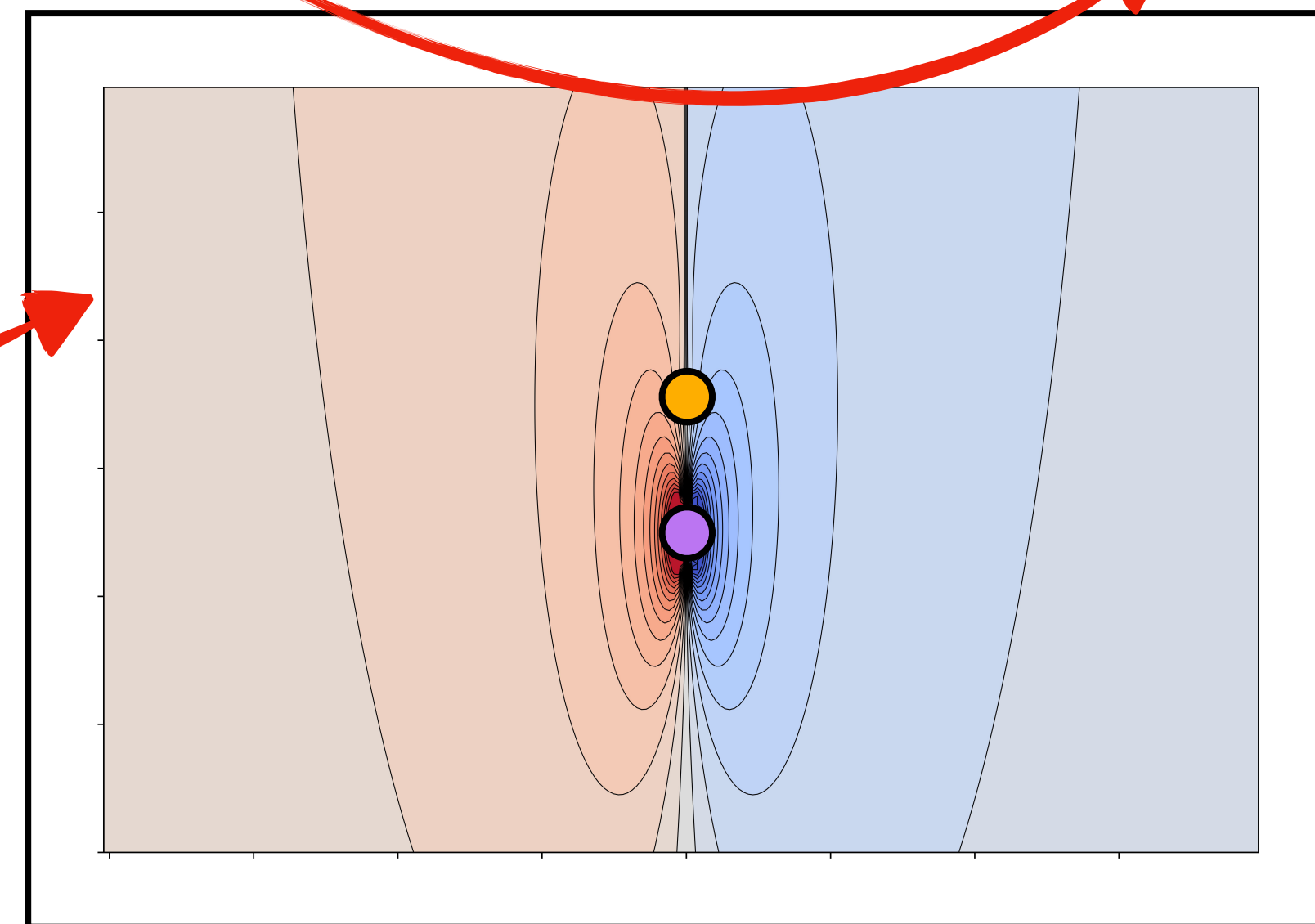
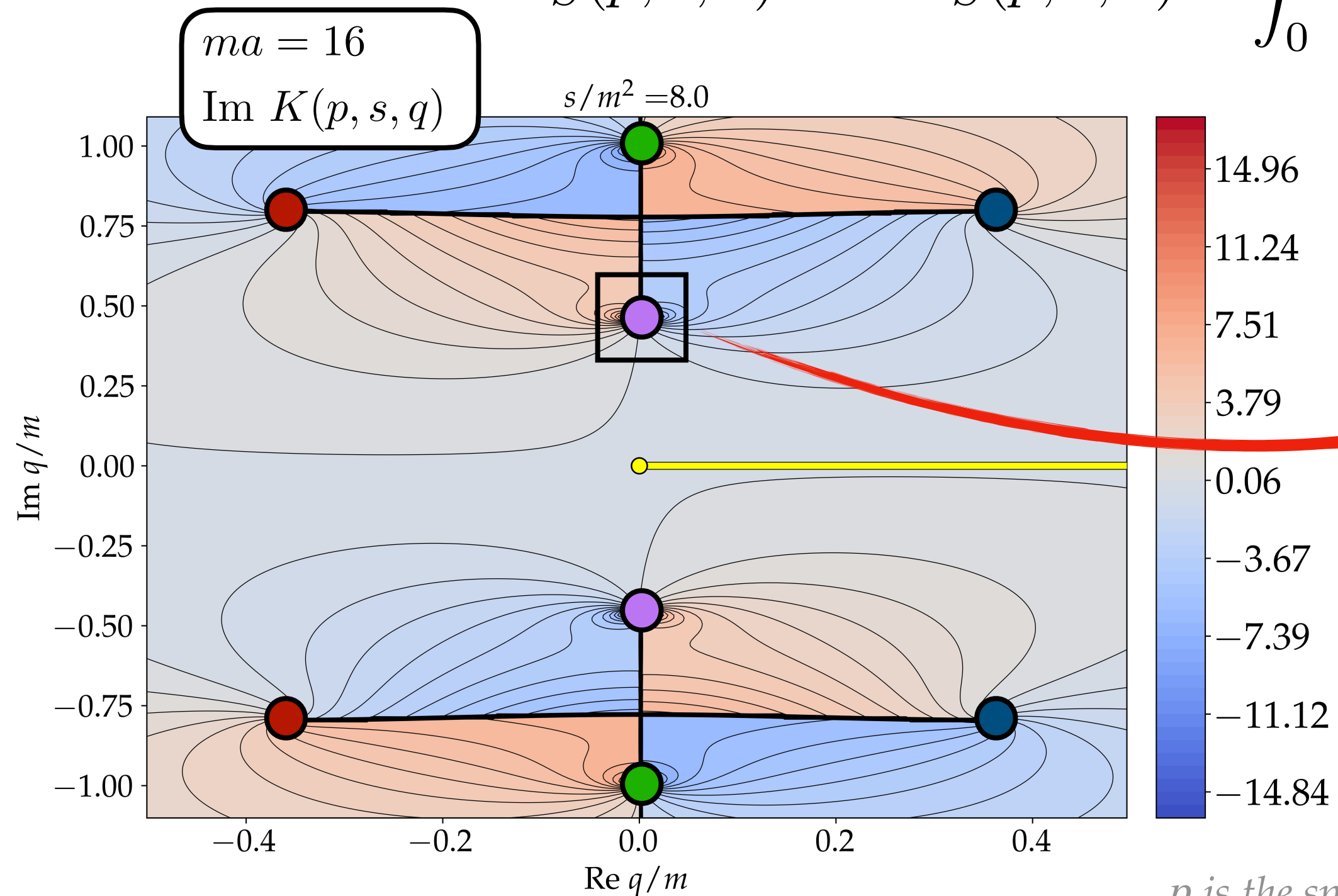


