The magnetized Gross-Neveu model at finite chemical potential

[based on arXiv:2304.14812]

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Gross-Neveu (GN) model

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$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi + \frac{g^2}{2N_f}(\bar{\psi}\psi)^2$$

- N_f flavors
- no mass term

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Ward identity	discrete chiral symmetry
$\langle\bar{\psi}\psi\rangle = \frac{iN_f}{g^2}\langle\sigma\rangle$	$\psi \to i\gamma_5\psi,\ \bar{\psi} \to i\bar{\psi}\gamma_5,\ \sigma \to -\sigma$

Motivation & Goals

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Why GN model?

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- **Toy model for QCD**

  e.g., chiral symmetry, . . .

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In this talk

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In this talk

- Study influence of magnetic field on GN phase structure using a mean-field approach and Lattice Field Theory.

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- Study influence of magnetic field on GN phase structure using a mean-field approach and Lattice Field Theory.
- 2+1 dimensions.

Large-N_f results

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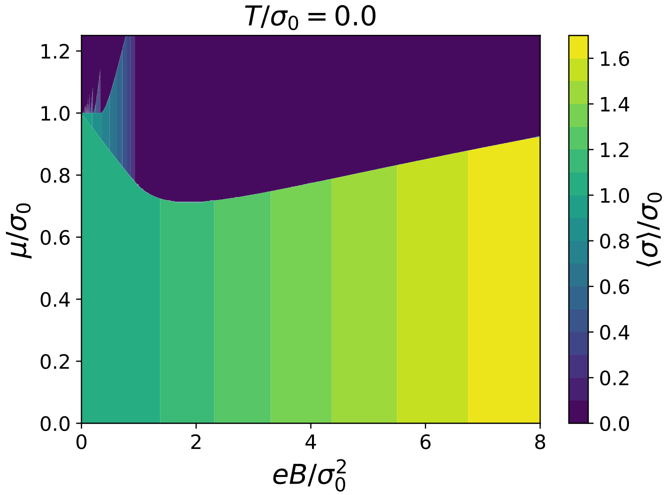
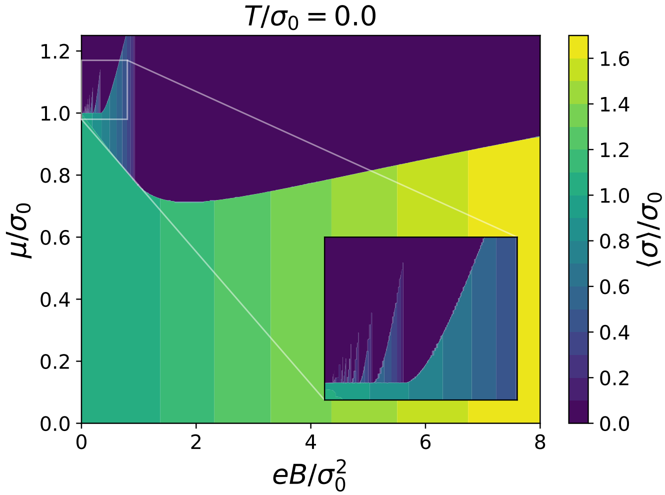


Large-N_f results



Large-N_f results

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- In contrast to simulations at $N_f = 4$ [Kogut, Strouthos; Phys. Rev. D **63** (2001)] backed by OPT study [Kneur et al.; Phys. Rev. D **76** (2007)].

Finite-volume large-N_f results

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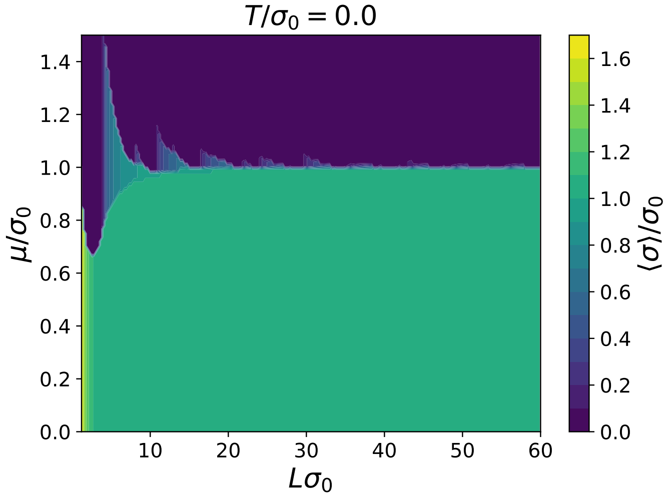

