The magnetized Gross-Neveu model at finite chemical potential

(handout version)

[based on arXiv:2304.14812]

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Handout version*

This handout is a slightly modified version of the talk given at Lattice 2023.
Some additional comments have been added in order to give context to the slides shown.
Slides marked by an asterisk (*) were not part of the original talk.

Gross-Neveu (GN) model

$$\mathcal{L} = i\bar{\psi}\left(\partial\!\!\!/ + \mu\gamma_0 + ieA\!\!\!/\right)\psi + \frac{g^2}{2N_f}(\bar{\psi}\psi)^2$$

- N_f flavors
- no mass term
- chemical potential μ
- external field A_μ

Gross-Neveu (GN) model

$$\mathcal{L} = i\bar{\psi}\left(\partial\!\!\!/ + \mu\gamma_0 + ieA\!\!\!/\right)\psi + \frac{g^2}{2N_f}(\bar{\psi}\psi)^2$$

or, equivalently,

$$\mathcal{L} = i\bar{\psi}\left(\partial\!\!\!/ + \sigma + \mu\gamma_0 + ieA\!\!\!/\right)\psi + \frac{N_f}{2g^2}\sigma^2$$

Ward identity

$$\langle\bar{\psi}\psi\rangle = \frac{iN_f}{g^2}\langle\sigma\rangle$$

discrete chiral symmetry

$$\psi \to i\gamma_5\psi, \;\; \bar{\psi} \to i\bar{\psi}\gamma_5, \;\; \sigma \to -\sigma$$

Gross-Neveu (GN) model*

One commonly gets rid of the $(\bar{\psi}\psi)^2$ term by introducing the auxiliary scalar field σ into the Lagrangian. These two Lagrangians are equivalent, as can be seen via Gaussian integration over σ.
The vector field A_μ is included so as to study the influence of a background magnetic field on the model.
Notice that the GN model has a discrete $\mathbb{Z}_2$ symmetry.

Motivation & Goals

Why GN model?

- Toy model for QCD
- Solid State Physics
- ...

Why magnetic field?

- Heavy-ion collisions
- Neutron stars
- Early universe

In this talk

- Study influence of magnetic field on GN phase structure using a mean-field approach and Lattice Field Theory.
- 2+1 dimensions.

Motivation & Goals*

Variants of the Gross-Neveu model have been used successfully as toy models for QCD due to some intersting features they share with QCD, such as chiral symmetry and its spontaneous breakdown and in low dimensions renormalizability and asymptotic freedom.
In solid state physics they are used to describe planar and one-dimensional materials such as graphene, high-T_c superconductors or polymers.

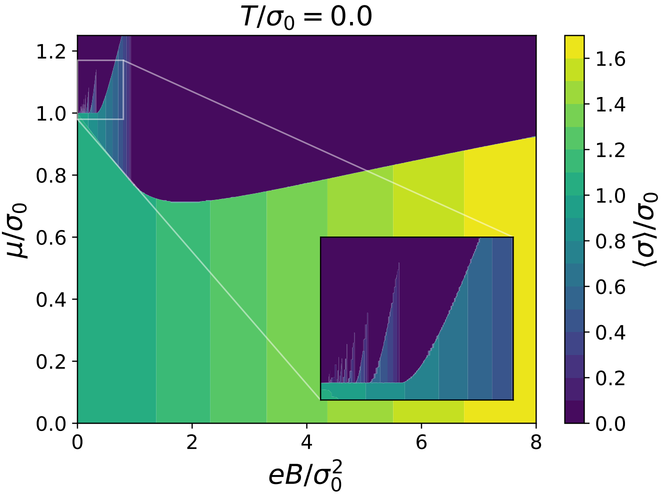
Very strong magnetic fields are present in heavy-ion collisions, nucleon stars and likely also were at the early stages of the universe. Thus, it is important to understand their potential influence on the structure of matter.

Large-N_f results*

Going to the limit of infinite flavor number, $N_\text{f} \to \infty$, often gives a qualitatively correct picture of the phase structure of Four-Fermi theories even at finite N_f. In this case, computing the path integral reduces to a simple minimization problem (see the Appendix). For $N_\text{f} \to \infty$ the mean-field limit becomes exact.

On the next slide we show the large-N_f phase diagram in the (B, μ) plane of the $(2+1)$-dimensional GN model at zero temperature. The magnetic field is assumed to be constant, homogenous and orthogonal to the spatial plane. We assume σ to be homogeneous in space and time.