Vitaly Efimov, 1970
Naidon and Endo, 2016

Singularities in Region of Interest

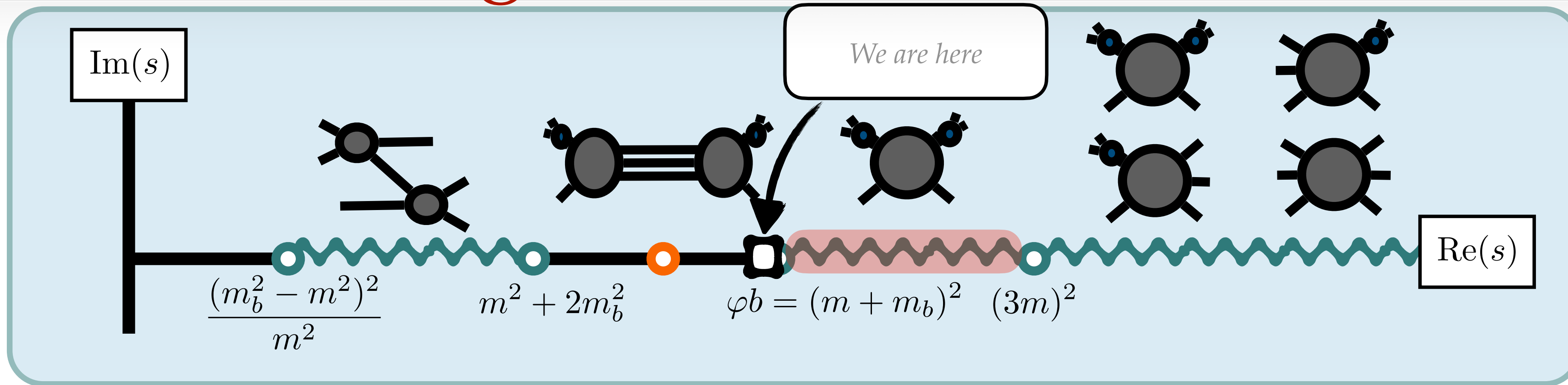


$$d_S(p, s, k) = -G_S(p, s, k) - \int_0^{q_{max}} dq \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s) d_S(q, s, k)$$

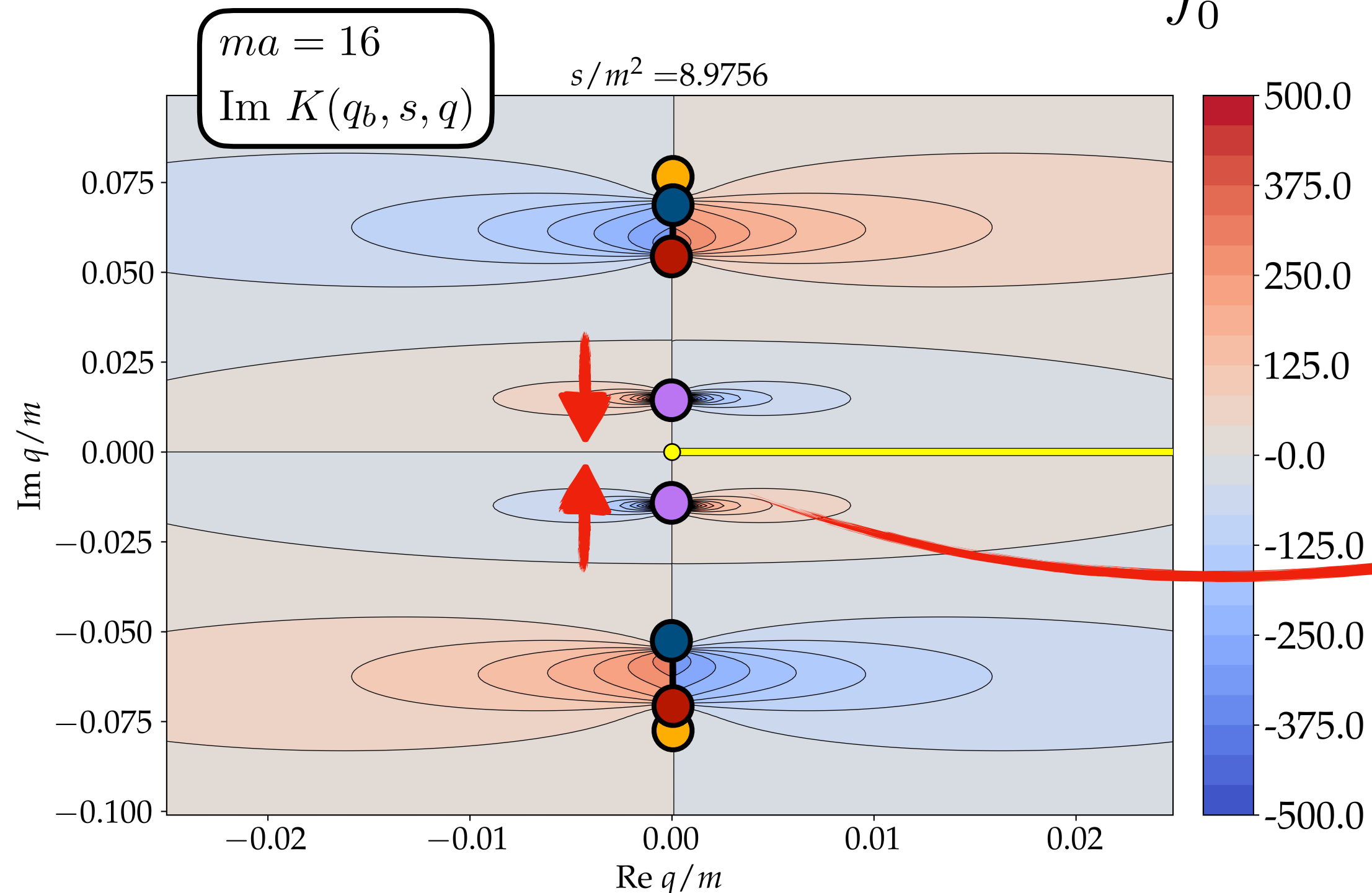


p is the spectator momenta corresponding to two-body subchannel energy, $\sigma_p = 2m^2$

Singularities in Region of Interest



$$d_S(p, s, k) = -G_S(p, s, k) - \int_0^{q_{max}} dq \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s) d_S(q, s, k)$$

