Spin-taste structure of minimally doubled fermions

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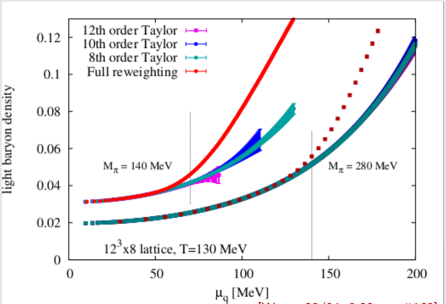
40th International Symposium on Lattice Field Theory, Fermilab, USA, 08/02/2023

Motivation: real chemical potential

For many years, **minimally doubled fermions** have been. . .

- A solution **waiting for a suitable problem**!
- Rooted staggered quarks fail at **real chemical potential** [Pasztor, 08/01, 9:30am, #312]



[Wong, 08/04, 9:20pm, #169]

- **No clear path forward** with rooted staggered fermions [Golterman:2006rw].
- **Real chemical potential** may be this suitable problem: let's take it on!

Overview

Universal progenitor: naive fermions

- **Naive fermions** obtained via symmetric discretization of derivative

$$S_{\text{nai}}[\psi,\bar{\psi}] = a^d \sum_{n,m\in\Lambda} \bar{\psi}_n \Big[\sum_{\mu} \gamma_\mu D^{(s)}_\mu[U] \ + \ m_0 \Big](n,m)\psi_m \ .$$

- **Symmetries** of naive theory: discrete translations, discrete rotations (W_4), parity, time reflection, charge conj., chiral, γ_5-hermiticity.
- 2^d**-fold degeneracy** (free theory): $\geq 2^d$ sites for full **spin-taste** content.
- 1st set of $2^{\frac{d}{2}}$ **single-site taste generators**: **spinor rotations with phases**:

$$\psi_n \overset{\tau_\mu}{\to} \tau_\mu(n)\,\psi_n\ , \qquad \bar{\psi}_n \overset{\tau_\mu}{\to} \bar{\psi}_n\,\tau_\mu(n)\ , \qquad \tau_\mu(n) \equiv \mathrm{S}_\mu \mathrm{e}^{\mathrm{i}\pi p_\mu(n)} \equiv \mathrm{i}\gamma_\mu\gamma_5(-1)^{p_\mu(n)}\ ,$$

  where $p_\mu(n) = n_\mu \bmod 2$. $D_{\text{nai}}[U]$ **is invariant** between the generators τ_μ.
- 2nd set of $2^{\frac{d}{2}}$ **extended generators**: **translations** within one hypercube:

$$\psi_n \overset{t_\mu}{\to} \psi_{n\pm\hat{\mu}}\ , \qquad \bar{\psi}_n \overset{t_\mu}{\to} \bar{\psi}_{n\pm\hat{\mu}}\ , \qquad t_\mu(n,m) = U_{\pm\mu}(n)\delta(n\pm\hat{\mu},m),\ \pm = (-1)^{p_\mu(n)}\ .$$

- **Doubling problem**: extra tastes contribute to the fermion determinant.

Nielsen-Ninomiya No-Go Theorem

Key strategies for alleviating the doubling problem

- **Universality**: same $a \to 0$ limit for theories differing only at $\mathcal{O}(a^n)$, $n > 0$.
- Drop unwanted d.o.f. directly: **(reduced) Kogut-Susskind fermions**.
- Add or modify $\mathcal{O}(a^n)$ terms to **break the taste symmetries**: Wilson, twisted-mass (Wilson), or **minimally doubled fermions**.

!! **Sums of lower dim. operators that cancel at $\mathcal{O}(a^n)$, $n < 0$** @ survivors.

- Or construct an operator that satisifies the **Ginsparg-Wilson relation**.

