QED in external EM fields

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Introduction

We simulate lattice QED in (large) external electromagnetic(EM) fields using methods developed for lattice QED.

Our first project is simulating lattice QED in large constant (in space and time) magnetic fields using the RHMC method. We shall assume that the external magnetic field B is oriented in the z (3) direction so that the external vector potential A_{ext} lies in the $x-y$ (1-2) plane.

Classically electrons(positrons) in such a field traverse helical orbits around magnetic field lines. In quantum mechanics the motion in the $x-y$ plane is in discrete energy levels – Landau levels – while that in the z direction is free.

$$E_n(p_z) = \pm\sqrt{m^2 + 2eBn + p_z^2}$$

where $n = 0, 1, 2, ...$, and the degeneracy of the lowest Landau level ($n = 0$) is half that of the higher levels.

When QED is taken into account, for large enough eB, we expect the energy levels to behave similarly, and the lowest Landau level LLL to give the dominant contribution to the functional integral.

One of the more interesting predictions of approximate analyses of QED in large B using a truncated Schwinger-Dyson approach is Magnetic Catalysis of dynamical symmetry breaking, giving a dynamical (non-perturbative) mass to the electron $\propto \sqrt{eB}$ and a non-vanishing chiral condensate $\propto (eB)^{3/2}$ when the input electron mass vanishes, associated with a dimensional reduction from $3+1$ to $1+1$ dimensions for charged particles.

For fine-structure constant $\alpha = 1/137$ the Schwinger-Dyson prediction for the dynamical electron mass at our chosen $eB \approx 0.4848...$, $m_{dyn} \approx 2 \times 10^{-35}$. Since this is far below anything we could measure on the lattice, we choose a stronger electron charge $\alpha = 1/5$ where the predicted $m_{dyn} \approx 3 \times 10^{-4}$.

Our simulations show clear evidence that the chiral condensate $\langle\bar{\psi}\psi\rangle$ remains non-zero as $m \to 0$. Hence chiral symmetry is broken dynamically by the magnetic field.

Extracting chiral condensate from lattice simulations

We simulate lattice QED in a strong magnetic field using the RHMC algorithm. A non-compact gauge action is used for the internal electromagnetic fields. We use staggered fermions and a rational approximation to tune to 1 electron flavour. We use a compact interaction between the electromagnetic fields (internal and external) and the fermions to render the action gauge-invariant. [See appendix for more details.]

As mentioned above we choose $\alpha = 1/5$ to give a measurable signal. Since for free fermions in an external magnetic field, our approach fails when $eB \gtrsim 0.63$ we choose $eB = 2\pi \times 100/36^2 \approx 0.4848...$ on lattices with $N_x = N_y = 36$ or $N_x = N_y = 18$.