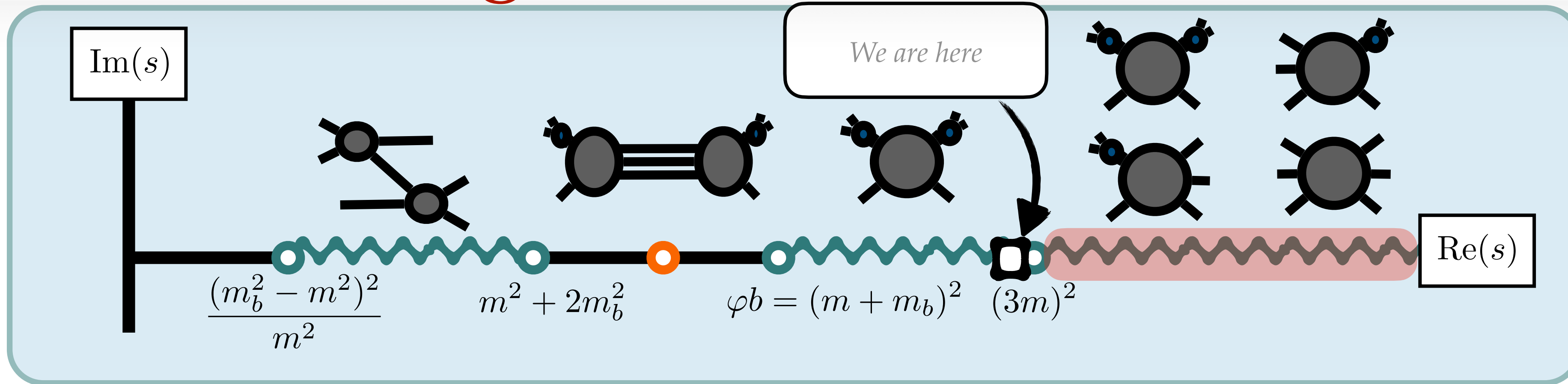


Two-body poles enclosing the end-point of integration

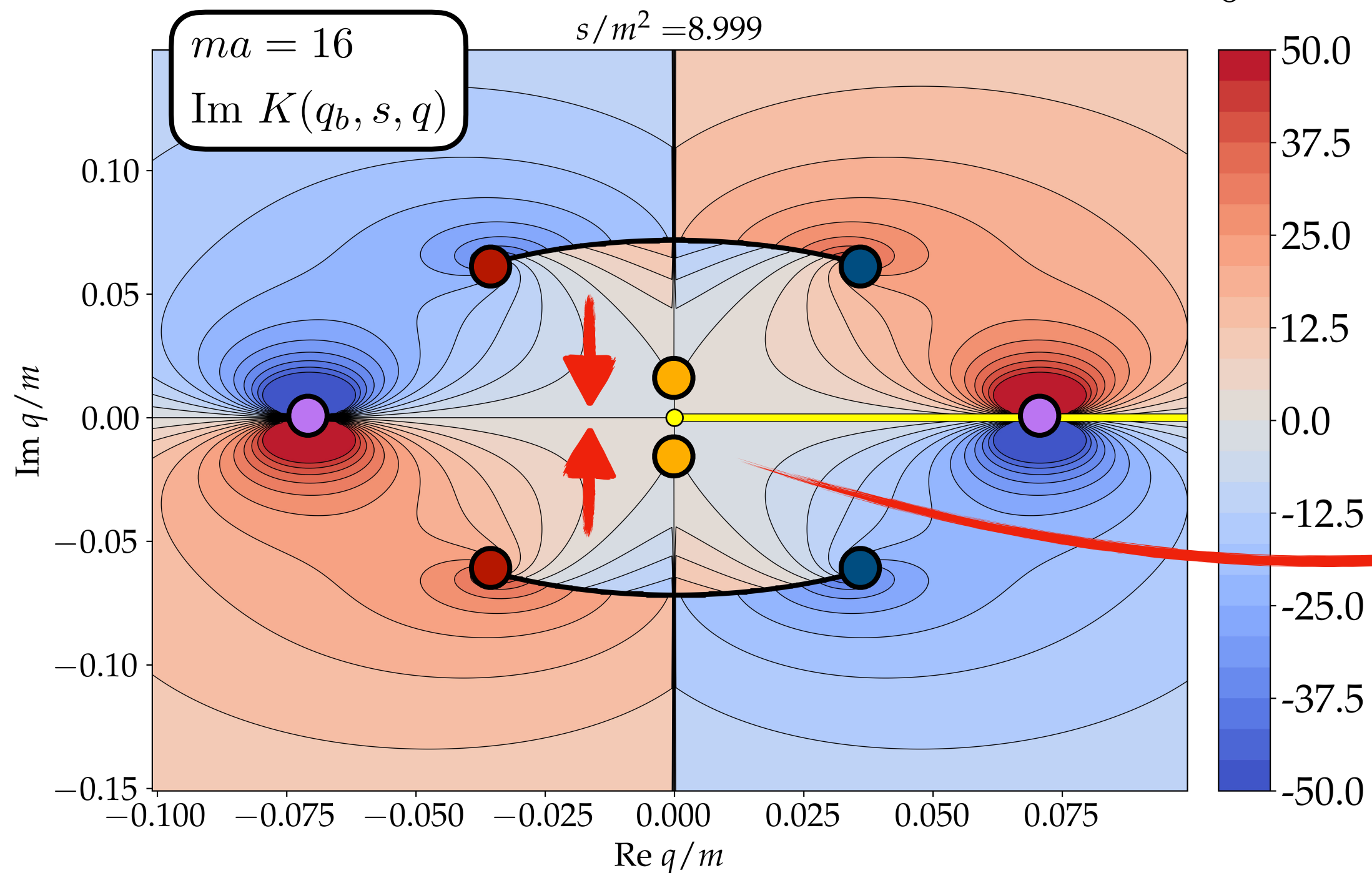
$$K(p, s, q) = \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s)$$

q_b is the spectator momenta corresponding to two-body subchannel energy, $\sigma_q = m_b^2$

Singularities in Region of Interest



$$d_S(p, s, k) = -G_S(p, s, k) - \int_0^{q_{max}} dq \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s) d_S(q, s, k)$$