No-Go Theorem for properties of lattice fermions in conjuction

1. **Dirac fermion(s)** in the continuum limit as $D(k) = \mathrm{i}\not{k} + \mathcal{O}(a)$.
2. **Odd number of equivalent solutions** $D(k) = 0$ (aka odd # of doublers).
3. **Locality**, i.e. *ultra-locality* $D(n, m) = 0 \; \forall \|n - m\| \geq$ const, or less strict *locality* $D(x, y) \leq \Gamma \mathrm{e}^{-\gamma\|x-y\|}$ w. γ indep. of a (distance metric $\|a - b\|$).
4. **Chiral symmetry** as $D\gamma_5 + \gamma_5 D = 0$ for $m_0 = 0$.

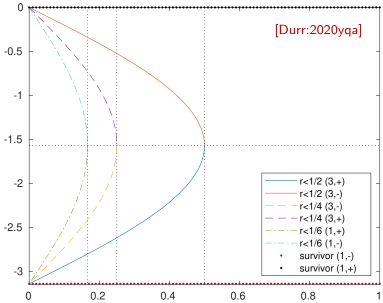
Karsten-Wilczek (KW) fermions

- Protect **vector-chiral symmetry @ two naive tastes** [Karsten:1981gd], [Wilczek:1987kw].
- Add **anisotropic taste-symmetry breaking** term $\propto \mathrm{i}\gamma_d$:

$$S_{\mathrm{KW}}[\psi,\bar{\psi}] = a^d \sum_{n,m\in\Lambda} \bar{\psi}_n \Big[D_{\mathrm{nai}}[U] \;+\; m_0 \;-\frac{ra}{2}\mathrm{i}\gamma_d \sum_{j=1}^{d-1} \Delta_j[U] \Big](n,m)\; \psi_m \;,$$

w. lattice Laplacian $\Delta_\mu[U](n,m) = \frac{U_{\mu,n}\delta(n+\hat{\mu},m)+U^\dagger_{\mu,n-\hat{\mu}}\delta(n-\hat{\mu},m)-2\delta(n,m)}{a^2}$.

- **Symmetries** of this theory: disc. transl., **discrete spatial rotations (W_2)**, **parity**, **time refl.** $\times$ **charge conj.**, **chiral**, γ_5-herm. [Pernici:1994yj], [Bedaque:2008xs].
- **Broken symmetry @ tree level**: time refl., charge conj. $r \to -r$.
- **Anisotropy** of time d vs space j.
- Extra tastes @ spatial momenta $3\pi, 2\pi, 1\pi$ and **net chirality zero** annihilate at $|r| = \frac{1}{6}, \frac{1}{4}, \frac{1}{2}$.
- **Survivors @** $ak_d = 0, \pi$, spatial $\boldsymbol{k} = 0$, opposite chiralities.
- **KW hypersite** is split by $t_d[U]$.



Spin-taste algebra: requirements and KW taste representation

- Representation à la staggered [Kluberg-Stern:1983lmr] known [Kimura:2011ik]: not helpful.
- **Mirror fermion symmetry: time refl.** $\times \ \tau_d$ [Pernici:1994yj].

$\Leftrightarrow$ equivalent **isovector symmetry: charge conj.** $\times \ \tau_d$ (cf. **Boriçi-Creutz**).

- $\mathfrak{su}(2)$ **representation** $\{\rho_i\} \equiv \{\rho(\sigma_i)\}$ must satisfy $[\rho_i, \rho_j] = 2\mathrm{i}\epsilon_{ijk}\rho_k$.
- **Taste-singlet** Dirac operator in the continuum limit: $[D_{\mathrm{nai}}, \rho_j] = \mathcal{O}(a)$.
- **Every part** of D_{KW} is (can be taken apart or combined) such that it has a **well-defined pattern under any spin-taste generator**.

