Muon $g-2$: Lattice calculations of the hadronic vacuum polarization

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Fermilab
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2023 LATTICE

HIM HELMHOLTZ Helmholtz-Institut Mainz

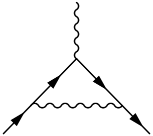
The muon $g-2$: A probe for new physics

- Magnetic moment of charged leptons $l \in \{e, \mu, \tau\}$:

$$\vec{\mu}_l = g_l \cdot \frac{e}{2m_l} \cdot \vec{s}$$

- Quantum corrections lead to deviations from the classical value $g = 2$ (Dirac), the anomalous magnetic moment,

$$a_l = \frac{g_l - 2}{2} = \frac{\alpha}{2\pi} + \mathrm{O}(\alpha^2) \quad \text{(Schwinger)}$$

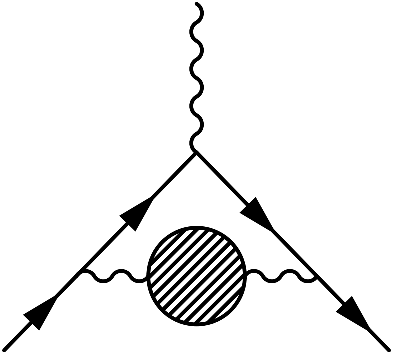


- Contributions from new physics at the scale Λ_{NP} enter a_l via

$$a_l - a_l^{\mathrm{SM}} \propto \frac{m_l^2}{\Lambda_{\mathrm{NP}}^2}$$

with $m_\mu / m_e \approx 207$.

THE MUON $g-2$: A PROBE FOR NEW PHYSICS



(leading order)
hadronic vacuum polarization

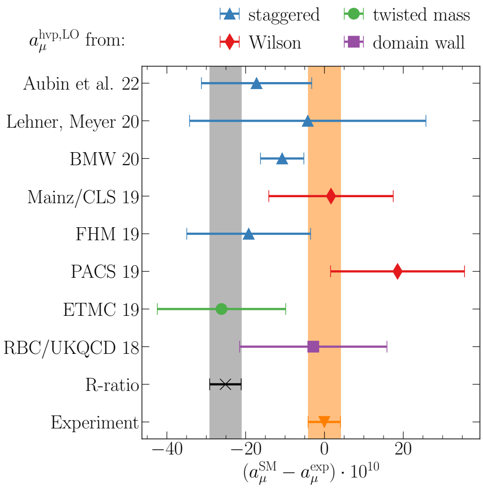
- Standard Model prediction from QED, electroweak and hadronic contributions:

$$a_l^{\text{SM}} = a_l^{\text{QED}} + a_l^{\text{EW}} + a_l^{\text{had}}$$

where $a_l^{\text{had}} = a_l^{\text{hvp}} + a_l^{\text{hlbl}}$.

- Δa_μ^{SM} is dominated by $\Delta a_\mu^{\text{hvp}}$.

Compute the hadronic contributions to a_μ^{hvp} from lattice QCD.



[BNL $g-2$, hep-ex/0602035]
[FNAL $g-2$, 2104.03281] [new results to come]

- There is a $4.2\,\sigma$ discrepancy between the current experimental average and the White Paper average [2006.04822].
- Based on data-driven evaluation of the LO HVP contribution ("R-ratio") with 0.6% precision [Alex Keshavarzi's talk].
- One sub-percent determination of a_μ^{hvp} from the lattice [BMWc, 2002.12347]:
  In tension with the dispersive result.

Goal

Several lattice results at $< 0.5\%$ precision.

a_μ^{hvp} ON THE LATTICE

- Compute $a_\mu^{\rm hvp}$ via [Laurup et al.] [Blum, hep-lat/0212018]

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty {\rm d}Q^2 f(Q^2)\hat{\Pi}(Q^2)\,, \qquad \text{with} \quad \hat{\Pi}(Q^2) = 4\pi^2\left[\Pi(Q^2) - \Pi(0)\right]$$

from a known QED kernel function $f(Q^2)$ and the polarization tensor

$$\Pi_{\mu\nu}(Q) = \int {\rm d}^4x\, {\rm e}^{iQ\cdot x} \langle j_\mu^{\rm em}(x)\, j_\nu^{\rm em}(0)\rangle = (Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)\Pi(Q^2)\,.$$