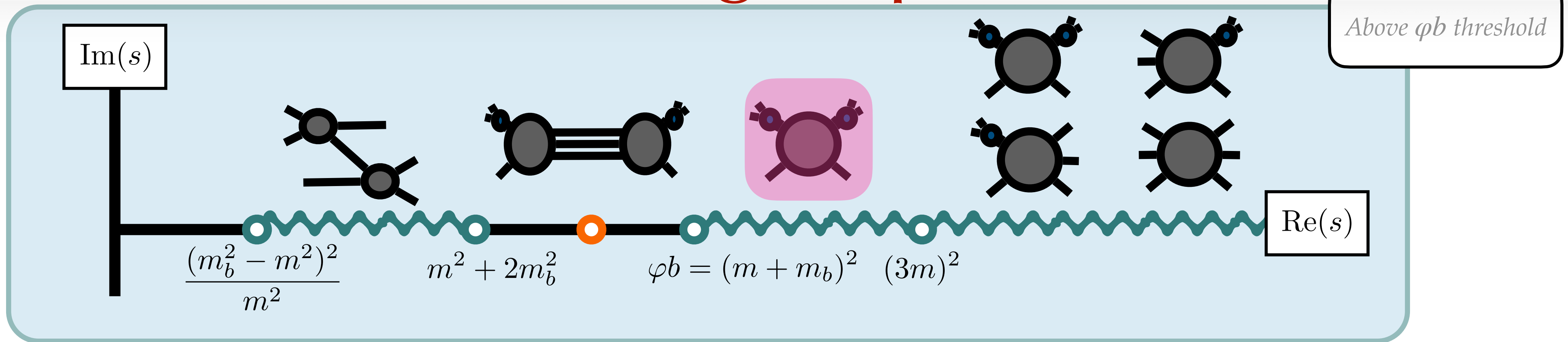


Two-body branch points enclosing one end-point of the integration interval

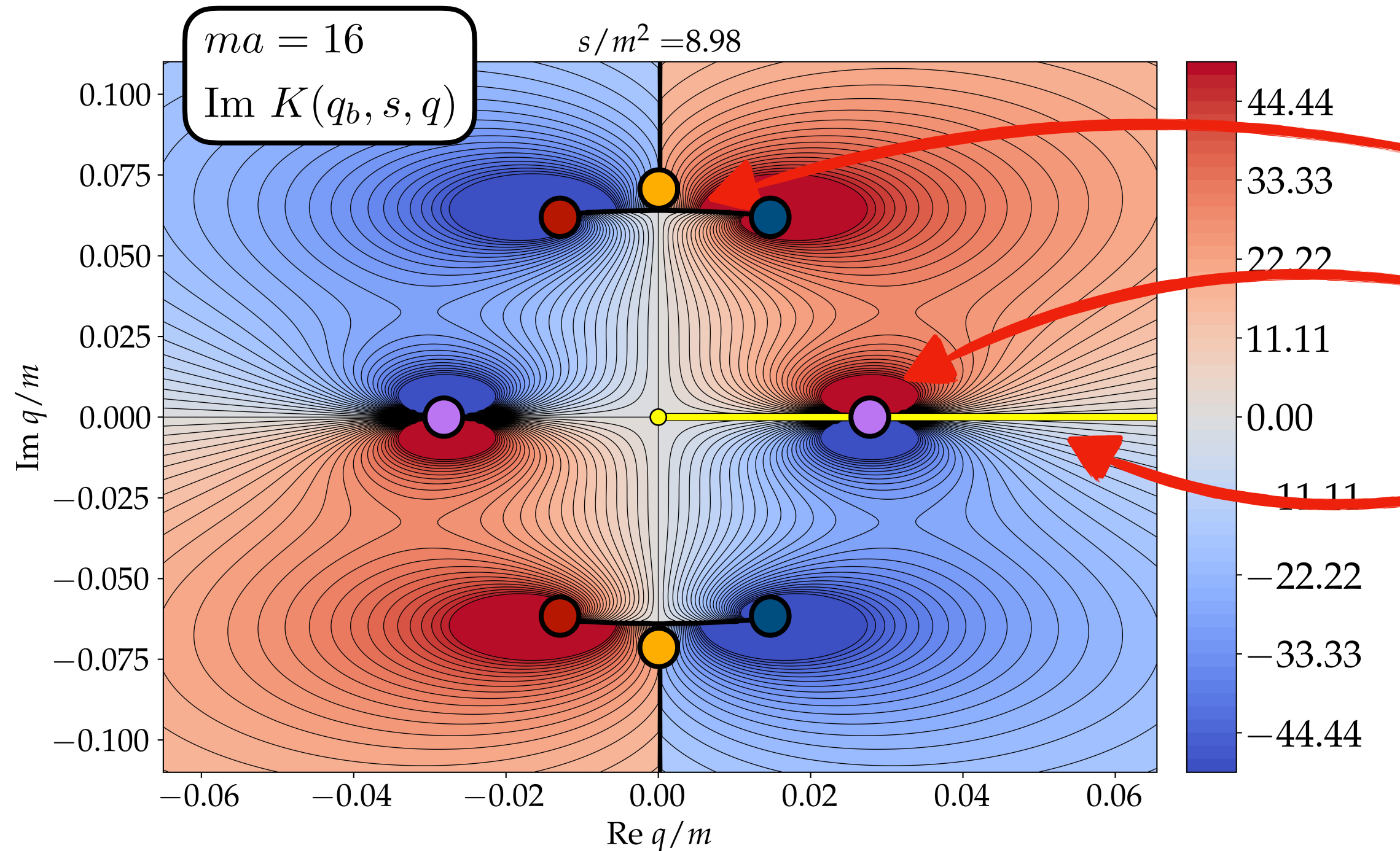
$$K(p, s, q) = \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s)$$

q_b is the spectator momenta corresponding to two-body bound-state energy, $\sigma_q = m_b^2$

Complications in obtaining amplitude



$$d_S(p, s, k) = -G_S(p, s, k) - \int_0^{q_{max}} dq \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s) d_S(q, s, k)$$



Set $p, k = q_b$ spectator momenta corresponding to two body bound state

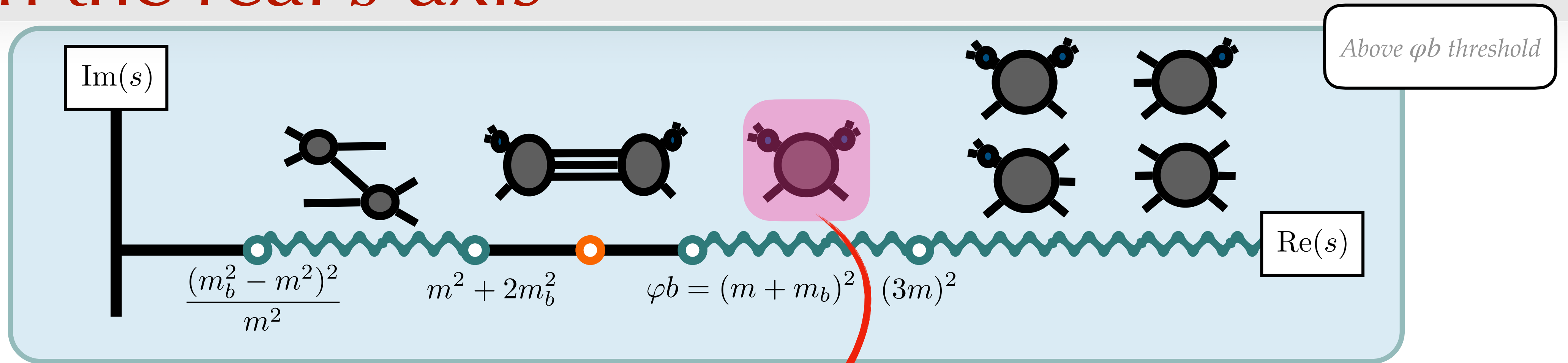
Singularity cut from the one particle exchange function