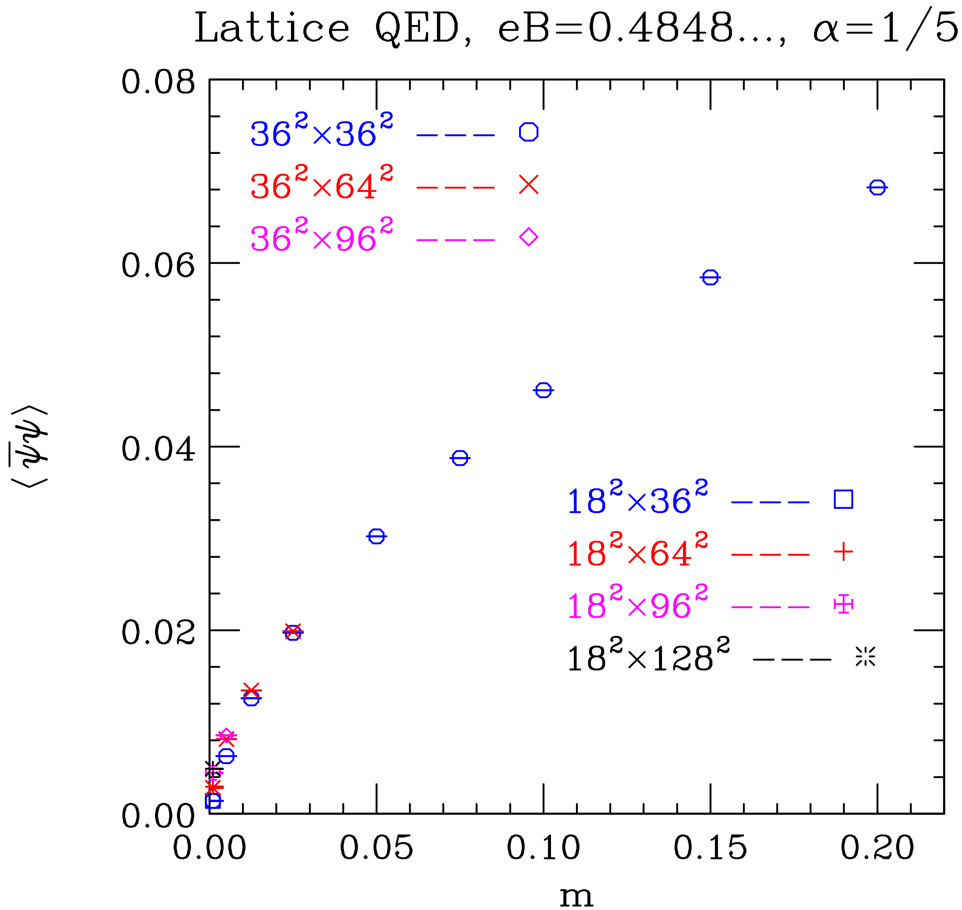
Since we are interested in the limit $m \rightarrow 0$, we need to perform simulations down to rather small m (We use masses as small as $m = 0.001$).

For the smallest masses we will need to measure the chiral condensate on a series of lattice sizes until increasing the lattice size does not increase $\langle\bar{\psi}\psi\rangle$.

Because we expect the functional integral will be dominated by the LLLs, then provided N_x and N_y are appreciably greater than the size of the LLL in the $x-y$ plane ($\sim 1/\sqrt{eB}$) we do not need to increase these dimensions in our simulations. Hence we run our simulations for a series of $N_z = N_t$ values with $N_x = N_y = 36$ or $N_x = N_y = 18$.

We simulate at a selection of masses covering the range $0.001 \leq m \leq 0.2$ In order to remove finite size effects, we need to simulate on lattices with $N_z = N_t$ as large as 128 (in particular $18^2 \times 128^2$) at $m = 0.001$. We simulate for a total of at least 1250 trajectories for each choice of parameters.

Figures 1,2,3, show the mass dependence of the chiral condensates $\langle\bar{\psi}\psi\rangle$ for the lattice sizes we use. Note that, for each m value, we consider the measurement for the lattice with the largest $N_z = N_t$ to be our best estimate for the infinite lattice