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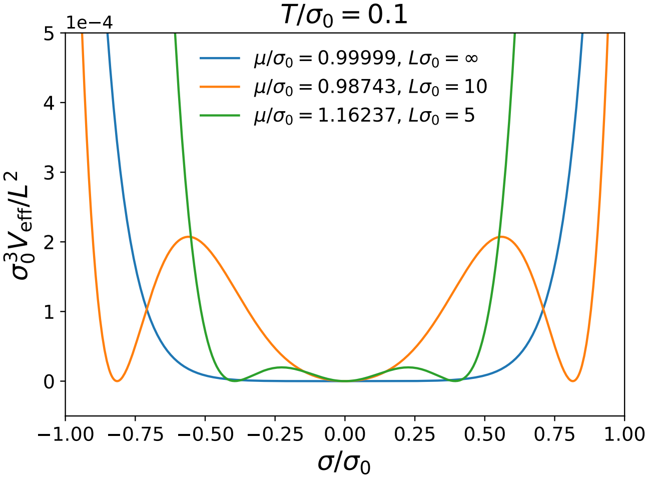

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- Complex-action problem for $\mu \neq 0 \neq B$ mild.

Lattice results at $B = 0$

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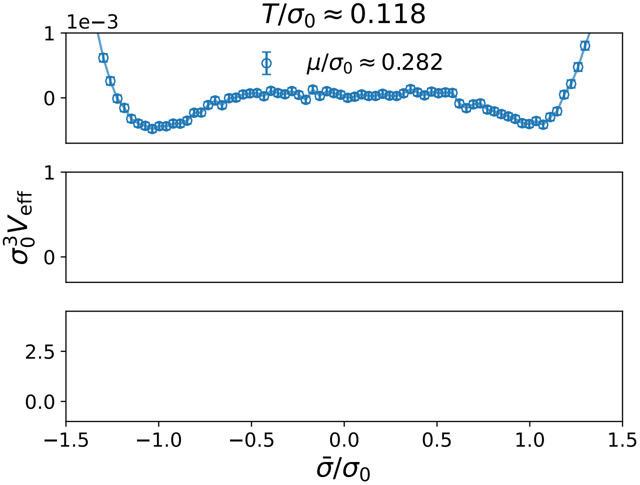

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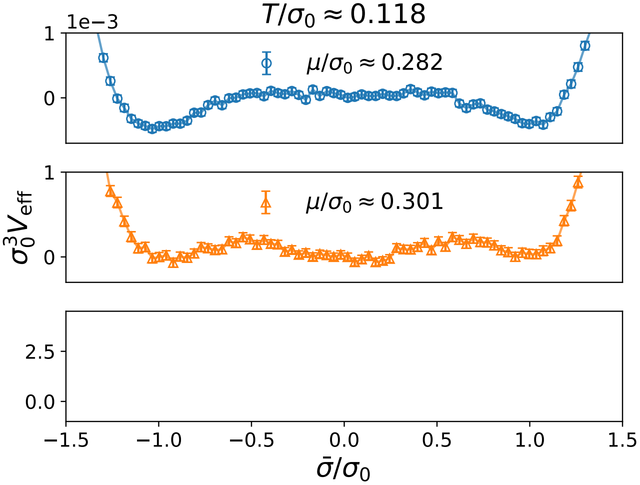