Two body amplitude has a pole in the integration region

Momentum integration interval

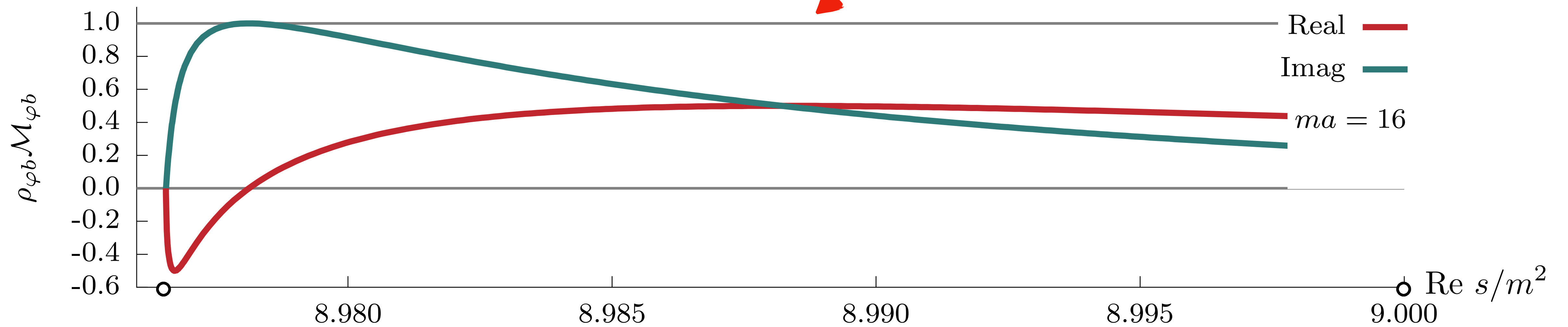
$$K(p, s, q) = \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s)$$

$\mathcal{M}_{\varphi b}$ on the real s axis

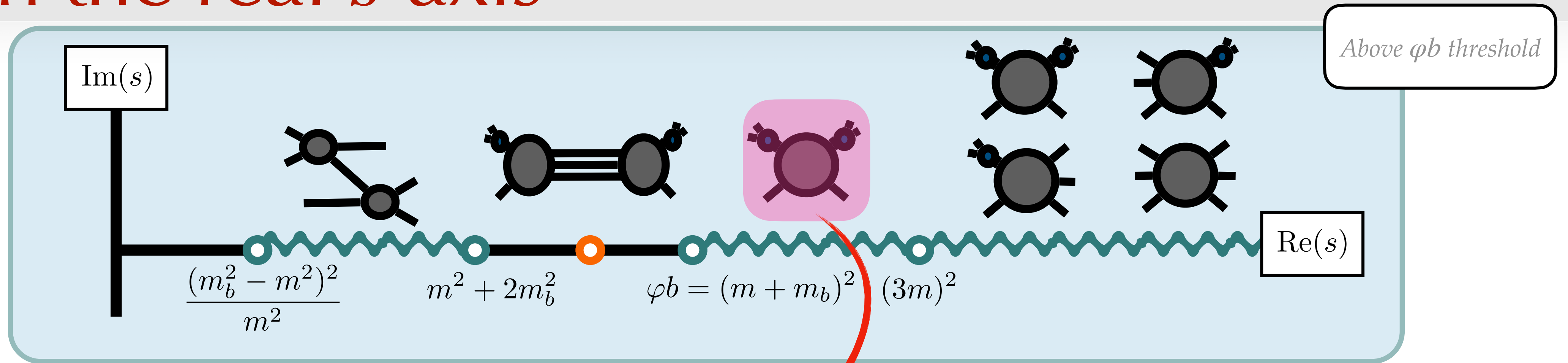


Follows two-body unitarity condition:

$$|\rho_{\varphi b} \mathcal{M}_{\varphi b}| \leq 1$$

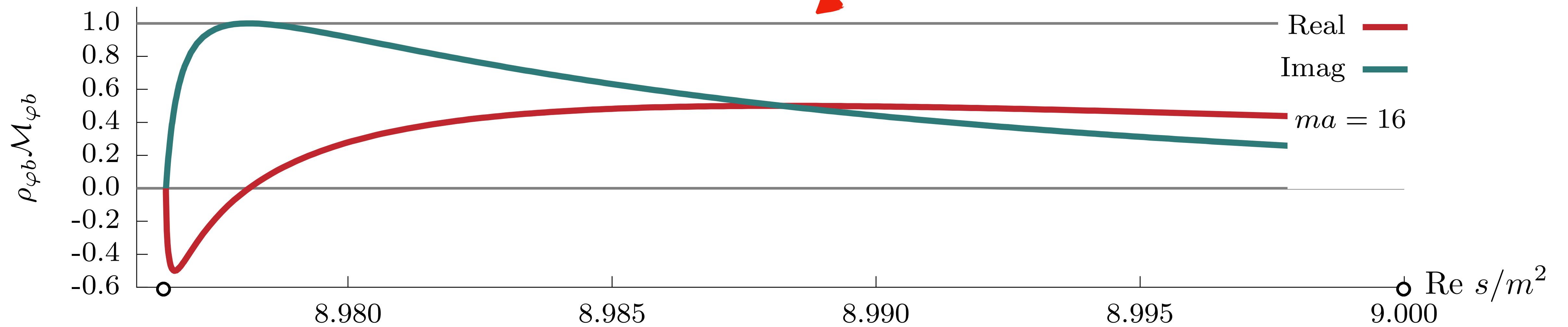


$\mathcal{M}_{\varphi b}$ on the real s axis

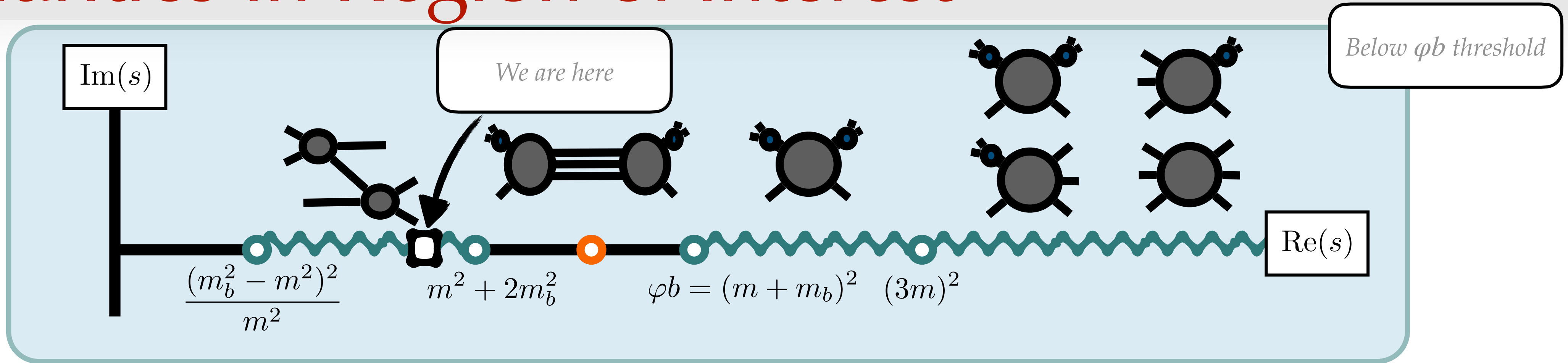


Follows two-body unitarity condition:

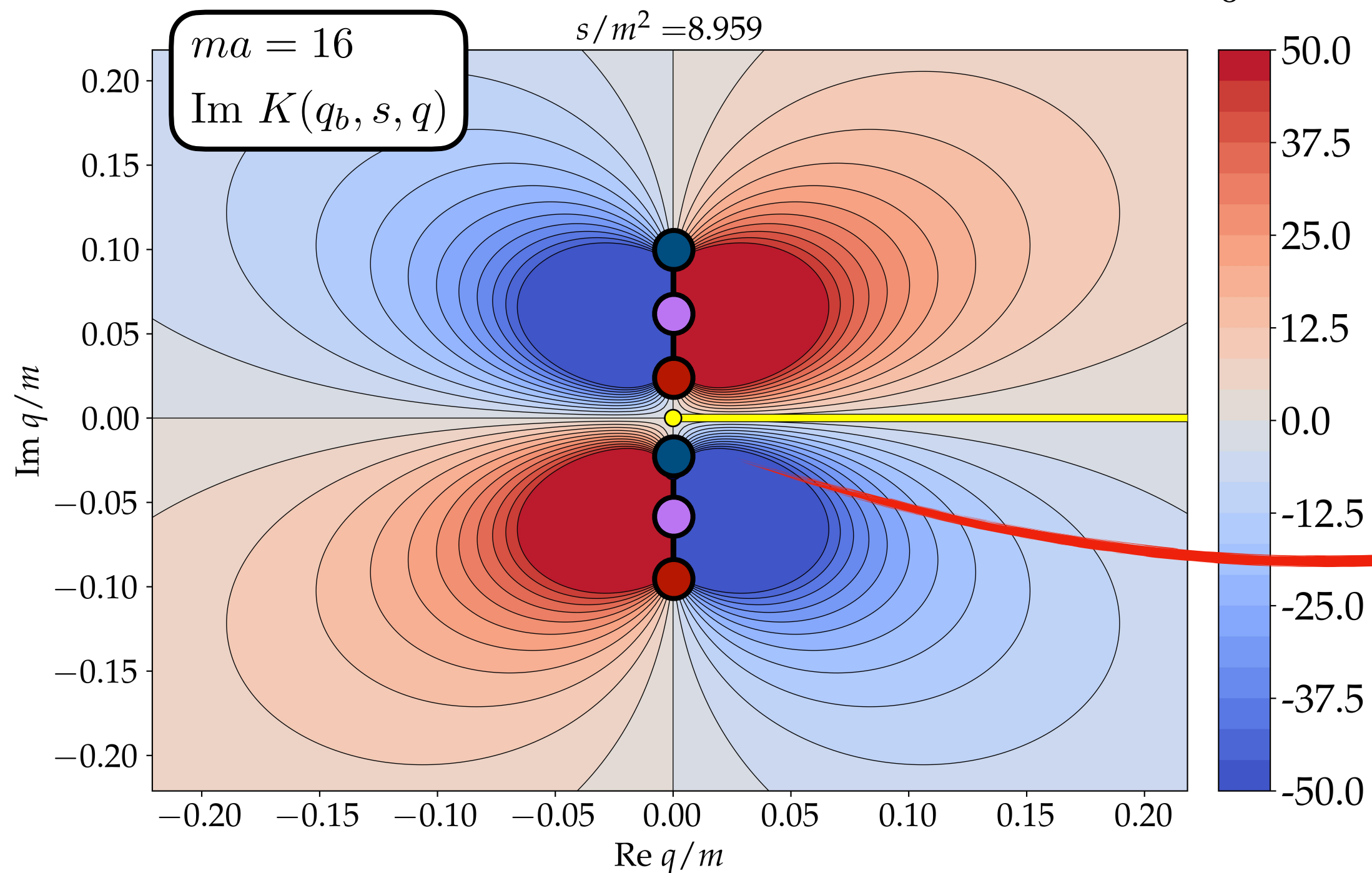
$$|\rho_{\varphi b} \mathcal{M}_{\varphi b}| \leq 1$$



Singularities in Region of Interest



$$d_S(p, s, k) = -G_S(p, s, k) - \int_0^{q_{max}} dq \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s) d_S(q, s, k)$$