Large-N_f results



Large-N_f results*

We observe that there a two major regions in the (B, μ) phase diagram:

1. The magnetic catalysis region for low μ, where the magnetic field enhances chiral symmetry breaking and
2. the inverse magnetic catalysis region for higher μ and low eB, where the magnetic field acts against symmetry breaking.

The most striking feature of the plot is the pattern of multiple (first-order) phase transitions in μ occurring at small eB. It is caused by the presence of discrete Landau levels, which are crossed successively when increasing μ.

Large-N_f results

Comments:

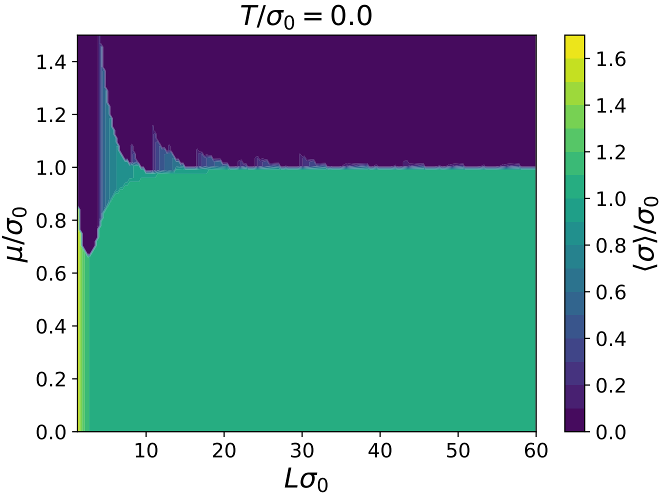
- No first-order transition at $B = 0$, $T \neq 0$.
- In contrast to simulations at $N_f = 4$ [Kogut, Strouthos; Phys. Rev. D **63** (2001)] backed by OPT study [Kneur et al.; Phys. Rev. D **76** (2007)].

Large-N_f results*

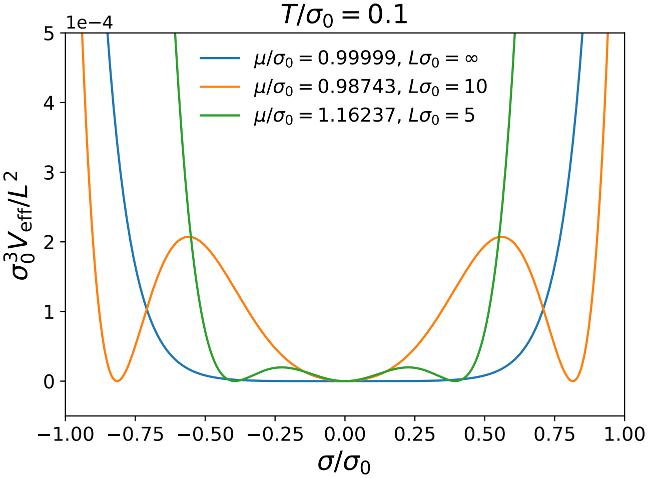
While not shown one also observes that at $B = 0$ and non-zero temperature the model does not show any first-order transition in the mean-field limit. This is in contrast to previous simulations of the $B = 0$ model using staggered fermions, where a first-order transition has been conjectured and later supported by analytical calculations using the Optimized Perturbation Theory (OPT) method.

However, even in mean-field one finds a first-order transition at $T \neq 0$ provided that one studies the system on a finite volume. On the next slide we first show the phase diagram of the model for $B = 0 = T$ in the (L, μ) plane, L denoting the spatial extent (assumed equal in both directions). Notice the resemblance to the (B, μ) diagram shown before. The slide that follows shows the effective potential of the model at finite T for three different volumes. It is apparent that one finds a first-order transition for both finite volumes due to the triple-minimum structure of V_{eff}.

Finite-volume large-N_f results



Finite-volume large-N_f results



Lattice study*

In order to test the validity of these mean-field results at finite N_f, we have performed extensive lattice studies of the model. The most important simulation details are shown on the next slide. More information on the complex-action problem can be found in the appendix.