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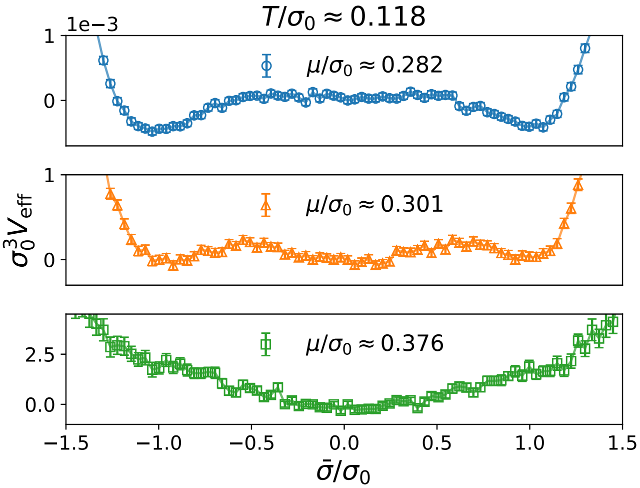

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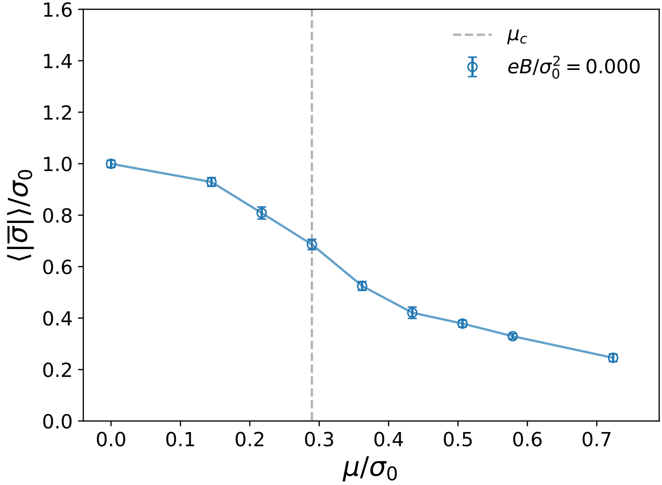

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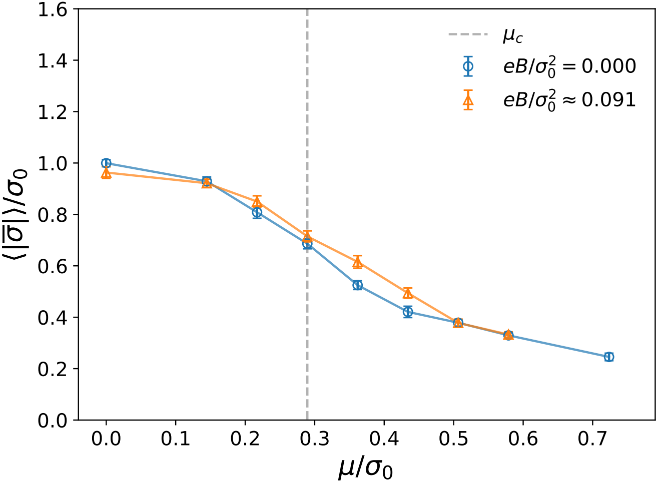

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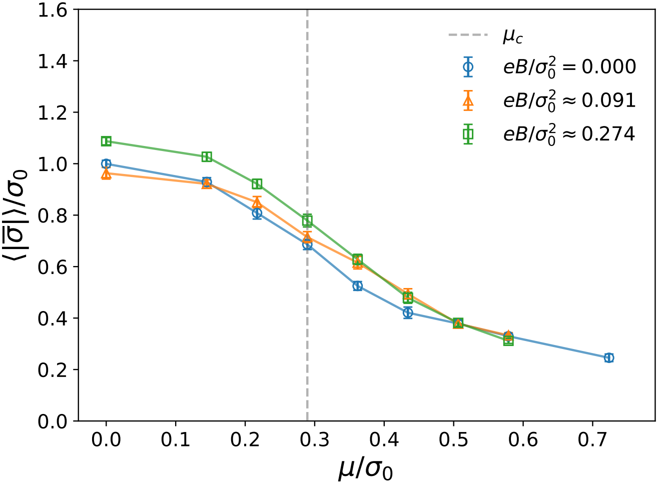