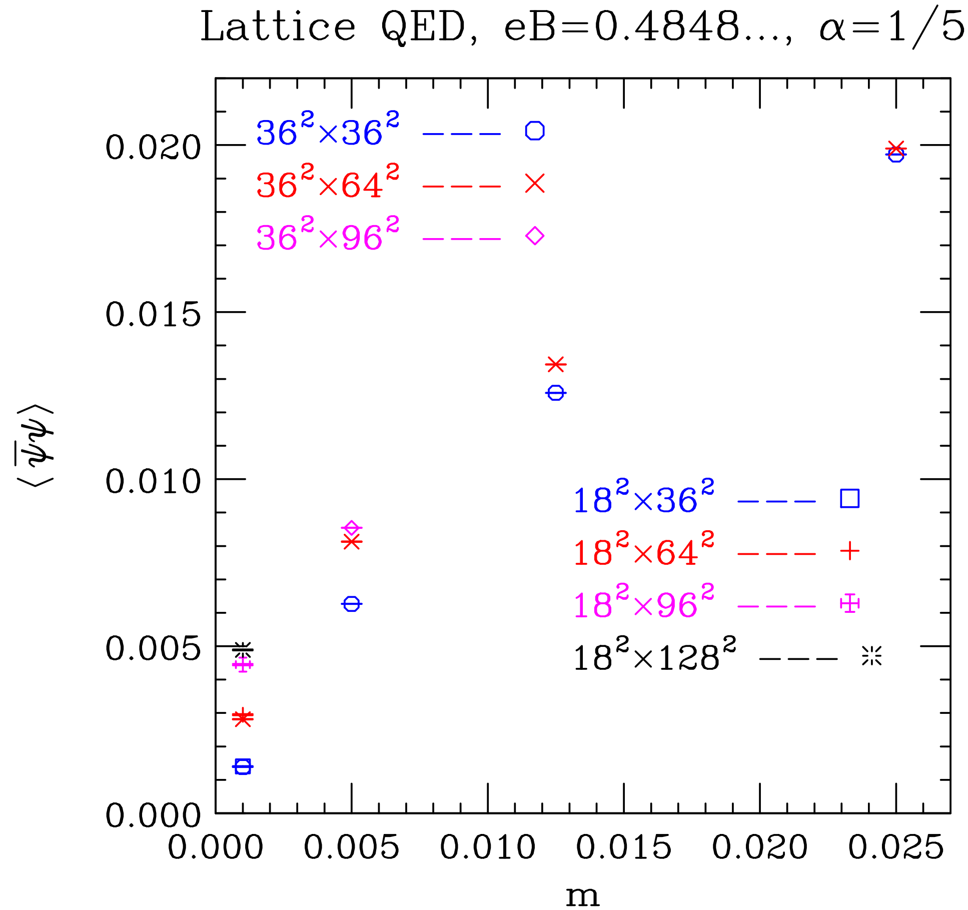


Figure 1: $\langle\bar{\psi}\psi\rangle$ as a function of mass at $eB = 2\pi \times 100/36^2$,showing dependence on lattice size in the z and t directions.