Lattice study

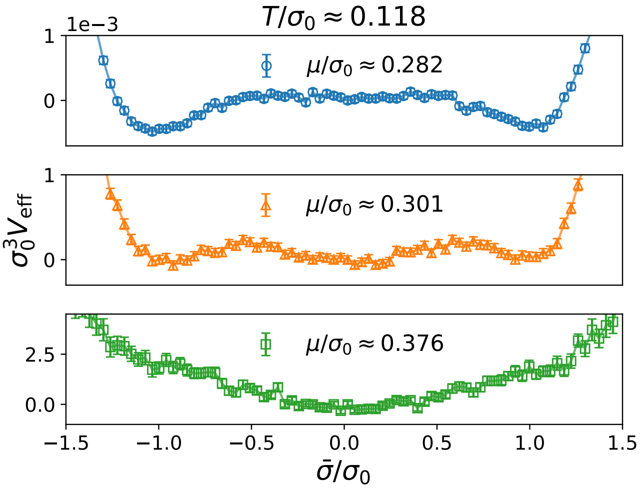
Simulation details:

- $N_{\mathrm{f}} = 1$ overlap fermions.
- 8^3, 12^3 and 16^3 lattices with different lattice spacings.
- Complex-action problem for $\mu \neq 0 \neq B$ mild.

Lattice results at $B = 0$*

The next slide shows results obtained at vanishing magnetic field. We have studied the probability distribution of $\bar{\sigma} := \frac{1}{V} \sum_{x\in\Lambda} \sigma(x)$, i.e. the space-time average of σ. The resulting (constraint) effective potential is shown for three values of μ on an 8^3 lattice on the next slide.
In the upper panel one sees two minima of V_{eff}, corresponding to spontaneous symmetry breaking. In the middle panel a third minimum arises, which is striking evidence for the presence of a first-order transition in the system. Lastly, the lower plot shows only a single minimum, i.e., chiral symmetry is restored.

Lattice results at $B = 0$

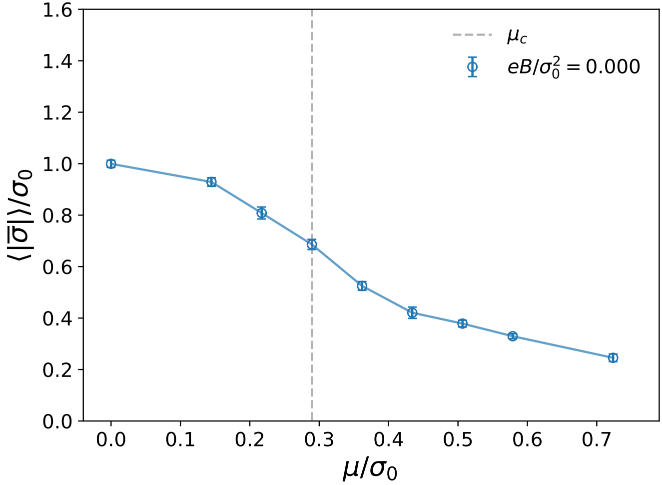


Lattice results at $B \neq 0$*

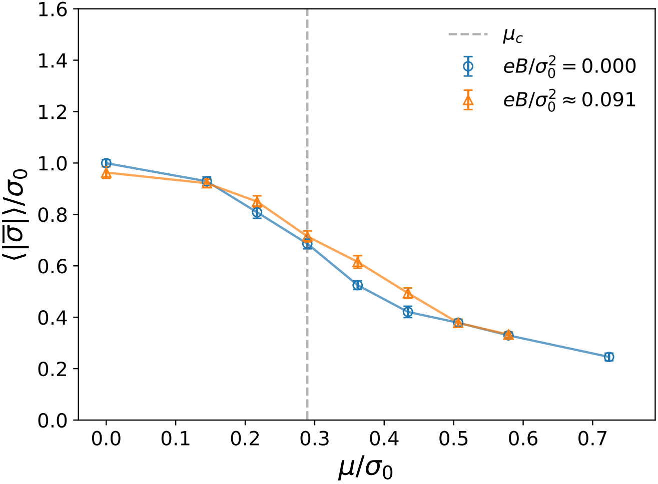
Next we show results for the chiral condensate (we measure the absolute value of $\bar{\sigma}$ (defined before) in order to avoid cancellations) as a function of μ for different values of B. The vertical line gives a rough estimate of the critical chemical potential of the transition (without errors). We show results for a 12^3 lattice but we have studied different volumes and lattice spacings as well.