KW taste representation (gamma matrices in chiral representation) [Weber:2016dgo]

$$\rho_1(n) = \sigma_1 \otimes \mathbb{1}\ (-1)^{p_s(n)} = \quad \gamma_d\ (-1)^{p_s(n)} = \mathrm{i}\prod_{j=1}^{d-1} \tau_j \equiv [\mathrm{i}\tau_5\tau_d](n)\ ,$$

$$\rho_2(n) = \sigma_2 \otimes \mathbb{1}\ (-1)^{p_d(n)} = \mathrm{i}\gamma_d\gamma_5\ (-1)^{p_d(n)} = \tau_d(n)\ ,$$

$$\rho_3(n) = \sigma_3 \otimes \mathbb{1}\ (-1)^{p(n)} = \quad \gamma_5\ (-1)^{p} = \prod_{\rho=1}^{d} \tau_\rho \equiv \tau_5(n)\ ,$$

using $p_s = \left(\sum_{j=1}^{d-1} p_j\right) \bmod 2$, $p = \left(\sum_{\mu=1}^{d} p_\mu\right) \bmod 2$. Staggered $\epsilon_n = (-1)^p$.

KW spin-taste symmetry pattern [Weber:2016dgo]

- **Extended operators encode non-trivial taste patterns**:

$$t_d[U](n,m) \sim \rho_1(n,m)\ , \quad t_j[U](n,m) \sim \rho_2(n,m)\ , \quad t_d t_j[U](n,m) \sim \rho_3(n,m)\ .$$

- Split up Laplacian $\frac{a^2}{2}\Delta_\mu[U](n,m) \equiv c_\mu[U](n,m) - \delta(n,m)$.
- KW operator w. isovector mass broken into chiral/spin/taste sectors:

$$D_{\mathrm{KW}}[U] = D_{\mathrm{nai}}[U] + \frac{\mathrm{i}r\gamma_d}{a}(d-1) - \frac{\mathrm{i}r\gamma_d}{a}\sum_{j=1}^{d-1} c_j[U] + m_0 + \frac{m_3}{N_{\mathrm{perm}}}\prod_{\mathrm{perm}}(c_j c_d)[U]\ .$$

Symmetry patterns

	$D_{\mathrm{nai}}[U]$	$\mathrm{i}r\gamma_d$	$\mathrm{i}r\gamma_d c_j[U]$	$\mathbb{1}$	$c_j[U]C_d[U]$	$c_j[U]$	$c_d[U]$
$\mathcal{P}$	+	+	+	+	+	+	+
$\mathcal{T}$	+	-	-	+	+	+	+
$\mathcal{C}$	+	-	-	+	+	+	+
ρ_1	+	+	-	+	-	-	+
ρ_2	+	-	-	+	-	+	-
ρ_3	+	-	+	+	+	-	-
γ_5	-	-	-	+	+	+	+

KW fermions as sea or valence: automatic $\mathcal{O}(a)$ improvement

- KW fermions need **three counterterms**, @ one loop since [Capitani:2010nn]

$$O^{F3} = d_3(r^2)\ a^d \sum_{n\in\Lambda} \bar{\psi}_n \frac{\mathrm{i} r\gamma_d}{a} \psi_n \ ,$$

$$O^{F4} = d_4(r^2)\ a^d \sum_{n,m\in\Lambda} \bar{\psi}_n \gamma_d D_d^{(s)}[U](n,m)\psi_m \ ,$$

$$O^{G4} = d_{4p}(r^2)\ \frac{1}{g^2} \sum_{n\in\Lambda} \sum_{j=1}^{d-1} \mathrm{Re}\,\mathrm{Tr}\left[1 - U_{jd}(n)\right] \ .$$

- Non-perturbative tuning: [Weber:2015oqf], [Godzieba, 08/01, 4:40pm, #264], [Vig, 08/01, 4:20pm, #168].
- **KW fermion determinant**: only even powers of $\{a, m_0, m_3, r\}$:

$$\det(D_{\mathrm{KW}}[U]) = C\left(a^2; m_0^2, m_3^2; r^2, d_3(r^2), d_4(r^2)\right) \geq 0 \ .$$

- $\det(D_{\mathrm{KW}}[U])$ **automatically $\mathcal{O}(a)$ improved** due to spin-taste structure.
- $\det(D_{\mathrm{KW}}[U])$ **equal for $\pm r$: may average $\pm r$ in valence sector**

$\rightarrow$ tuning-free **automatic $\mathcal{O}(a)$ improvement of observables** is possible!