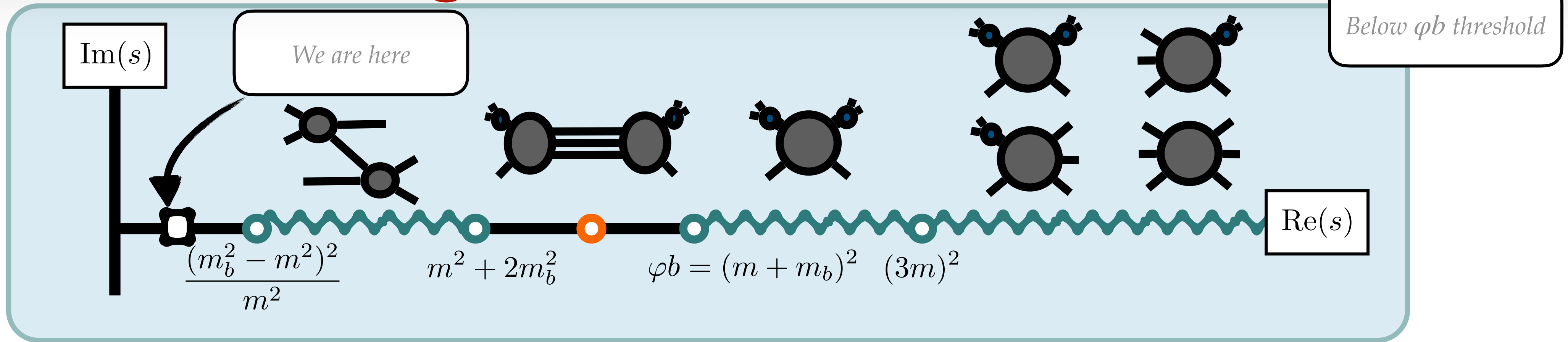


Two-body pole lies on top of the logarithmic cut

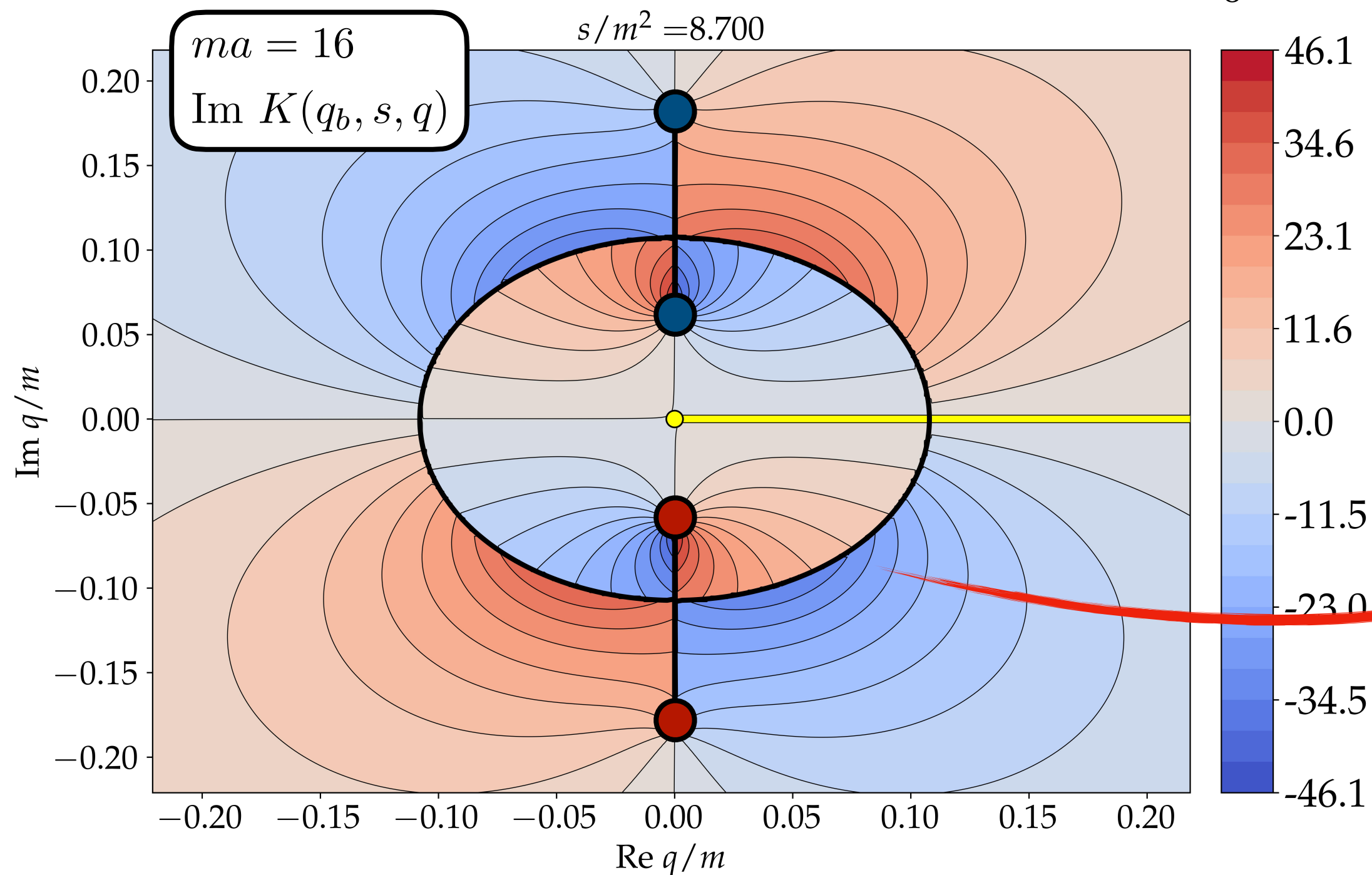
$$K(p, s, q) = \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s)$$

q_b is the spectator momenta corresponding to two-body bound-state energy, $\sigma_q = m_b^2$

Singularities in Region of Interest



$$d_S(p, s, k) = -G_S(p, s, k) - \int_0^{q_{max}} dq \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s) d_S(q, s, k)$$