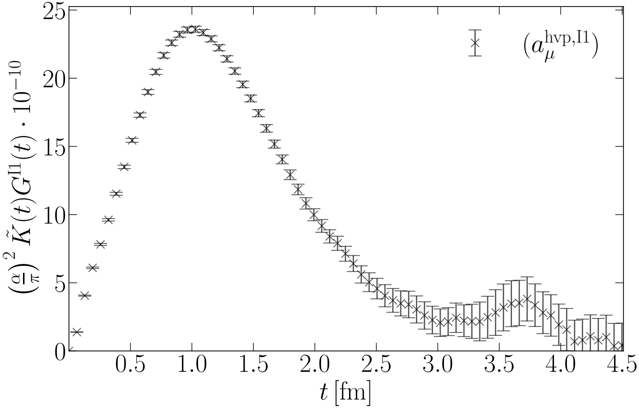
- $a_\mu^{\rm hvp}$ in the time-momentum representation [Bernecker, Meyer, 1107.4388],

$$a_\mu^{\rm hvp} := \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\widetilde{K}(t) \quad \text{with the known QED kernel function } \widetilde{K}(t)\,,$$

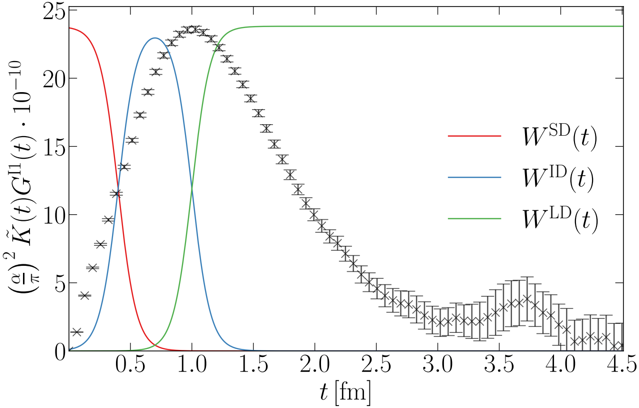
in terms of the zero-momentum vector correlator $G(t)$ (de facto standard).

- Alternative: coordinate space method [Meyer, 1706.01139] [Chao et al., 2211.15581].

$$(a_\mu^{\rm hvp}) := \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\widetilde{K}(t), \qquad G(t) = -\frac{a^3}{3}\sum_{k=1}^{3}\sum_{\vec{x}} \langle j_k^{\rm em}(t,\vec{x})\, j_k^{\rm em}(0)\rangle$$



$$(a_\mu^{\text{hvp}})^i := \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\widetilde{K}(t)\; W^i(t;t_0;t_1)\,, \qquad G(t) = -\frac{a^3}{3}\sum_{k=1}^{3}\sum_{\vec{x}} \langle j_k^{\text{em}}(t,\vec{x})\, j_k^{\text{em}}(0)\rangle$$



- Windows in the TMR: separate short- from long-distance effects [RBC/UKQCD, 1801.07224].

$a_\mu^{\rm hvp}$ ON THE LATTICE: EUCLIDEAN TIME WINDOWS

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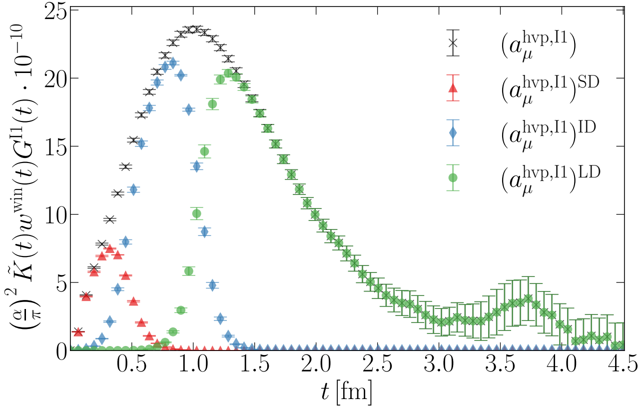


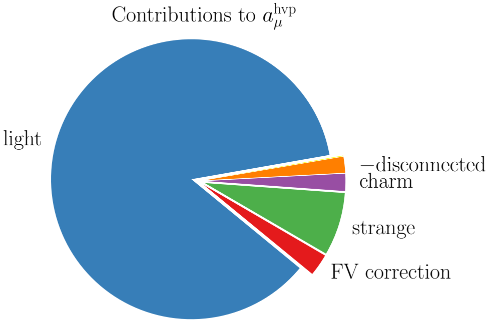
- Windows in the TMR: separate short- from long-distance effects [RBC/UKQCD, 1801.07224].
- Intermediate window $a_\mu^{\rm win}$:
  - Cutoff effects suppressed.
  - No signal-to-noise problem.
  - Finite-volume effects small.