KW fermion bilinears

Single-site operators

$\otimes$	Γ_5	Γ_d	Γ_j	Γ_{jk}	Γ_{jd}	Γ_{j5}	Γ_{d5}	$\mathbb{1}$
ρ_1	γ_d	γ_d	$\gamma_{j5}\epsilon$	γ_j	γ_{j5}	γ_{j5}	$\gamma_d\epsilon$	–
ρ_2	–	$\gamma_{d5}\epsilon$	γ_j	γ_{jk}	–	$\gamma_j\epsilon$	γ_{d5}	γ_{d5}
ρ_3	γ_5	γ_5	–	$\gamma_{jd}\epsilon$	γ_{jd}	γ_{jd}	–	$\gamma_5\epsilon$
$\mathbb{1}$	ϵ	–	–	γ_{jk}	$\gamma_{jd}\epsilon$	–	$\mathbb{1}$	$\mathbb{1}$

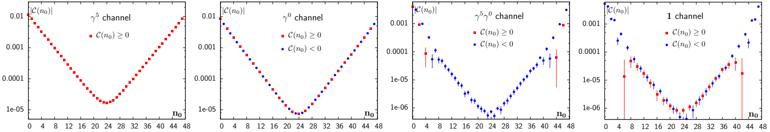
- ϵ **bilinears cannot excite low-energy states** due to wrong Fermi point.
- **Two-link operators** replace single-site ϵ-bilinears as $\epsilon \to c_j c_d \gamma_5$.
- Further spin-taste states accessible as **parity partners** in oscillating part.

One-link operators

$\otimes$	Γ_5	Γ_d	Γ_j	Γ_{jk}	Γ_{jd}	Γ_{j5}	Γ_{d5}	$\mathbb{1}$
ρ_1	$c_j\gamma_5$	$c_j\gamma_5$	$c_d\gamma_{jk}$	$c_d\gamma_{jk}$	$c_j\gamma_{jd}$	$c_j\gamma_{jd}$	$c_d\mathbb{1}$	$c_d\mathbb{1}$
ρ_2	$c_d\gamma_5$	$c_d\gamma_5$	$c_j\gamma_{jk}$	$c_j\gamma_{jk}$	$c_d\gamma_{jd}$	$c_d\gamma_{jd}$	$c_j\mathbb{1}$	$c_j\mathbb{1}$
ρ_3	$c_j\gamma_d$	$c_j\gamma_d$	$c_d\gamma_j$	$c_d\gamma_j$	$c_j\gamma_{j5}$	$c_j\gamma_{j5}$	$c_d\gamma_{d5}$	$c_d\gamma_{d5}$
$\mathbb{1}$	$c_d\gamma_d$	$c_d\gamma_d$	$c_j\gamma_j$	$c_j\gamma_j$	$c_d\gamma_{j5}$	$c_d\gamma_{j5}$	$c_j\gamma_{d5}$	$c_j\gamma_{d5}$

KW pion tastes from single-site bilinears [Weber:2015oqf]

SU(3) plaquette action, $\beta = 6.0$, $m_\pi \approx 655$ MeV, $d_3 = -0.45$, $d_4 = -0.001$