Apart from the weakest non-vanishing magnetic field, all of our results are consistent with magnetic catalysis below the critical μ, i.e. the magnetic field enhances the chiral condensate for every value of $\mu < \mu_c$. There appears to be no sign of inverse magnetic catalysis as well as of the multiple phase transitions found in the mean-field limit. The reason that the weakest non-zero magnetic field appears to be different is a finite-size effect.

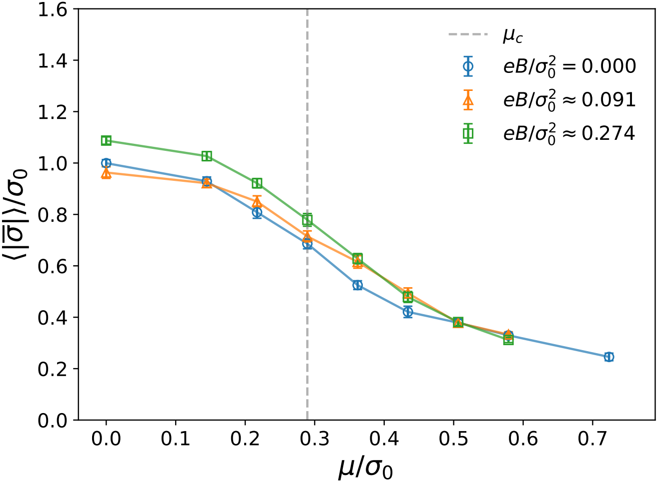
Lattice results at $B \neq 0$



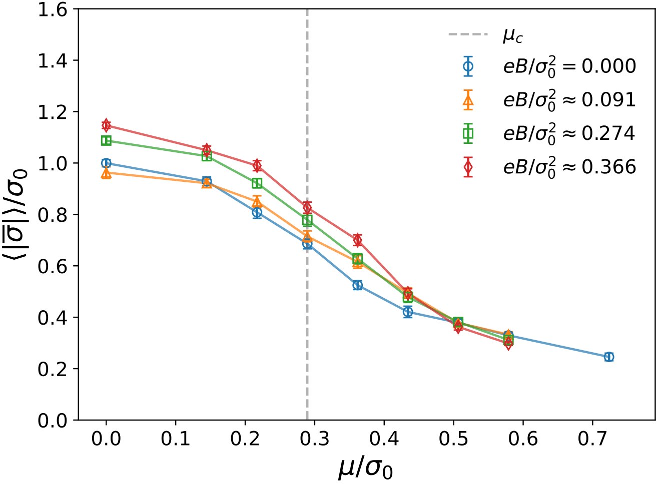
Lattice results at $B \neq 0$



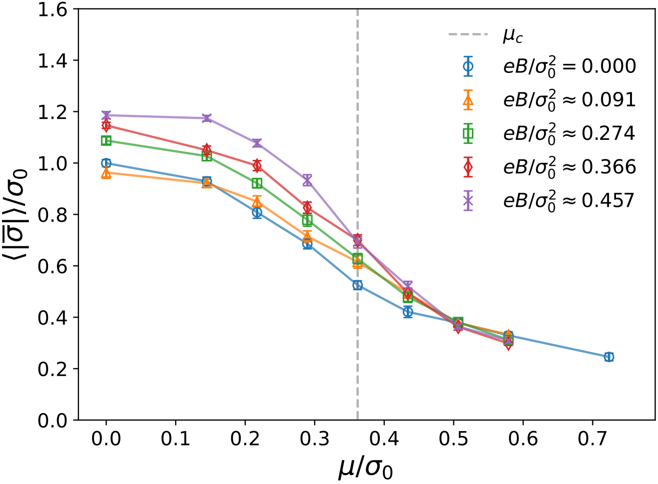
Lattice results at $B \neq 0$



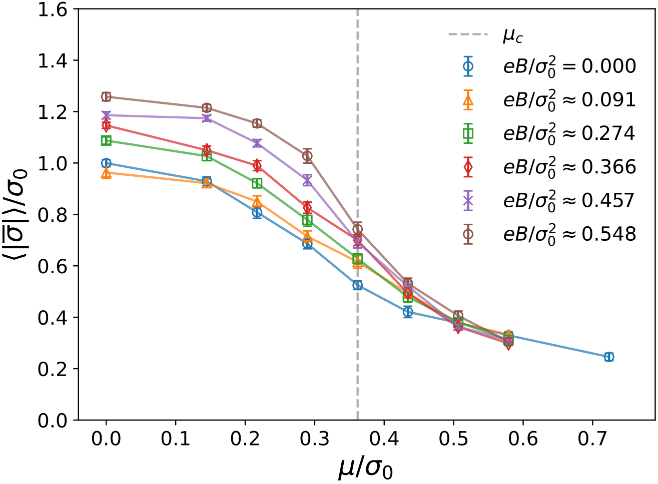
Lattice results at $B \neq 0$



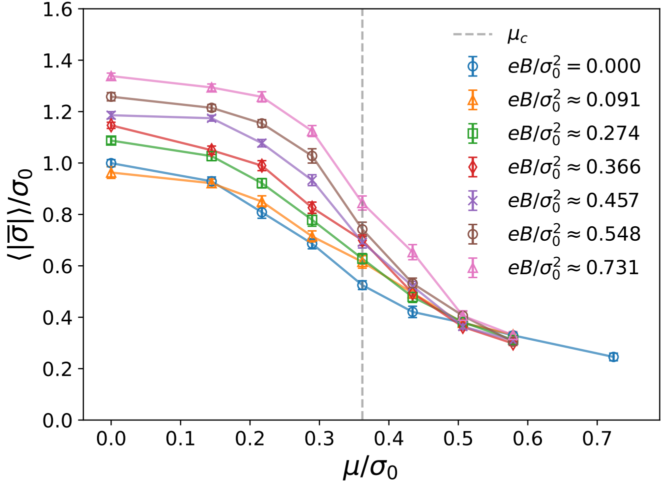
Lattice results at $B \neq 0$



Lattice results at $B \neq 0$



Lattice results at $B \neq 0$



Summary & Outlook

Summary

- First-order phase transition for $B = 0$ and $T \neq 0$ on a finite volume.