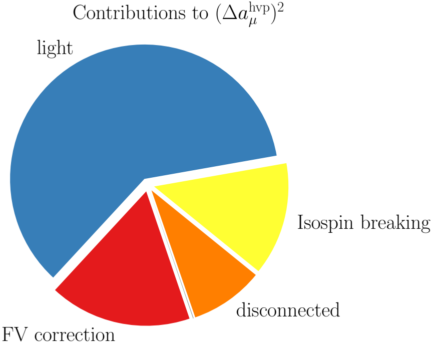
The electromagnetic current

$$j_\mu^{\rm em} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c + \ldots = j_\mu^{I=1} + j_\mu^{I=0}$$

from zero-momentum vector-vector correlation functions

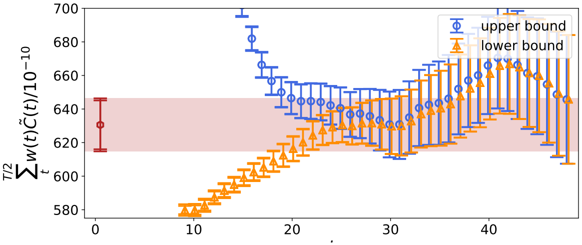
$$G^{\rm isoQCD}(t) = \frac{5}{9}G^{\rm light}(t) + \frac{1}{9}G^{\rm strange}(t) + \frac{4}{9}G^{\rm charm}(t) + G^{\rm disc}(t) + \ldots$$





Based on [BMWc, 2002.12347]: $a_\mu^{\rm hvp} = 707.5\,(5.5)\cdot 10^{-10}$

Dominant sources of uncertainty

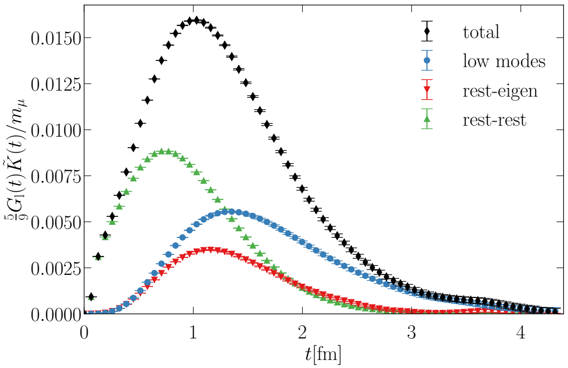


Exponential deterioration of the signal-to-noise ratio.

Improve the signal at large t via:

- Bounds on the correlator.
- Noise reduction methods:
  - Truncated Solver Method
  - Low Mode Averaging
  - All Mode Averaging
- Spectral reconstruction of the $\pi\pi$ contributions.
- Multi-level integration.
  [Dalla Brida et al., 2007.02973]

[RBC/UKQCD, 1910.11745]



[Mainz, Lattice 2022]

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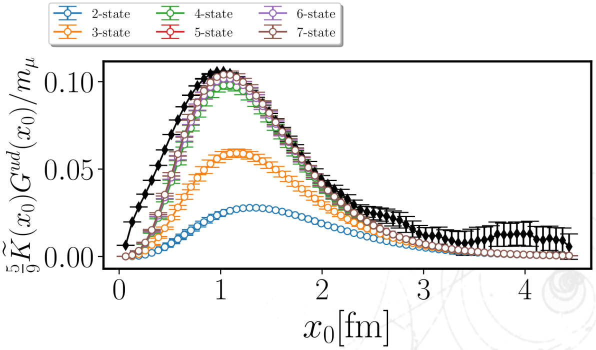

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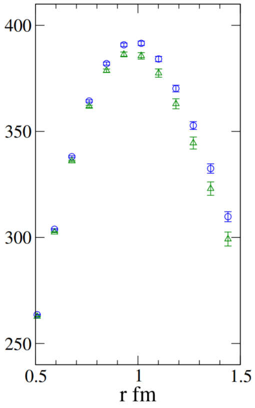
[Mainz]

FINITE-VOLUME EFFECTS

3% finite-L corrections for $a_\mu^{\rm hvp}$ at $m_\pi L = 4$, mostly in the **isovector channel**.