Taste-ρ_3 pion: $am_{\pi_3} = 0.299(2)$

Taste-ρ_1 partner: $am_{\pi_1} = 0.319(2)$

Taste-ρ_2 pion: $am_{\pi_2} = 0.333(12)$

Singlet partner: $am_{\pi_s} = 0.332(23)$

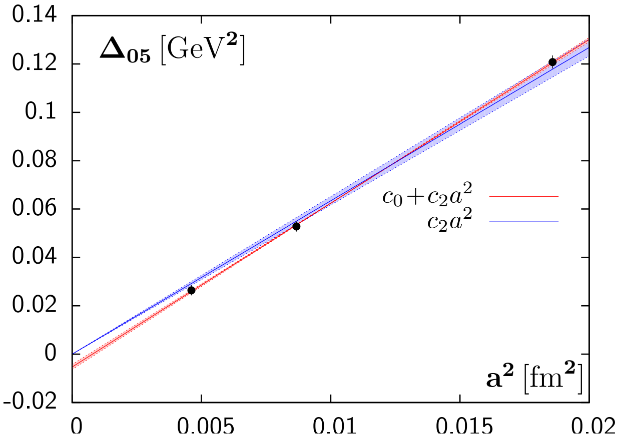
- (taste-ρ_3 pion, taste-ρ_1 partner) or (taste-ρ_2 pion, taste-singlet partner):
- **Within each pair degeneracy for naive, but for KW broken**, at least for the pair with the lone Goldstone (ρ_3) pion.
- **No degeneracy between the two pairs for either, smaller splitting** for KW (cf. $am_{\pi_2} - am_{\pi_3} = 0.381 - 0.339$ for naive)!

Two distinct sources of taste-symmetry violation

- **By terms that are $\mathcal{C}/\mathcal{T}/\rho_2$ odd, i.e. odd in r (already at tree level).**
- **Due to local gauge-field fluctuations like naive/staggered fermions.**

Understanding their interplay requires a **dedicated research program**!

Chiral behavior and taste-symmetry restoration [Weber:2015oqf]



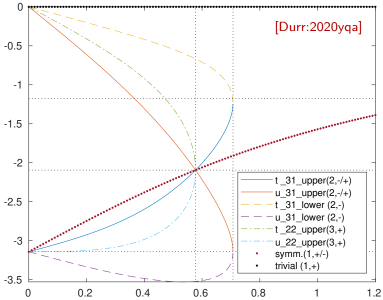
- Tuned KW on quenched, **bare links in** $D_{KW}[U]$, $m_{\pi_3} \in (240, 650)$ MeV.
- Quenched **chiral logs** in $m_{\pi_3}(m_0)$ consistent w. phenomenology [Wittig:2002ux].
- **Taste splitting** $\Delta_{05} \equiv r_0^2(m_{\pi_1}^2 - m_{\pi_3}^2)$ only mildy quark mass dependent, **extrapolates to (nearly) zero**. Likely indicaties $\mathcal{O}(a^4)$ effects. . . ?

Boriçi-Creutz (BC) fermions

- Inspired by **graphene, extended to 4D**: [Creutz:2007af],[Borici:2007kz].
- Gist: **idempotent** $\Gamma \equiv \frac{1}{\sqrt{d}} \sum_{\mu=1}^{d} \gamma_\mu$ **picks hypercubic diagonal (hcd)**.
- Taste-symmetry breaking term using **dual gamma basis** $\gamma'_\mu \equiv \Gamma\gamma_\mu\Gamma$:

$$S_{\mathrm{BC}}[\psi,\bar\psi] = a^d \sum_{n,m\in\Lambda} \bar\psi_n \Big[D_{\mathrm{nai}}[U] \;+\; m_0 - \frac{ra}{2} \sum_{\mu=1}^{d} \mathrm{i}\gamma'_\mu \Delta_\mu[U] \Big] (n,m)\; \psi_m \;.$$

- **Symmetries**: disc. transl., **discrete rotations around hcd, axis exch.**, **parity** $\times$ **time refl.** $\times$ **charge conj.**, **chiral**, γ_5-herm. [Bedaque:2008xs].
- **Broken symmetry @ tree level**: $(\mathcal{P}\times\mathcal{T})$, **charge conj.** $r \to -r$.
- **Extra tastes annihilate** in two waves at $|r| = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}$ with major **chirality exchange** event.
- **Survivors @** $ak_\mu = -2\mathrm{atan}\left(\frac{1}{r}\right)$ **or** $0 \in$ **hcd**, opposite chiralities.
- Symmetry between **fw/bw projections on hcd** broken.