- (Inverse) magnetic catalysis and multiple transitions in large-N_f.
- Only magnetic catalysis for $N_\mathrm{f} = 1$.
- Contradicts OPT calculations [Kneur, Pinto, Ramos; Phys. Rev D **88** (2013)].

Outlook

- Spectral analysis to study fate of Landau levels.
- More realistic models.

Contact*

For questions/discussion please do not hesitate to contact the author of this talk via
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Backup

The Large-N_f limit

The GN Lagrangian reads

$$\mathcal{L} = i\bar{\psi}\mathbb{1}_{N_f}\left(\partial\!\!\!/ + \sigma + \mu\gamma_0 + ieA\!\!\!/\right)\psi + \frac{N_f}{2g^2}\sigma^2 \ .$$

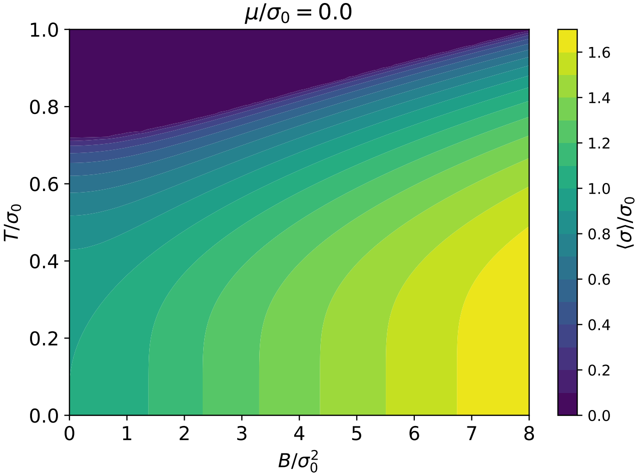
In the limit $N_f \to \infty$, after integrating out the fermions in the path integral, the chiral condensate $\langle\sigma\rangle \propto \langle\bar{\psi}\psi\rangle$ is given by the minimum of

$$S_{\text{eff}}[\sigma] = -\ln\det(D[\sigma]) + \frac{1}{2g^2}\int d^3x\,\sigma^2(x) \ ,$$

