Amplifier Noise

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1 Introduction

While this note includes a general treatment of amplifier noise and electrical circuit theory, the main purpose is to compute the correlated noise in the Tianlai array due to amplifier noise in one feed coupling into another feed.

2 Conventional amplifier noise model

In this section I review the conventional amplifier noise model. The amplifier is modeled as a linear 2-port device. A key assumption is *linearity* — virtually all the manipulations below assume a linear system. Any arbitrary linear 2-port device can be represented by 2 ideal sources and a network without any internal sources, which can be represented by a 2x2 matrix. The equivalent circuit of the amplifier connected to a source of impedance Z_s is shown in Figure 1. The antenna produces a current i_a across a shunt impedance Z_s. Since we are only concerned with amplifier noise, I will set $i_a = 0$ for this analysis. The amplifier noise generates an equivalent voltage (e_n), which is added to the amplifier input voltage, and an equivalent current (i_n), which is added to the input current.

The amplifier 2x2 matrix can be represented as an impedance matrix where I_i and I_o are the input and output currents respectively and V_i and V_o are the input and output voltages respectively.

$$\begin{pmatrix} V_i \\ V_o \end{pmatrix} = \begin{pmatrix} Z_{ii} & Z_{io} \\ Z_{oi} & Z_{oo} \end{pmatrix} \begin{pmatrix} I_i \\ I_o \end{pmatrix} \tag{1}$$

Normally, an amplifier will be designed so that Z_{io} is negligible. Approximating $Zio = 0$

$$V_i = Z_{in} I_i \tag{2}$$

where

$$Z_{in} = Z_{ii} \tag{3}$$

However, equation (2) is generally valid for any amplifier output load impedance (Z_L) except that Z_{in} will not longer be equal to Z_{ii}, but will be a function of

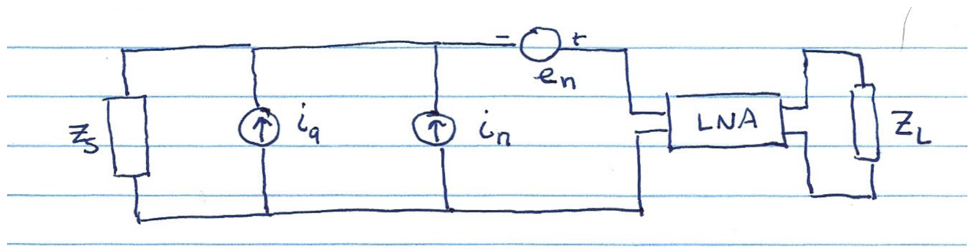

Figure 1: Amplifier Equivalent Circuit. The antenna is represented by an ideal current source (i_a in parallel with a shunt impedance (Z_s). The amplifier noise is represented by an equivalent ideal voltage source (e_n) which is added to the input voltage and an equivalent ideal current source (i_n) which is added to the input current. The LNA is represented by a 2-port network whose output is terminated in a load (Z_L).

the full impedance matrix and Z_L. The full impedance matrix for the amplifier can be written as

$$\mathbf{Z} = \begin{pmatrix} Z_{in} & 0 \\ gZ_{in} & Z_{out} \end{pmatrix} \tag{4}$$

where $Z_{oi} = gZin$ is the amplification term and $Z_{22} = Z_{out}$ is the output impedance.

It is convenient to convert the noise voltage into an equivalent noise current. Thevenin's theorem states the current source $e_n/Z_s = e_nY_s$ will produce the same currents in Z_s and Z_{in} as the original voltage source. The total (rms) current in the input circuit is therefore

$$i_t i_t^* = (Y_s e_n)(Y_s e_n)^* + 2\mathrm{Re}(Y_s e_n i_n^*) + i_n i_n^* \tag{5}$$

We define the correlation

$$Y_c = e_n^* i_n / (e_n^* e_n) \tag{6}$$

We can always divide sources into two or even more components because of the superposition principle.

$$i_n = i_u + i_c \tag{7}$$

In this analysis it is customary to divide the noise currents into two components: one that is correlated with e_n and one that is uncorrelated with e_n. If we choose $i_c = Y_c e_n$ that will produce the correct correlation and it is easily verified that $i_u = i_n - I_c$ is uncorrelated with e_n. We can now write the noise current in its standard form:

$$i_t = i_u + (Y_s + Y_c)e_n \tag{8}$$

and

$$|i_t|^2 = |i_u|^2 + |Y_s + Y_c|^2 |e_n|^2 \tag{9}$$

It should be noted that for fixed Y_s the noise can be described as a single noise current. However, the noise current is a function of Y_s. A typical application of this analysis is to study the dependence of the noise on Y_s. To do that analysis, it is necessary to know the correlation between the noise voltage and the noise current. However, for fixed Y_s, a knowledge of i_t provides a complete description of the circuit.