- EFT and model calculations.
  - NNLO χPT
  - Two-pion spectrum in finite-volume and the timelike pion form factor [Meyer, 1105.1892] [Lellouch and Lüscher, hep-lat/0003023] [Giusti et al., 1808.00887].
  - Pions winding around the torus and the electromagnetic pion form factor [Hansen, Patella, 1904.10010, 2004.03935].
  - Rho-pion-gamma model [Sakurai] [Jegerlehner, Szafron, 1101.2872] [HPQCD, 1601.03071].
- Simulations at $L > 10\,\mathrm{fm}$ [PACS, 1902.00885] [BMWc, 2002.12347].
  - Uncertainty statistics dominated.



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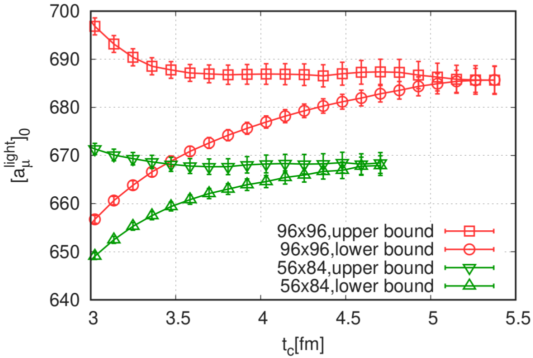

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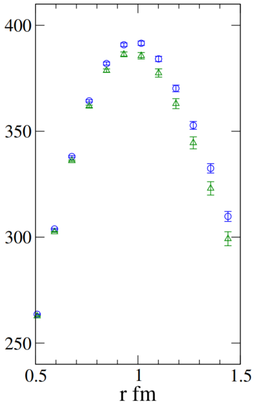


- Simulations at $L > 10\,\mathrm{fm}$ [PACS, 1902.00885] [BMWc, 2002.12347].
  - Uncertainty statistics dominated.

Systematic uncertainties from the continuum extrapolation may be dominant.

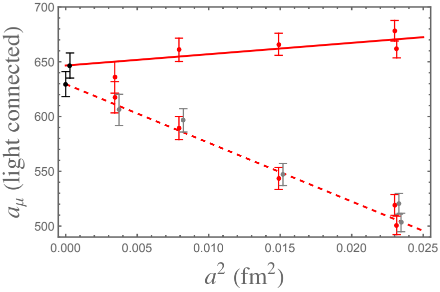
- Log-enhanced cutoff effects $\mathrm{O}(a^2 \log(a))$ from very short distances in the TMR integral [Della Morte et al., 0807.1120] [Cè et al., 2106.15293].
  $\rightarrow$ Removed by computing the high energy contribution in perturbative QCD [1107.4388, Bernecker and Meyer] [Sommer et al., 2211.15750].

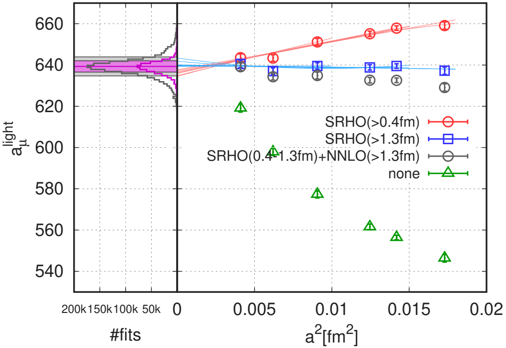
- Have to expect the leading asymptotic behavior $\sim [\alpha_\mathrm{s}(1/a)]^{\hat{\Gamma}} a^2$ with unknown $\hat{\Gamma} \gtrapprox 0$ [1912.08498, Husung et al.] [Husung].

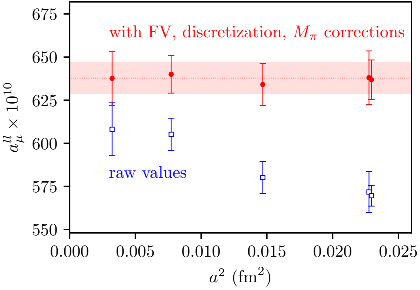
- Mandatory to include fine resolutions $\leq 0.05\,\mathrm{fm}$ for per-mil uncertainties.

- Staggered quarks: taste violations distort the pion spectrum.
  - Taste breaking may introduce non-linear effects (in a^2).

  $\rightarrow$ Corrections applied at finite lattice spacing.







- Continuum extrapolations of a_μ^{hvp} computed with staggered quarks.
- Compare raw and corrected data.