Figure 2: As in figure 1, but on an expanded scale

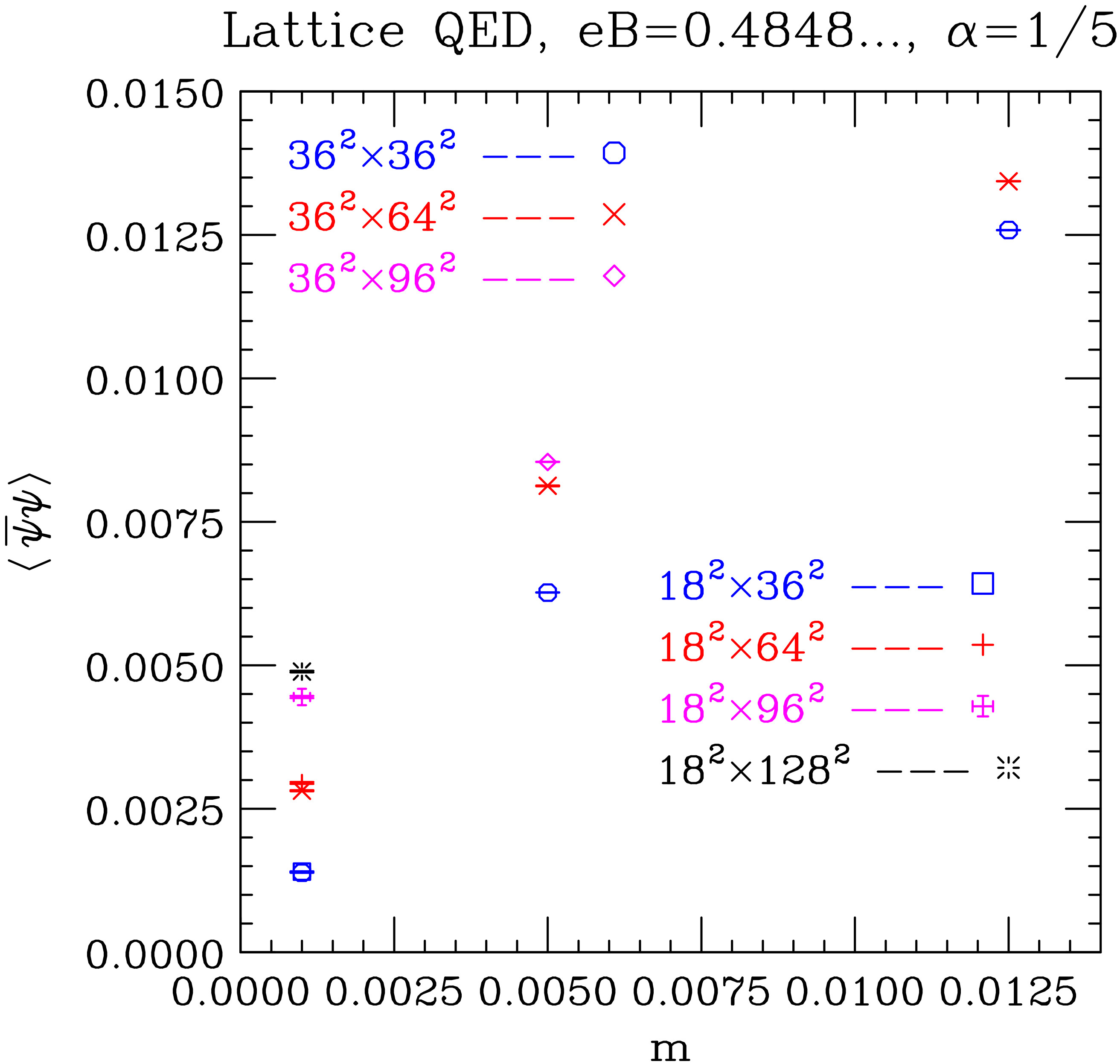


Figure 3: As in figure 2, but on an expanded scale.

value.

- From these plots, it would appear that the chiral condensate remains finite as $m \to 0$, and that the $m = 0$ value ≈ 0.004, indeed a selected set of fits to the simple form

$$\langle\bar{\psi}\psi\rangle = a + b\,m + c\,m\log(m)$$

gives good qualitative fits with

$$0.0035 \lesssim \langle\bar{\psi}\psi\rangle(m=0) \lesssim 0.0046$$

Comparison of lattice QED with Schwinger-Dyson results

Our lattice QED simulations at $\alpha = 1/5$, $eB = 0.4848...$, $N_f = 1$ are consistent with a chiral condensate $\langle\bar{\psi}\psi\rangle \approx 4 \times 10^{-3}$ in the limit $m \rightarrow 0$. This result is gauge invariant.

The Schwinger-Dyson estimates at $\alpha = 1/5$, $eB = 0.4848...$, $N_f = 1$ give $\langle\bar{\psi}\psi\rangle \approx 1.2 \times 10^{-4}$ in the massless limit. This is for what is considered to be the optimal gauge for this truncation. A generic covariant gauge choice yields $\langle\bar{\psi}\psi\rangle \approx 2.4 \times 10^{-3}$ in the limit $m \rightarrow 0$.

However, if we ignore the fact that the $3+1 \rightarrow 1+1$ dimensional reduction only applies to charged particles and apply it to all fields, the dynamics reduces to that of the Schwinger model ($1+1$ dimensional QED) for which we know the chiral condensate. Relating this to the $3+1$ dimensional condensate gives $\langle\bar{\psi}\psi\rangle = 5.43... \times 10^{-3}$, suspiciously close to what we measured.