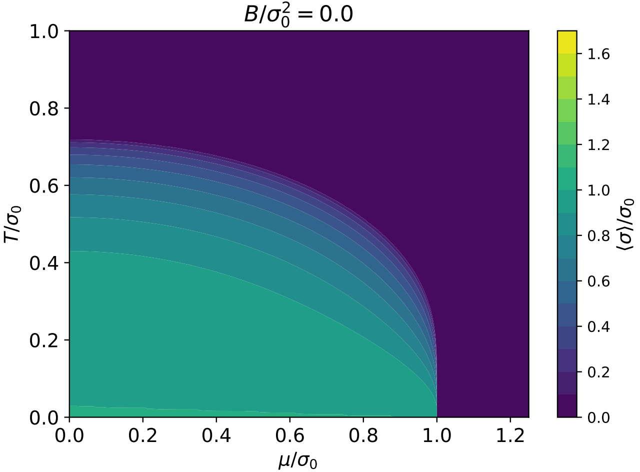
with

$$D[\sigma] = \partial\!\!\!/ + \sigma + \mu\gamma_0 + ieA\!\!\!/ \ .$$

Mean-field phase diagrams



Mean-field phase diagrams



Reducible representation of γ_μ

To allow for a notion of chirality in $(2+1)$ dimensions, we combine the two irreducible representations of the Dirac algebra into a reducible one:

$$\gamma_0 = \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix} , \quad \gamma_1 = \begin{pmatrix} \tau_3 & 0 \\ 0 & -\tau_3 \end{pmatrix} , \quad \gamma_2 = \begin{pmatrix} \tau_1 & 0 \\ 0 & -\tau_1 \end{pmatrix} ,$$

where τ_μ are the usual Pauli matrices.

There are now two γ matrices which anti-commute with all the others:

$$\gamma_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} 0 & i\mathbb{1}_2 \\ -i\mathbb{1}_2 & 0 \end{pmatrix} .$$

Chiral symmetry in the continuum

The 1-flavor massless $(2+1)$-dimensional GN model in a reducible representation of the Dirac algebra has the following symmetries:

$$\begin{aligned} U_{\mathbb{1}}(1) &: \psi \to e^{i\alpha}\psi \,, \\ U_{\gamma_{45}}(1) &: \psi \to e^{i\alpha\gamma_{45}}\psi \,, \quad \gamma_{45} = i\gamma_4\gamma_5 \,, \\ \mathbb{Z}_2 &: \psi \to i\gamma_5\psi . \end{aligned}$$

The $\mathbb{Z}_2$ symmetry generated by γ_4 is not independent.

A mass term induces the breaking pattern

$$U_{\mathbb{1}}(1) \times U_{\gamma_{45}}(1) \times \mathbb{Z}_2 \to U_{\mathbb{1}}(1) \times U_{\gamma_{45}}(1) \,.$$

Chiral symmetry on the lattice

On the lattice the symmetries look as follows:

$$
\begin{aligned}
U_{\mathbb{1}}(1) &: \psi \to e^{i\alpha}\psi \ , \\
U_{\gamma_{45}}(1) &: \psi \to e^{i\alpha\gamma_{45}}\psi \ , \\
\mathbb{Z}_2 &: \psi \to i\gamma_5(1 - D_{\mathrm{ov}})\psi \ , \quad \bar{\psi} \to i\bar{\psi}\gamma_5 \ ,
\end{aligned}
$$

where D_{ov} is the massless overlap operator, in our case.

There is another $\mathbb{Z}_2$ generated by γ_4, which is, again, independent.

A mass term $\bar{\psi}\left(1 - \frac{D_{\mathrm{ov}}}{2}\right)\psi$ again breaks the $\mathbb{Z}_2$ symmetry, but leaves both $U(1)$'s intact.

Overlap operator in the GN model

We use Neuberger's overlap operator [Neuberger; Phys. Lett B **417** (1998)]

$$D_{\mathrm{ov}} = \mathbb{1} + A/\sqrt{A^\dagger A}\,, \quad A = D_{\mathrm{W}} - \mathbb{1}\,,$$

where D_W is the standard Wilson operator. The full operator, including σ and μ, reads [Gavai, Sharma; Phys. Lett B **716** (2012)]

$$D_{\mathrm{full}} = \left(1 - \frac{\sigma + \mu\gamma_0}{2}\right) D_{\mathrm{ov}} + \sigma + \mu\gamma_0\,.$$

With the Ginsparg-Wilson chiral condensate $\Sigma_{\mathrm{GW}} = \langle \bar\psi \left(\mathbb{1} - \frac{D_{\mathrm{ov}}}{2}\right) \psi \rangle$ we have a Ward identity in analogy to the continuum theory:

$$\langle \sigma \rangle = \Sigma_{\mathrm{GW}}\,.$$

The observable

In order to avoid cancellation of contributions from the two minima of the effective action ($\pm\sigma$) in the broken phase we measure the absolute value

$$\langle|\sigma|\rangle = \langle|\sum_{x\in\Lambda}\sigma(x)|\rangle \ ,$$

where the sum runs over the whole lattice.