Finding the BC spin-taste representation

- Representation à la staggered [Kluberg-Stern:1983lmr] known [Kimura:2011ik]: not helpful.
- **Taste structure far more intricate than claimed in [Bedaque:2008xs]. Define:**

$$\psi_n \xrightarrow{T_\Gamma} T_\Gamma(n)\,\psi_n\ , \qquad \bar\psi_n \xrightarrow{T_\Gamma} \bar\psi_n\, T_\Gamma^\dagger(n)\ , \qquad T_\Gamma(n) \equiv \mathrm{i}\Gamma\gamma_5\xi_r^{q(n)}\ ,$$

where $\xi_r = \mathrm{i}s_r = \mathrm{i}\,\mathrm{sign}\,(r)$ and $q(n) \equiv \big(\sum_{\rho=1}^{d} n_\rho\big) \bmod 4$. So

$$T_\Gamma(n)T_\Gamma^\dagger(n) = \mathbb{1}\ , \quad T_\Gamma(n)\gamma_\mu T_\Gamma^\dagger(m) = -\gamma'_\mu \xi_r^{q(n-m)} \Rightarrow T_\Gamma(n)\Gamma T_\Gamma^\dagger(n) = -\Gamma\ .$$

- Split up Laplacian and **transform BC operator** as

$$T_\Gamma(n)\Big[D_{\mathrm{nai}}[U] + \frac{\mathrm{i}r}{a}\Gamma - \frac{\mathrm{i}r}{a}\sum_{\mu=1}^{d}\gamma'_\mu c_\mu[U]\Big](n,m)T_\Gamma^\dagger(m)$$

$$= \Big[\frac{\mathrm{i}s_r}{a}\sum_{\mu=1}^{d}\gamma'_\mu c_\mu[U] - \frac{\mathrm{i}r}{a}\Gamma + |r|D_{\mathrm{nai}}[U]\Big](n,m)\ .$$

- Finally, charge conjugate to see the **invariance of $S_{\mathrm{BC}}[U]$ for $r^2 = 1$:**

$$-\mathcal{C}\Big(T_\Gamma(n)\Big[D_{\mathrm{nai}}[U] + \frac{\mathrm{i}r}{a}\Gamma - \frac{\mathrm{i}r}{a}\sum_{\mu=1}^{d}\gamma'_\mu c_\mu[U]\Big](n,m)T_\Gamma^\dagger(m)\Big)\mathcal{C}^{-1}$$

$$= -\Big[|r|D_{\mathrm{nai}}[U^{\mathcal{C}}] + \frac{\mathrm{i}r}{a}\Gamma - \frac{\mathrm{i}s_r}{a}\sum_{\mu=1}^{d}\gamma'_\mu c_\mu[U^{\mathcal{C}}]\Big]^T(m,n)\ .$$

Completing the BC spin-taste representation

- $r^2 = 1$ **isovector symmetry: charge conj.** $\times$ T_Γ (cf. **Karsten-Wilczek**).
- **Taste operator** $T_\Gamma(n)$ is hermitian/antihermitian on even/odd sites n.
- Always **paired with hermitian conjugate** in fermion bilinears (cf. action).
- Block decomposition of Γ in chiral representation as

$$\Gamma = \begin{pmatrix} 0 & R \\ R^\dagger & 0 \end{pmatrix}, \quad R = \sqrt{\frac{2}{d}} \begin{pmatrix} \varrho^{-1} & \varrho^{-3} \\ \varrho^{-1} & \varrho^{+1} \end{pmatrix}, \quad \varrho = \frac{1+\mathrm{i}}{\sqrt{2}} = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}} .$$

BC taste representation

$$\rho_1(n) = \begin{pmatrix} 0 & R \\ R^\dagger & 0 \end{pmatrix} \xi_r^{-q(n)} = \Gamma \; \xi_r^{-q(n)} \equiv [\mathrm{i}\, T_5 T_\Gamma](n) ,$$

$$\rho_2(n) = \mathrm{i} \begin{pmatrix} 0 & -R \\ R^\dagger & 0 \end{pmatrix} \xi_r^{+q(n)} = \mathrm{i}\Gamma\gamma_5 \; \xi_r^{+q(n)} = T_\Gamma(n) ,$$

$$\rho_3(n) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \xi_r^{2q(n)} = \gamma_5 \; \xi_r^{2q(n)} \equiv T_5(n) .$$

Note: $\rho_3(n) = T_5(n) = \tau_5(n)$ **is real and the same as for Karsten-Wilczek**.