If the Schwinger-Dyson results are correct, this suggests that our simulations use a LLL whose xy projection covers too few lattice sites to be able to distinguish that our photons are really $3+1$ dimensional, i.e. the lattice is too coarse. So we need to simulate at lower eB where the LLL has a larger xy projection in lattice units to test if the Schwinger-Dyson results are correct.

Of course, the lattice α is the bare value measured at lattice spacing 1 while the Schwinger-Dyson α is the running α measured at the dynamical mass, which should be smaller. This would make the disagreement even larger, but difficult to estimate.

The main sources of systematic errors for the lattice calculation are the continuation to $m=0$, and those related to the 'flavour' symmetry violations associated with using (rooted) staggered fermions. Other sources include the momentum cutoff provided by the lattice, which is required, since QED is presumably only an effective field theory and does not have an ultraviolet completion.

The approximations used to justify the truncations used to make the Schwinger-Dyson approach tractable can only be justified for small α. It is quite possible that $\boldsymbol{\alpha = 1/5}$ is large enough that these approximations have broken down.

Although the lattice and Schwinger-Dyson estimates of the chiral condensate differ by between 1 and 2 orders of magnitude at $\boldsymbol{\alpha = 1/5}$ this should be compared with the difference between the value of the dynamical electron masses at $\boldsymbol{\alpha = 1/137}$ and at $\boldsymbol{\alpha = 1/5}$, which is more than 30 orders of magnitude.

Discussions and Conclusions

- Our simulations of Lattice QED in a constant magnetic field using the RHMC method show evidence of a non-zero chiral condensate in the $m \to 0$ limit at a relatively strong coupling ($\alpha = 1/5$). However, the chiral condensate appears to be about 1.5 orders of magnitude larger than the best estimate using a truncated Schwinger-Dyson approach, and closer to an earlier estimate which used an unimproved rainbow approximation.
- The lattice simulations admit systematic improvements to understand the source of any disagreements between lattice and continuum methods.
- The stored configurations allow for further measurements of properties of QED in an external magnetic field, beyond those that are measured during the simulations.
- We will measure the effect that QED in an external magnetic field has on the coulomb field of a point charge using (large) Wilson loops on stored configurations. This is expected to be

a combination of partial screening and distortion of this electric field, and should be visible even at physical charge $\alpha \approx 1/137$.

- Other quantities we plan to calculate on stored configurations include the electron and photon propagators and the fermion effective action (proportional to $\log(D+m)$) in a small constant external electric field.

- We are currently extending our simulations to a weaker external magnetic field to check the dependence of the chiral condensate on eB.

- QED in an external electric field is of interest because of the Sauter-Schwinger effect (production of electron-positron pairs from the vacuum). We will check lattice calculations against known results in the absence of QED.

- We are now investigating how we might simulate the Sauter-Schwinger effect, including QED, on the lattice. This is a much more difficult problem, since the action with an external electric field becomes complex. A first attempt will be to simulate with external magnetic and electric fields where this external field configuration comes from boosting a system with only an external magnetic field.

- Future plans include to attempt to check the validity of our simulations with an external magnetic field, using improved lattice actions.

These simulations were performed on the Bebop Cluster at ANL, Cori at NERSC using an ERCAP(DOE) allocation, Perlmutter at NERSC using early-user access and using ACCESS(NSF) allocations on Expanse at UCSD, Bridges-2 at PSC and Stampede-2 at TACC.

One of us (DKS) would like to thank G. T. Bodwin for helpful discussions, while JBK would like to acknowledge conversations with A. Shovkovy and V. Yakimenko.