[Aubin et al., 2204.12256] [BMWc, 2002.12347]
[Fermilab, HPQCD, MILC, 1902.04223]

- Need (few) per-mill precision scale setting [Mainz, 1705.01775]:

$$\frac{\delta a}{a} = 1\,‰ \rightarrow \frac{\delta_a a_\mu^{\mathrm{hvp}}}{a_\mu^{\mathrm{hvp}}} = 1.8\,‰ \qquad \text{whereas} \qquad \frac{\delta_a a_\mu^{\mathrm{win}}}{a_\mu^{\mathrm{win}}} = 0.5\,‰$$

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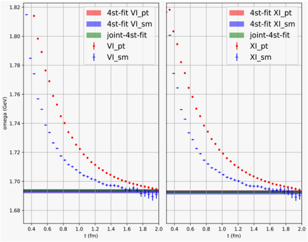
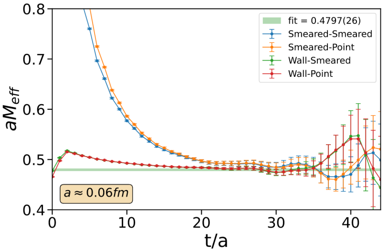
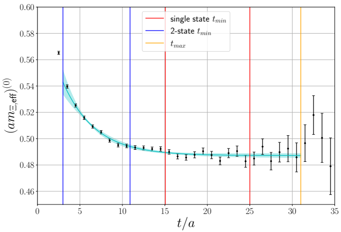
- Pseudoscalar decay constants, baryons (Ω, Ξ), gradient flow scales (t_0, w_0)



[Wang]



[Bazavov]



[Segner]

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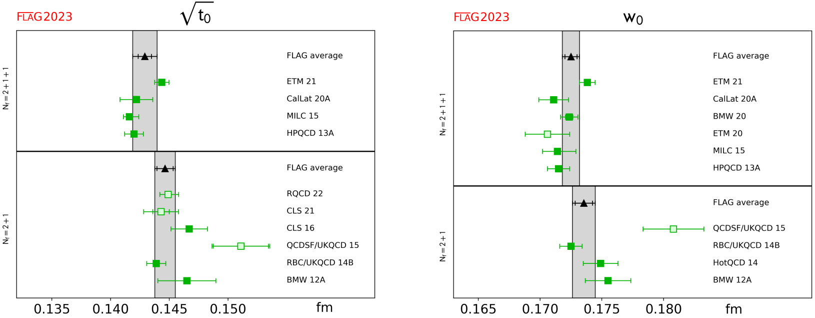
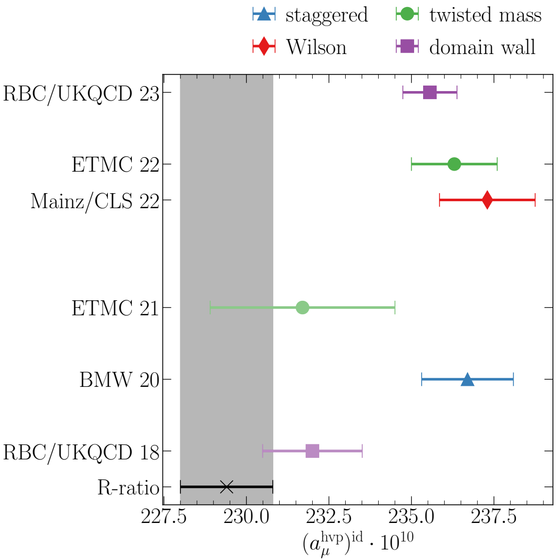


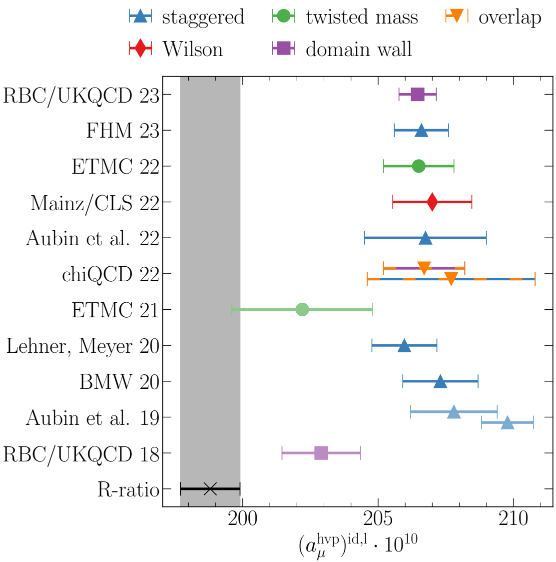
[FLAG21]