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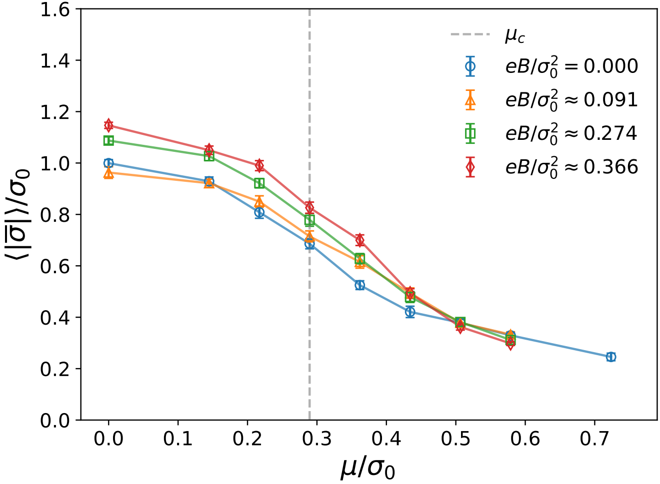

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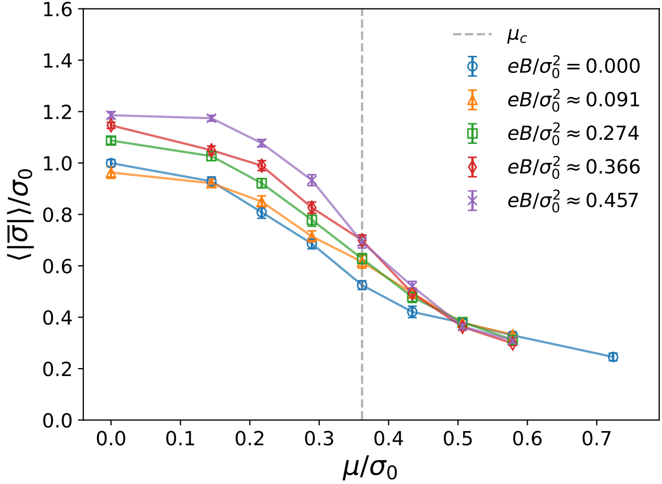

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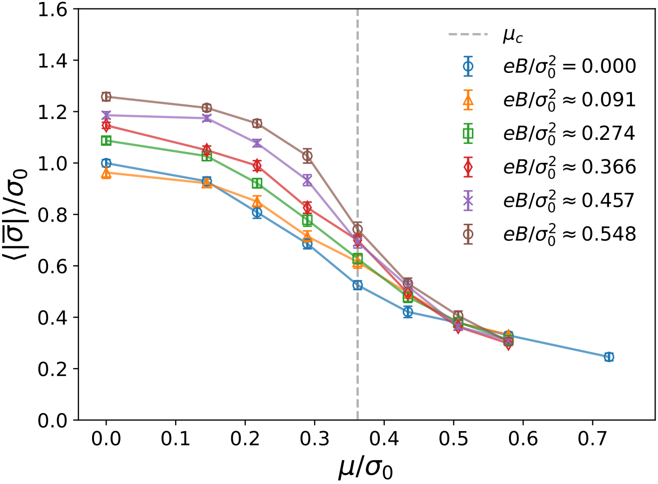

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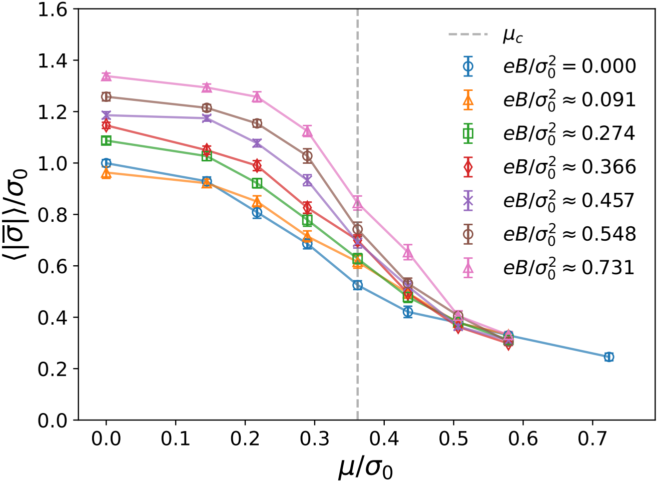

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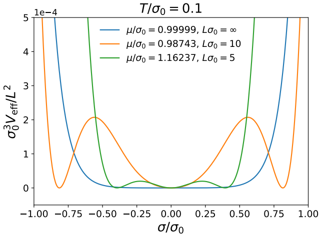
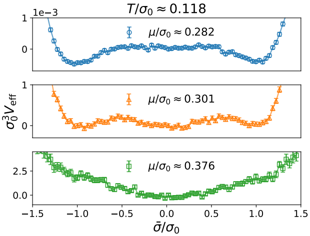
- First-order phase transition for $B = 0$ and $T \neq 0$ on a finite volume.