- **BC representation of $\mathfrak{su}(2)$ algebra** (hermitian conj. correspondingly):

$$[\rho_1^\dagger, \rho_2] = [\rho_1, \rho_2^\dagger] = 2\mathrm{i}\rho_3 , \quad [\rho_2, \rho_3] = 2\mathrm{i}\rho_1 , \quad [\rho_3, \rho_1] = 2\mathrm{i}\rho_2 .$$

BC spin-taste symmetry pattern

- **Two-link bilinears encode taste diagonal patterns (real evals):**

$$t_{\pm(\mu+\nu)}[U](n,m) \sim \rho_3(n,m)\ , \quad t_{\pm(\mu-\nu)}[U](n,m) \sim \mathbb{1}(n,m)\ .$$

- $c_{\mu+\nu}[U](n,m) = \frac{t_{+\mu+\nu}+t_{-\mu-\nu}}{2}[U](n,m)$ commutes with $(\mathcal{P}\times\mathcal{T})\ /\ \mathcal{C}$: **simultaneous eigenbasis** exists, isovector-mass can be defined!
- BC operator w. isovector mass broken into chiral/spin/taste sectors:

$$D_{\mathrm{BC}}[U] = D_{\mathrm{nai}}[U] + \frac{\mathrm{i}r\Gamma}{a} - \frac{\mathrm{i}r}{a}\sum_{\mu=1}^{d}\gamma'_\mu c_\mu[U] + m_0 + \frac{m_3}{N_{\mathrm{perm}}}\prod_{\mathrm{perm}} c_{\mu+\nu}[U]\ .$$

Symmetry patterns

	$D_{\mathrm{nai}}[U]$	$\mathrm{i}r\Gamma$	$\mathrm{i}r\gamma'_\mu c_\mu[U]$	$\mathbb{1}$	$c_{\mu+\nu}[U]$
$\mathcal{P}\times\mathcal{T}$	$+$	-	-	$+$	$+$
$\mathcal{C}$	$+$	-	-	$+$	$+$
ρ_1	$+\mathrm{i}s_r\gamma'_\mu c_\mu[U]$	$+$	$-\lvert r\rvert D_{\mathrm{nai}}[U]$	$+$	-
ρ_2	$+\mathrm{i}s_r\gamma'_\mu c_\mu[U]$	-	$-\lvert r\rvert D_{\mathrm{nai}}[U]$	$+$	-
ρ_3	$+$	-	$+$	$+$	$+$
γ_5	-	-	-	$+$	$+$

BC one-link fermion bilinears

- r^2 **isovector symmetry** $(\mathcal{C} \times \rho_2)$ **constrains one-link terms** in BC action:

$$\left(\varrho^{\mp s_r} t_{+\mu}(n,m) - \varrho^{\pm s_r} t_{-\mu}(n,m)\right) \begin{matrix} \xrightarrow{\rho_1} & \pm \\ \xrightarrow{\rho_2} & \mp \end{matrix} \left(\varrho^{\pm s_r} t_{+\mu}(n,m) - \varrho^{\mp s_r} t_{-\mu}(n,m)\right) .$$

Thus, we must **revise the dim.-4 fermion counterterm** [Capitani:2009yn]:

$$O^{F4} = c_4(r^2)\ a^d \sum_{n,m\in\Lambda} \bar{\psi}_n \Gamma \sum_{\mu=1}^{d} \left(D_\mu^{(s)} - \frac{\mathrm{i}r}{a}\Delta_\mu\right)[U](n,m)\psi_m \ .$$

- **One-link bilinears encode taste off-diagonal patterns (imag. evals):**

$$\rho_1(n) t_{\pm\mu}[U](n,m)\rho_1^\dagger(m) = \pm \mathrm{i} s_r t_{\pm\mu}[U](n,m) \ ,$$

$$\rho_2(n) t_{\pm\mu}[U](n,m)\rho_2^\dagger(m) = \mp \mathrm{i} s_r t_{\pm\mu}[U](n,m) \ .$$