Window observables

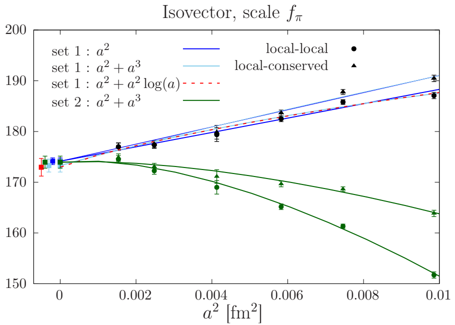
The intermediate-distance window

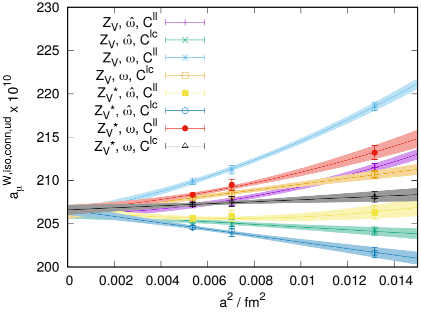


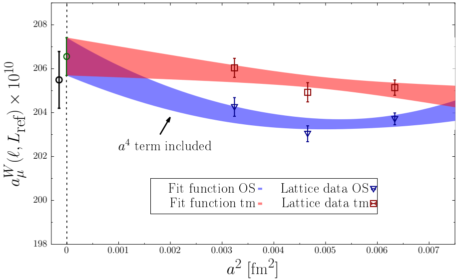
- $3.8\,\sigma$ tension between lattice QCD and data-driven evaluation [Colangelo et al., 2205.12963].
- This accounts for 50% of the difference between BMW 20 and the White Paper average for a_μ^{hvp}.



- $3.8\,\sigma$ tension between lattice QCD and data-driven evaluation [Colangelo et al., 2205.12963].
- This accounts for 50% of the difference between BMW 20 and the White Paper average for a_μ^{hvp}.
- Agreement across many actions for the light-connected contribution (87%).
- Data-driven estimate: [Benton et al., 2306.16808] [Golterman]



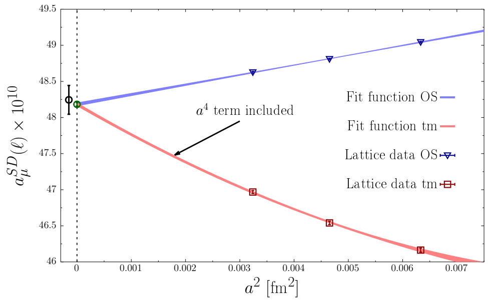


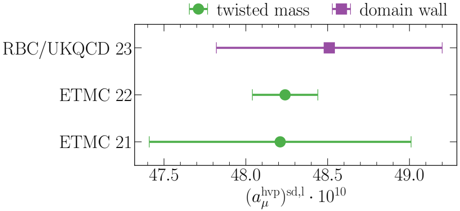


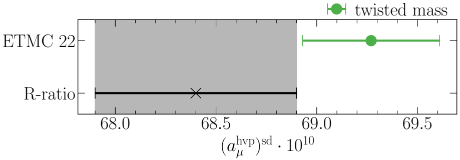
- Different discretization prescriptions have to agree in the continuum.
- May perform combined extrapolations.

[Mainz, 2206.06582] [RBC/UKQCD, 2301.08696] [ETMC, 2206.15084]

- Short-distance window dominated by perturbative QCD.
- Systematic uncertainties from the continuum extrapolation dominant but subleading with respect to a_μ^{hvp}.
- More results to come?



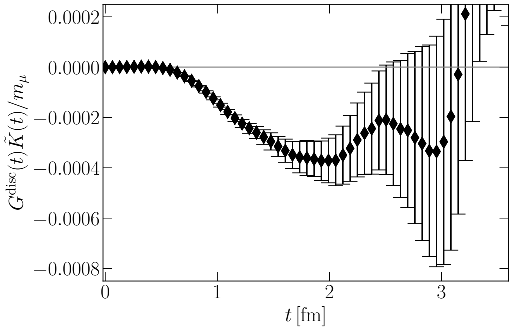




[ETMC, 2206.15084]

SUBLEADING CONTRIBUTIONS TO a_μ^{hvp}

- Signal-to-noise problem:
  How far can we integrate?