Appendix: Lattice QED in an external Magnetic Field

We simulate using the non-compact gauge action

$$S(A) = \frac{\beta}{2}\sum{}_{n,\mu<\nu}[A_\nu(n+\hat{\mu}) - A_\nu(n) - A_\mu(n+\hat{\nu}) + A_\mu(n)]^2$$

where n is summed over the lattice sites and μ and ν run from 1 to 4 subject to the restriction. $\beta = 1/e^2$. The functional integral to calculate the expectation value for an observable $\mathcal{O}(A)$ is then

$$\langle\mathcal{O}\rangle = \frac{1}{Z}\int_{-\infty}^{\infty}\Pi_{n,\mu}dA_\mu(n)e^{-S(A)}[\det\mathcal{M}(A+A_{ext})]^{1/8}\mathcal{O}(\mathcal{A})$$

where $\mathcal{M} = M^\dagger M$ and M is the staggered fermion action in the presence of the dynamic photon field A and external photon field A_{ext} describing the magnetic field B (or rather eB). M is defined by

$$M(A + A_{ext}) = \sum{}_\mu D_\mu(A + A_{ext}) + m$$

where the operator D_μ is defined by

$$[D_\mu(A + A_{ext})\psi](n) = \frac{1}{2}\eta_\mu(n)\{e^{i(A_\mu(n)+A_{ext,\mu}(n))}\psi(n+\hat{\mu}) - e^{-i(A_\mu(n-\hat{\mu})+A_{ext,\mu}(n-\hat{\mu}))}\psi(n-\hat{\mu})\}$$

and η_μ are the staggered phases. Note that this treatment of the gauge-field–fermion interactions is compact and so has period 2π in the gauge fields.

We implement the RHMC simulation method of Clark and Kennedy, using a $(12,12)$ $[(15,15)]$ rational approximation to $\mathcal{M}^{-1/8}$ and $(20,20)$ $[(25,25)]$ rational approximations $\mathcal{M}^{\pm 1/16}$. To account for the range of normal modes of the non-compact gauge action we vary the trajectory lengths τ over the range,

$$\frac{\pi}{2\sqrt{\beta}} \le \tau \le \frac{4\pi}{\sqrt{2\beta(4 - \sum_\mu \cos(2\pi/N_\mu))}},$$

of the periods of the modes of this gauge action.

A_{ext} are chosen in the symmetric gauge as

$$A_{ext,1}(i,j,k,l) = -\frac{eB}{2} \quad (j-1) \quad i \neq N_1$$
$$A_{ext,1}(i,j,k,l) = -\frac{eB}{2}(N_1+1) \quad (j-1) \quad i = N_1$$
$$A_{ext,2}(i,j,k,l) = +\frac{eB}{2} \quad (i-1) \quad j \neq N_2$$
$$A_{ext,2}(i,j,k,l) = +\frac{eB}{2}(N_2+1) \quad (i-1) \quad j = N_2$$

while $A_{ext,3}(n) = A_{ext,4}(n) = 0$. In practice we subtract the average values of A_{ext},μ from these definitions. This choice produces a magnetic field eB in the $+z$ direction on every $1,2$ plaquette except that with $i = N_1$, $j = N_2$, which has the magnetic field $eB(1-N_1N_2)$. Because of the compact nature of the interaction, requiring $eBN_1N_2 = 2\pi n$ for some integer $n = 0, 1, ...N_1N_2/2$ makes the value of this plaquette indistinguishable from eB. Hence $eB = 2\pi n/(N_1N_2)$ lies in the interval

$[0, \pi]$.

One of the observables we calculate is the electron contribution to the effective gauge action per site $\frac{-1}{8V}\mathrm{trace}[\ln(\mathcal{M})]$. For $\ln(\mathcal{M})$ we use a $(30, 30)$ rational approximation to the logarithm. Here we use the Chebyshev method of Kelisky and Rivlin. While this has worse errors than a Remez approach, it preserves some of the properties of the logarithm itself, and is applicable on the whole complex plane cut along the negative real axis.