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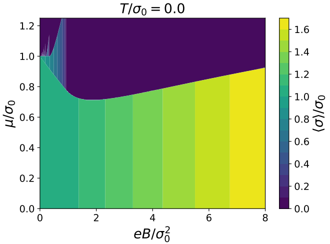

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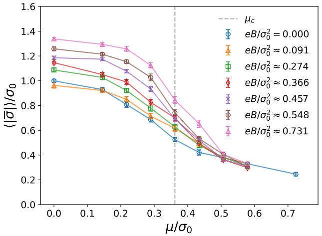

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Outlook

- Spectral analysis to study fate of Landau levels.
- More realistic models.

Backup

The Large-N_f limit

The GN Lagrangian reads

$$\mathcal{L} = i\bar{\psi}\mathbb{1}_{N_f}\left(\partial\!\!\!/ + \sigma + \mu\gamma_0 + ieA\!\!\!/\right)\psi + \frac{N_f}{2g^2}\sigma^2 \ .$$

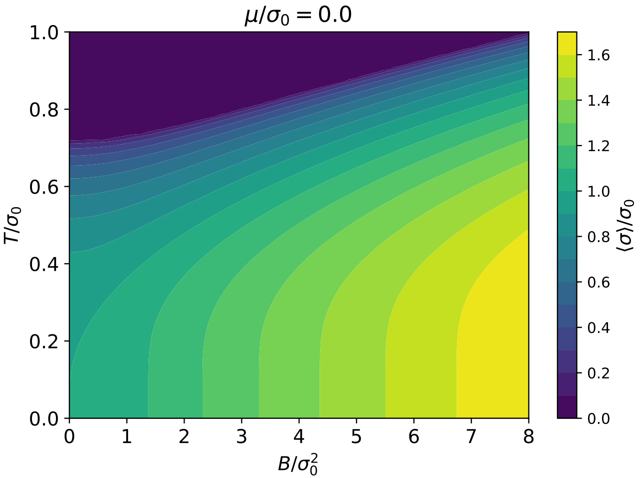
In the limit $N_f \to \infty$, after integrating out the fermions in the path integral, the chiral condensate $\langle\sigma\rangle \propto \langle\bar{\psi}\psi\rangle$ is given by the minimum of

$$S_{\text{eff}}[\sigma] = -\ln\det(D[\sigma]) + \frac{1}{2g^2}\int d^3x\,\sigma^2(x) \ ,$$

with

$$D[\sigma] = \partial\!\!\!/ + \sigma + \mu\gamma_0 + ieA\!\!\!/ \ .$$

Mean-field phase diagrams



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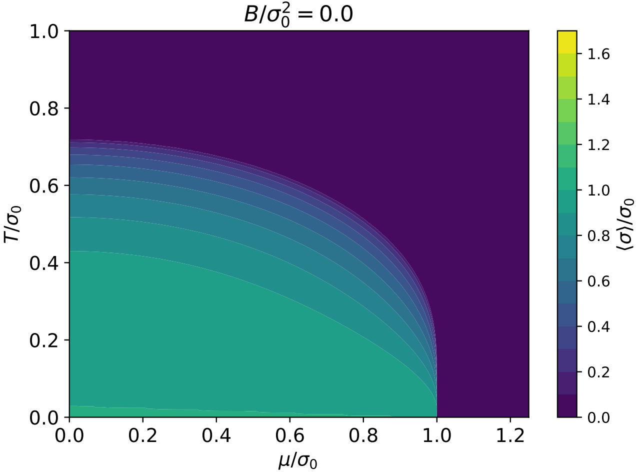