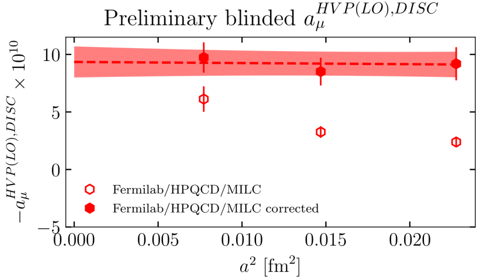


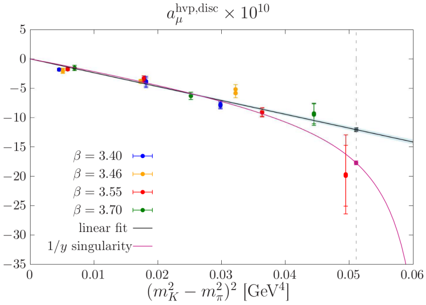
→ Bounding method for disconnected or isoscalar correlator

$$G^{I=0,\not{c}}(t) = G^{\mathrm{disc}}(t) + \frac{1}{18}G^{\mathrm{l}}(t) + \frac{1}{9}G^{\mathrm{s}}(t)$$

- Many algorithmic tricks:
  - One-end trick / Frequency splitting
    [McNeile, Michael, hep-lat/0603007]
    [Giusti et al., 1903.10447]
  - Low-mode averaging
    [Neff et al., hep-lat/0106016]
    [Giusti et al., hep-lat/0402002]
    [DeGrand et al., hep-lat/0401011]
  - Truncated solver method
    [Bali et al., 0910.3970]
  - Hierarchical probing
    [Stathopoulos et al., 1302.4018]
  - Hopping parameter expansion
    [Thron et al., hep-lat/9707001]
  - Randomized sparse grid
    [Blum et al., 1512.09054]

- $G^{\rm disc}(t) \to -\frac{1}{9}G^{\rm I=1}(t)$ at large t.
  - Finite-size correction.
  - Taste breaking.
  - Chiral dependence.





[FHM, 2112.11339] [Mainz 20 (prelim)]

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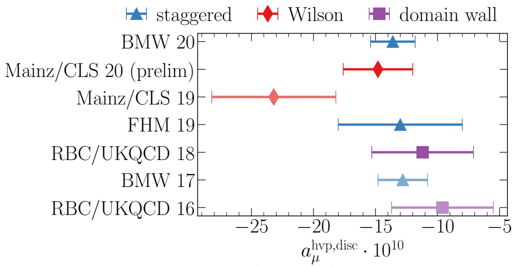
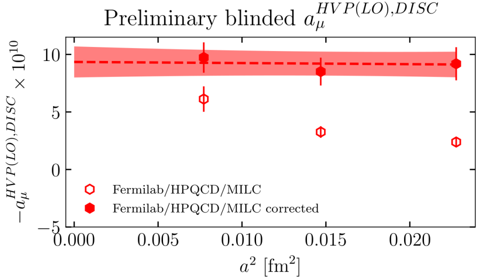
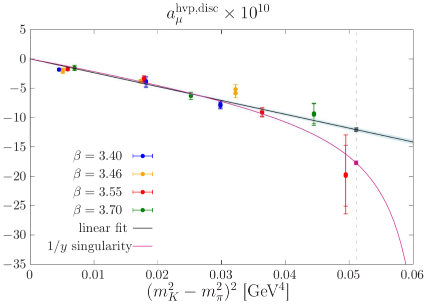
2% contribution sufficiently well determined?







[FHM, 2112.11339] [Mainz 20 (prelim)]

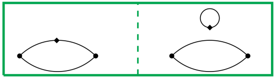
QED AND STRONG ISOSPIN BREAKING

Need to include $\mathrm{O}(\frac{m_\mathrm{u}-m_\mathrm{d}}{\Lambda_\mathrm{QCD}})$ and $\mathrm{O}(\alpha)$ effects for per-mil precision.

- Results in isospin symmetric QCD have to be compared in the same scheme.
  $\rightarrow$ Effort in FLAG to propose a scheme [Tantalo, 2301.02097] [Portelli].
- Various ways to compute these corrections:
  - **Perturbative expansion around isospin symmetric QCD** [RM123, 1303.4896].
  - Simulation of dynamical QCD+QED [CSSM/QCDSF/UKQCD] [RC*, 2212.11551].
  - Infinite volume QED [RBC/UKQCD, 1801.07224] [Biloshytskyi et al., 2209.02149][Parrino].
- A lot of work for a small correction:
  Low-mode averaging, truncated solves, non-unitary valence quarks, ...
- QED_L: Finite-volume corrections scale as $\mathrm{O}(1/L^3)$ [Bijnens et al., 1903.10591]
  $\rightarrow$ sufficient for the precision goal.