- $\rho_1(n)$ / $\rho_2(n)$ vs $(\mathcal{P} \times \mathcal{T})$ / $\mathcal{C}$: **no simultaneous eigenbasis!**

One-link point-split taste operators [Basak:2017oup], [Osmanaj, 03/09/23 @ MITP] **?**

$$u(n) = \frac{1}{2d} \sum_{\mu=1}^{d} \left(2\delta(n,m) + \mathrm{i}t_{+\mu}[U](n,m) - \mathrm{i}t_{-\mu}[U](n,m)\right)\psi(m) \ ,$$

$$d(n) = \Gamma \xi_r^{q(n)} \frac{1}{2d} \sum_{\mu=1}^{d} \left(2\delta(n,m) - t_{+\mu}[U](n,m) - t_{-\mu}[U](n,m)\right)\psi(m) \ ,$$

where $\xi_r = -\mathrm{i}$ (due to $r = -1$). **Not ρ_3 eigenstates, no taste projection!**

BC fermions as sea or valence: automatic $\mathcal{O}(a)$ improvement

- BC fermions need **three counterterms**, @ one loop since [Capitani:2009yn]

$$O^{F3} = c_3(r^2)\ a^d \sum_{n\in\Lambda} \bar{\psi}_n \frac{r}{a} \mathrm{i}\Gamma \psi_n \ ,$$

$$O^{F4} = c_4(r^2)\ a^d \sum_{n,m\in\Lambda} \bar{\psi}_n \Gamma \sum_{\mu=1}^{d} \left(D_\mu^{(s)} - \frac{\mathrm{i}r}{a}\Delta_\mu \right)[U](n,m)\psi_m \ ,$$

$$O^{G4} = c_{4p}(r^2)\ \frac{1}{2g^2} \sum_{n\in\Lambda} \sum_{\mu,\nu,\rho=1}^{d} \mathrm{Re}\,\mathrm{Tr}\left[(U_{\mu\rho} - 1)(U_{\rho\nu} - 1) \right] \ .$$

- Non-pert. tuning: more ([Basak:2017oup]) or less ([Osmanaj, 03/09/23 @ MITP]) dubious.
- **BC fermion determinant**: only even powers of $\{a, r\}$:

$$\det(D_{\mathrm{BC}}) = C\left(a^2; m_0, m_3; r^2, d_3(r^2), d_4(r^2) \right) \geq 0 \ .$$

Even powers of m_0 and m_3 require $m_0 = 0$, $m_3 = 0$, or $r^2 = 1$.

- **Automatic $\mathcal{O}(a)$ improvement of** $\det(D_{\mathrm{BC}})$ due to spin-taste structure.
- $\det(D_{\mathrm{BC}})$ is **independent of** $\mathrm{sign}\,(r)$: **may average $\pm r$ in valence sector**

$\rightarrow$ tuning-free **automatic $\mathcal{O}(a)$ improvement of observables** is possible!

Summary

- Two variants, i.e. **Karsten-Wilczek or Boriçi-Creutz**, explored so far.
- Both have an **isovector charge conjugation invariance** of the action.
- From those isovector generators both **taste** $\mathfrak{su}(2)$ **reps** constructed.
- Both algebras share **common taste generator** $\rho_3 = \gamma_5(-1)^p$.
- Two-link **scalar isovector operators** $\Rightarrow$ taste/chirality-splitting terms.
- Due to the spin-taste pattern: **automatic** $\mathcal{O}(a)$ **improvement**.
- First **quenched KW results on taste splittings** look promising.

With everything else said. . .

- **Karsten-Wilczek fermions** are rather simple and extensively tested, have a convenient symmetry pattern that aligns well with typical LQCD problems and setups, and may be a **leap forward for finite density QCD**. Their time may have come as **new beast in the zoo of lattice fermions**.
- **Boriçi-Creutz fermions** are very complicated and confusing, have a weird symmetry pattern at odds with typical LQCD problems and setups. They are fascinating oddballs, but any practical use is **strongly discouraged**.

Thank you for your attention!