Reducible representation of γ_μ

To allow for a notion of chirality in $(2+1)$ dimensions, we combine the two irreducible representations of the Dirac algebra into a reducible one:

$$\gamma_0 = \begin{pmatrix} \tau_2 & 0 \\ 0 & -\tau_2 \end{pmatrix} , \quad \gamma_1 = \begin{pmatrix} \tau_3 & 0 \\ 0 & -\tau_3 \end{pmatrix} , \quad \gamma_2 = \begin{pmatrix} \tau_1 & 0 \\ 0 & -\tau_1 \end{pmatrix} ,$$

where τ_μ are the usual Pauli matrices.

There are now two γ matrices which anti-commute with all the others:

$$\gamma_4 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} 0 & i\mathbb{1}_2 \\ -i\mathbb{1}_2 & 0 \end{pmatrix} .$$

Chiral symmetry in the continuum

The 1-flavor massless $(2+1)$-dimensional GN model in a reducible representation of the Dirac algebra has the following symmetries:

$$
\begin{aligned}
U_{\mathbb{1}}(1) &: \psi \to e^{i\alpha}\psi \;, \\
U_{\gamma_{45}}(1) &: \psi \to e^{i\alpha\gamma_{45}}\psi \;, \quad \gamma_{45} = i\gamma_4\gamma_5 \;, \\
\mathbb{Z}_2 &: \psi \to i\gamma_5\psi .
\end{aligned}
$$

The $\mathbb{Z}_2$ symmetry generated by γ_4 is not independent.

A mass term induces the breaking pattern

$$
U_{\mathbb{1}}(1) \times U_{\gamma_{45}}(1) \times \mathbb{Z}_2 \to U_{\mathbb{1}}(1) \times U_{\gamma_{45}}(1) \;.
$$

Chiral symmetry on the lattice

On the lattice the symmetries look as follows:

$$U_{\mathbb{1}}(1) : \psi \to e^{i\alpha}\psi \ ,$$
$$U_{\gamma_{45}}(1) : \psi \to e^{i\alpha\gamma_{45}}\psi \ ,$$
$$\mathbb{Z}_2 : \psi \to i\gamma_5(1 - D_{\text{ov}})\psi \ , \quad \bar{\psi} \to i\bar{\psi}\gamma_5 \ ,$$

where D_{ov} is the massless overlap operator, in our case.

There is another $\mathbb{Z}_2$ generated by γ_4, which is, again, independent.

A mass term $\bar{\psi}\left(1 - \frac{D_{\text{ov}}}{2}\right)\psi$ again breaks the $\mathbb{Z}_2$ symmetry, but leaves both $U(1)$'s intact.

Overlap operator in the GN model

We use Neuberger's overlap operator [Neuberger; Phys. Lett B **417** (1998)]

$$D_{\mathrm{ov}} = \mathbb{1} + A/\sqrt{A^\dagger A}\,, \quad A = D_{\mathrm{W}} - \mathbb{1}\ ,$$

where D_W is the standard Wilson operator. The full operator, including σ and μ, reads [Gavai, Sharma; Phys. Lett B **716** (2012)]

$$D_{\mathrm{full}} = \left(1 - \frac{\sigma + \mu\gamma_0}{2}\right) D_{\mathrm{ov}} + \sigma + \mu\gamma_0\ .$$

With the Ginsparg-Wilson chiral condensate $\Sigma_{\mathrm{GW}} = \langle \bar{\psi}\left(\mathbb{1} - \frac{D_{\mathrm{ov}}}{2}\right)\psi\rangle$ we have a Ward identity in analogy to the continuum theory:

$$\langle\sigma\rangle = \Sigma_{\mathrm{GW}}\ .$$

The observable

In order to avoid cancellation of contributions from the two minima of the effective action ($\pm\sigma$) in the broken phase we measure the absolute value

$$\langle|\sigma|\rangle = \langle|\sum_{x\in\Lambda}\sigma(x)|\rangle \ ,$$

where the sum runs over the whole lattice.

As a caveat, this definition makes it harder to determine a phase transition, as we cannot measure $\langle|\sigma|\rangle = 0$.