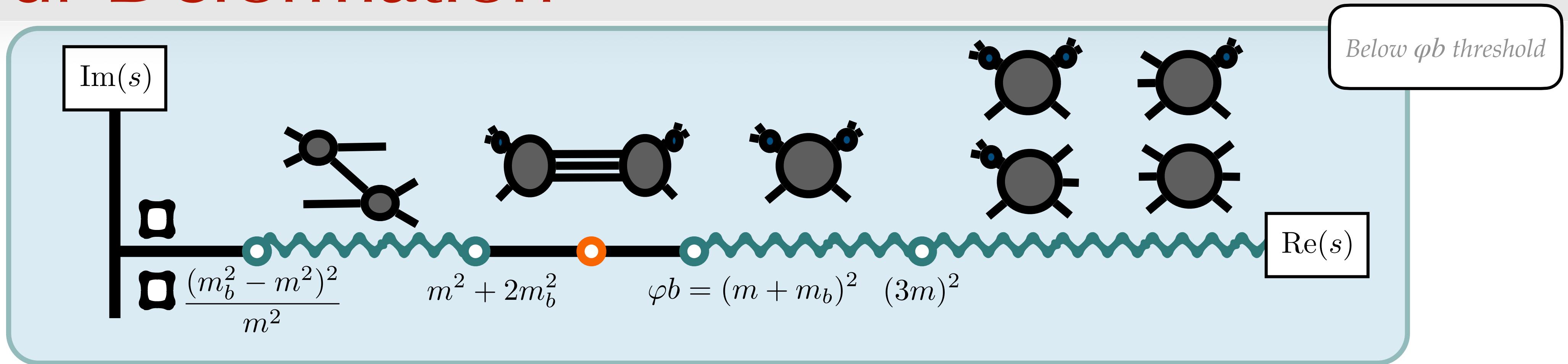


Logarithmic cut forms a circle around the end point of integration

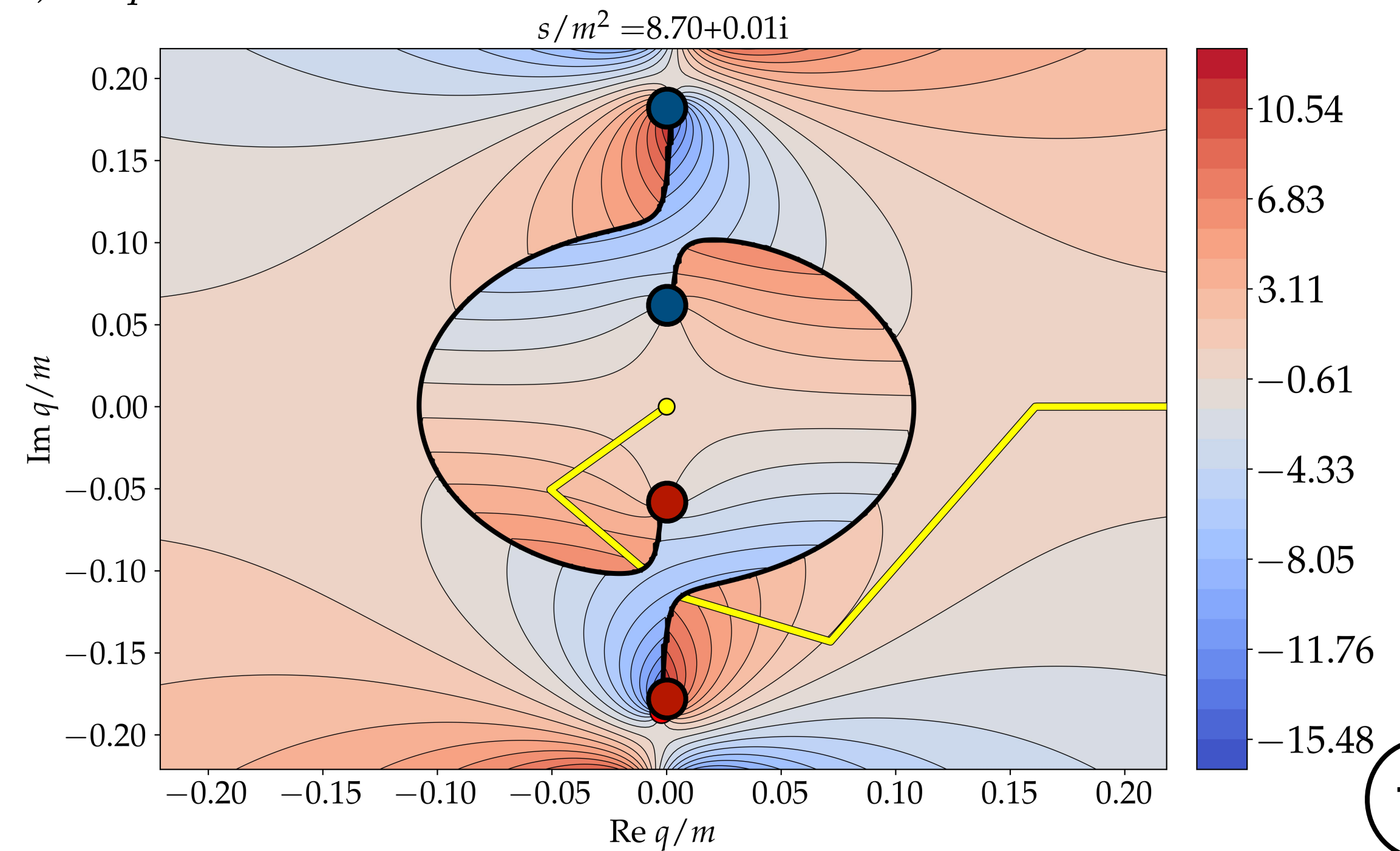
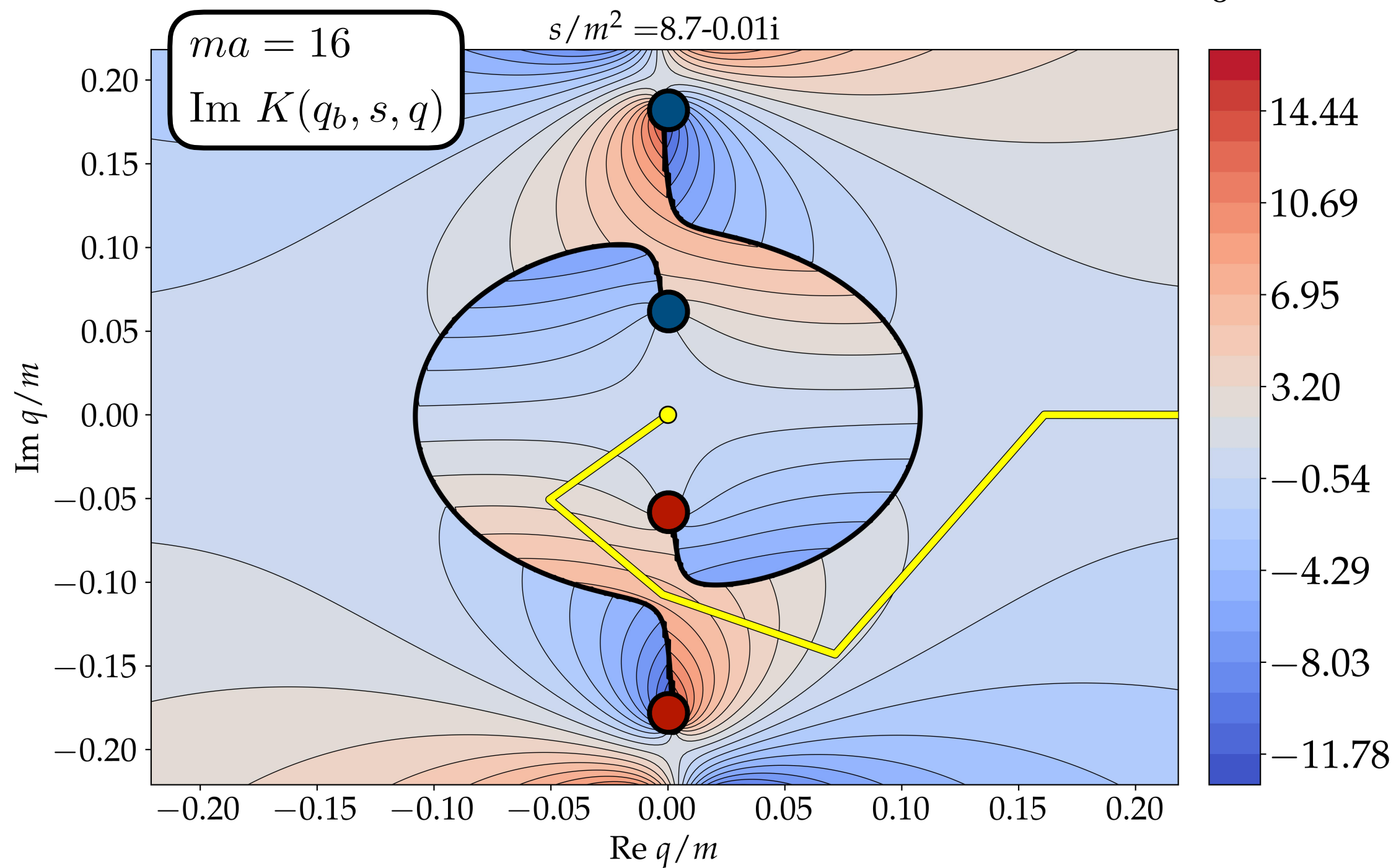
$$K(p, s, q) = \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s)$$

q_b is the spectator momenta corresponding to two-body bound-state energy, $\sigma_q = m_b^2$

Contour Deformation



$$d_S(p, s, k) = -G_S(p, s, k) - \int_0^{q_{max}} dq \frac{q^2}{(2\pi)^2 \omega_q} G_S(p, s, q) \mathcal{M}_2(q, s) d_S(q, s, k)$$



$\mathcal{M}_{\varphi b}$ on the real s axis