Overview of published results - contributions to $a_\mu \times 10^{10}$

- Strong isospin breaking:
  Five groups agree within $1\,\sigma$.

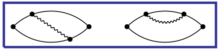


6.60(63)(53)		BMW
10.6(4.3)(6.8)		RBC/UKQCD
6.0(2.3)		ETM
7.7(3.7)	9.0(2.3)	FHM
9.0(0.8)(1.2)		LM

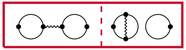
BMW [Nature 593 (2021) 7857, 51-55]
RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
ETM [Phys. Rev. D 99, 114502 (2019)]
FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]

Adapted from [V. Gülpers @ Lattice HVP workshop 2020]

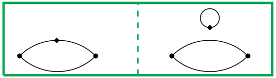
Overview of published results - contributions to $a_\mu \times 10^{10}$



BMW $-1.23(40)(31)$
RBC/UKQCD $5.9(5.7)(1.7)$
ETM $1.1(1.0)$



$-0.55(15)(10)$ BMW
$-6.9(2.1)(2.0)$ RBC/UKQCD



$6.60(63)(53)$ BMW
$10.6(4.3)(6.8)$ RBC/UKQCD
$6.0(2.3)$ ETM
$7.7(3.7)$ $9.0(2.3)$ FHM
$9.0(0.8)(1.2)$ LM

BMW [Nature 593 (2021) 7857, 51-55]
RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
ETM [Phys. Rev. D 99, 114502 (2019)]
FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]

- Strong isospin breaking: Five groups agree within $1\,\sigma$.
- QED: agreement on the total valence contribution.

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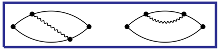
Overview of published results - contributions to $a_\mu \times 10^{10}$



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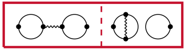
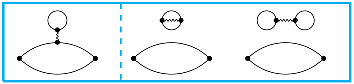
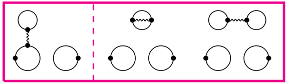


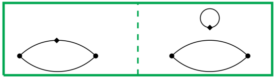
$-0.55(15)(10)$	BMW
$-6.9(2.1)(2.0)$	RBC/UKQCD



$-0.0093(86)(95)$ $0.37(21)(24)$ BMW



$0.011(24)(14)$ $-0.040(33)(21)$ BMW



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$10.6(4.3)(6.8)$		RBC/UKQCD
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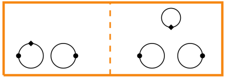

$-4.67(54)(69)$ BMW

BMW [Nature 593 (2021) 7857, 51-55]
RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
ETM [Phys. Rev. D 99, 114502 (2019)]
FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]

Adapted from [V. Gülpers @ Lattice HVP workshop 2020]

- Strong isospin breaking: Five groups agree within $1\,\sigma$.
- QED: agreement on the total valence contribution.
- One complete calculation [BMWc, 2002.12347]: $\delta a_\mu^{\text{hvp}} = 0.5(1.4) \cdot 10^{-10}$
- Work in progress:
  [Mainz, 2206.06582]
  [RBC/UKQCD, Lattice 2022]
  [BMWc, Lattice 2022]
  [FHM, 2212.12031]
  [Harris et al., 2301.03995]

Conclusions: Tensions

- The discrepancy between lattice and data-driven calculations in the **intermediate window** is firmly established.
- Further checks via $a_\mu^{\mathrm{hvp,SD}}$ and $a_\mu^{\mathrm{hvp,LD}}$ (to come) [Lehner].
- Other windows can be calculated to scrutinize the discrepancy [Lehner and Meyer, 2003.04177] [Colangelo et al., 2205.12963] [FHM, 2207.04765].
- More insights from direct comparison with the smeared R-ratio? [EMTC, 2212.08467].
- Similar tension in $\Delta\alpha_{\mathrm{had}}$ [BMWc, 1711.04980, 2002.12347] [Mainz, 2203.08676] [Lellouch].

Conclusions: The way ahead

- More and more precise lattice results for a_μ^{hvp} urgently needed (and expected).
- Improvements: In the last years and ongoing
  - Isovector contribution with sub-percent precision.
  - EFT and data based finite-size corrections.
  - Finer lattices, more lattice spacings.
  - More precise scale setting.
  - Isospin breaking effects (beyond the electroquenched approximation).
  - **Blinded analyses**.
- Perform lattice averages of sub-contributions to improve the accuracy of a_μ^{hvp}?
  - Relies on a common hadronic scheme for isospin symmetric QCD.
  - Correlations: Finite-size corrections, taste-breaking corrections, same ensembles...