The complex-action problem

With the standard reweighting approach

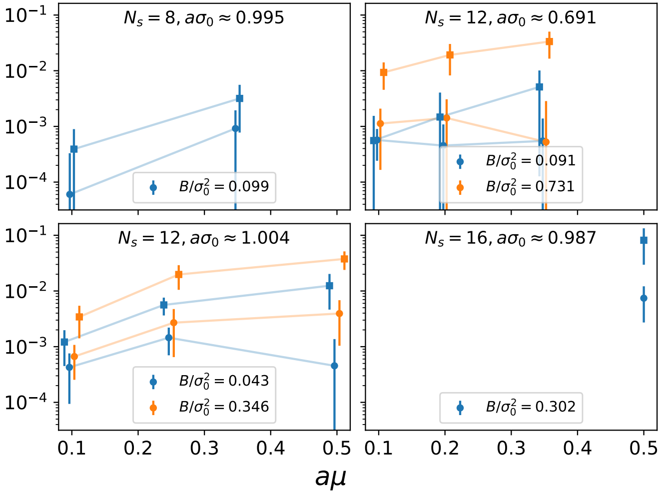
$$\langle \mathcal{O} \rangle = \frac{\left\langle e^{-\mathrm{i}S_I} \mathcal{O} \right\rangle_R}{\left\langle e^{-\mathrm{i}S_I} \right\rangle_R} = \langle \mathcal{O} \rangle_R + \frac{\mathrm{cov}_R\left(e^{-\mathrm{i}S_I}, \mathcal{O}\right)}{\left\langle e^{-\mathrm{i}S_I} \right\rangle} \;,$$

where the action is written as $S = S_R + \mathrm{i}S_I$,

$$\langle \mathcal{O} \rangle_R = \frac{\int \mathcal{D}\sigma \, e^{-S_R} \mathcal{O}[\sigma]}{\int \mathcal{D}\sigma \, e^{-S_R}} \;,$$

and cov_R is the covariance w.r.t S_R.
If $\mathrm{cov}_R\left(e^{-\mathrm{i}S_I}, \mathcal{O}\right)$ is small and $\left\langle e^{-\mathrm{i}S_I} \right\rangle$ is close to unity (as in our case for $\mathcal{O} = |\sigma|$) one may ignore the complex-action problem and only compute $\langle \mathcal{O} \rangle_R$.

The complex-action problem



Index theorem

An exact index theorem in our case would read

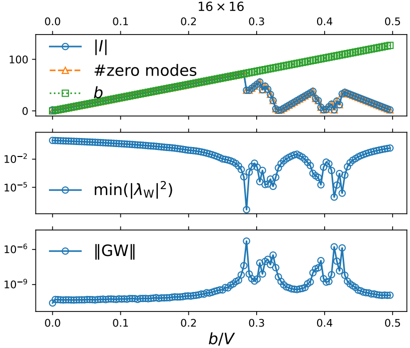
$$I := \mathrm{Index}[D_{\mathrm{ov}}] = \frac{1}{2}\mathrm{tr}\left[\gamma_5 D_{\mathrm{ov}}\right] = b\ ,$$

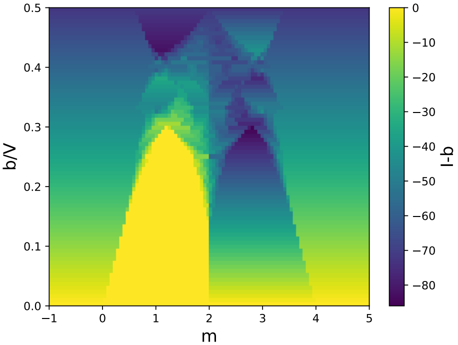
where b is proportional to the magnetic flux:

$$B = \frac{2\pi}{V} b\ .$$

Due to the vanishing theorem $|I|$ is equal to the total number of zero modes of D_{ov}. As the next slide shows the index theorem can be violated for large b and/or Wilson mass parameter m far from 1. λ_W is the smallest eigenvalue of the Wilson kernel used for the overlap and $||\mathrm{GW}||$ measures a possible violation of the Ginsparg-Wilson equation.

Index theorem





Hofstadter's butterfly