As a caveat, this definition makes it harder to determine a phase transition, as we cannot measure $\langle|\sigma|\rangle = 0$.

The complex-action problem

With the standard reweighting approach

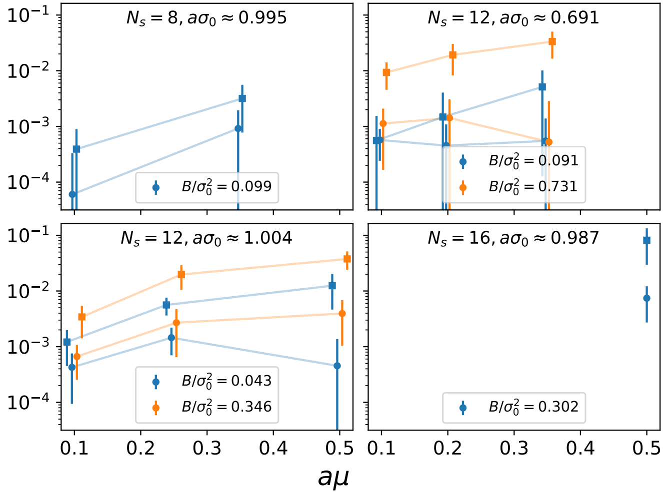
$$\langle \mathcal{O} \rangle = \frac{\left\langle e^{-\mathrm{i}S_I} \mathcal{O} \right\rangle_R}{\left\langle e^{-\mathrm{i}S_I} \right\rangle_R} = \langle \mathcal{O} \rangle_R + \frac{\mathrm{cov}_R \left(e^{-\mathrm{i}S_I}, \mathcal{O} \right)}{\left\langle e^{-\mathrm{i}S_I} \right\rangle} ,$$

where the action is written as $S = S_R + \mathrm{i}S_I$,

$$\langle \mathcal{O} \rangle_R = \frac{\int \mathcal{D}\sigma \, e^{-S_R} \mathcal{O}[\sigma]}{\int \mathcal{D}\sigma \, e^{-S_R}} ,$$

and cov_R is the covariance w.r.t S_R.
If $\mathrm{cov}_R \left(e^{-\mathrm{i}S_I}, \mathcal{O} \right)$ is small and $\left\langle e^{-\mathrm{i}S_I} \right\rangle$ is close to unity (as in our case for $\mathcal{O} = |\sigma|$) one may ignore the complex-action problem and only compute $\langle \mathcal{O} \rangle_R$.

The complex-action problem



Index theorem

An exact index theorem in our case would read

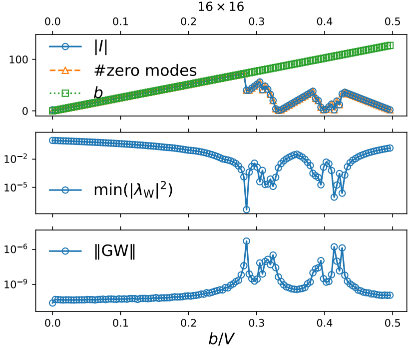
$$I := \mathrm{Index}[D_{\mathrm{ov}}] = \frac{1}{2}\mathrm{tr}\left[\gamma_5 D_{\mathrm{ov}}\right] = b\ ,$$

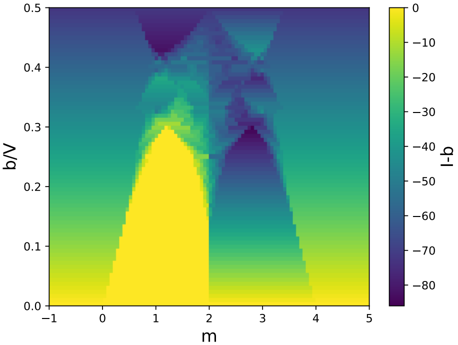
where b is proportional to the magnetic flux:

$$B = \frac{2\pi}{V} b\ .$$

Due to the vanishing theorem $|I|$ is equal to the total number of zero modes of D_{ov}. As the next slide shows the index theorem can be violated for large b and/or Wilson mass parameter m far from 1. λ_W is the smallest eigenvalue of the Wilson kernel used for the overlap and $||\mathrm{GW}||$ measures a possible violation of the Ginsparg-Wilson equation.

Index theorem





Hofstadter's butterfly