In more general terms, one needs two sources to describe a linear 2-port circuit. Specifying a value of Y_s converts the circuit to a 1-port, which can be described by a single source. In the Tianlai case Y_s in the antenna admittance, which is a fixed quantity.

The total current i_t will be divided between Z_s and Z_{in} according to:

$$i_{in} = \frac{Y_{in}}{Y_{in} + Y_s} i_t \tag{10}$$

$$i_s = \frac{Y_s}{Y_{in} + Y_s} i_t \tag{11}$$

3 S-matrix description

It might be a concern that the (V,I) parameterization is not valid at microwave frequencies. The connection between the (V,I) representation and a wave formalism is described below, with the goal of computing the current in one feed due to the noise in another feed..

.A typical S-matrix measurement configuration for a 2-port circuit is shown in Figure 2. The incident waves are a_1 and a_2 and the reflected waves are b_1 and b_2, from ports 1 and 2 respectively. The waves are carried in a single mode transmission line, which is commonly a $Z_0 = 50\omega$ coaxial cable. The relationship between incident and scattered waves is given by the S-matrix:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{12}$$

The waves in the S-matrix formalism can be obtained from the (V,I) desciption via a linear transformation. So the S-matrix description is equivalent to the Z-matrix description or any other representation involving V and I. Looking at the transmission line, the voltages and currents are related by the characteristic impedance, namely,

$$v_f = i_f Z_0 \tag{13}$$

$$v_r = i_r Z_0 \tag{14}$$

where f refers to the forward wave and r refers to the backwards wave. The equations (13) and (14) apply separately to each port; I have suppressed the port label to simplify the notation. I will identify $a = v_f$ and $b = v_r$, but I could have used any quantity that depends linearly on the fields in the coaxial cable. One common choice is to take $a = v_f/\sqrt{(}Z_0)$ so that the power in the

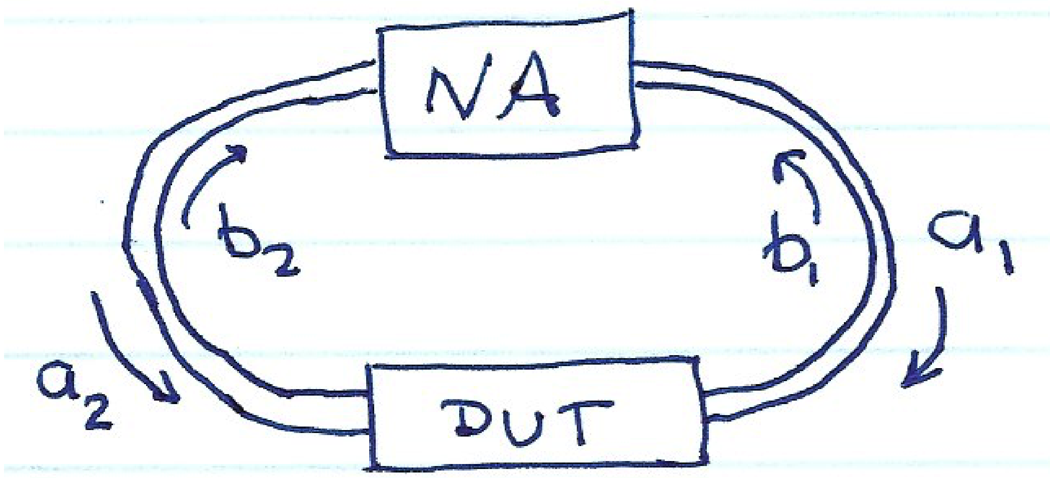

Figure 2: Network Test Setup. The network analyzer (NA) launches forward waves a_1 and a_2 and measures the reflected wave b_1 and b_2. The results can be expressed as a 2x2 scattering matrix.

wave is simply $|a|^2$. At the interface between the coaxial cable and the device under test (DUT), the following boundary conditions must hold:

$$V = v_f + v_r \tag{15}$$

$$I = i_f - i_r \tag{16}$$

The solution of these equations is

$$a = (V + IZ_0)/2 \tag{17}$$

$$b = (V - IZ_0)/2 \tag{18}$$

If the network has only a single port or S_{12} is zero

$$b_1 = S_{11} a_1 \tag{19}$$

This is approximately true for the Tianlai feeds since S_{12} is small. In this approximation the ratio of

$$Z_s = V/I \tag{20}$$

is the input impedance of the DUT, the feed in in the case of Tianlai. Combining (17), (18), and (20), we obtain the standard result for reflection

$$b/a = S_{11} = \frac{Z_s - Z_0}{Z_s + Z_0} \tag{21}$$

We are now in position to solve for the noise current flowing into port 2 because of the LNA noise at port 1. We start with (12) expressing the forward and backward waves at ports 1 and 2 in terms of V and I. The resulting equations are

$$(v_1 - i_1 Z_0) = S_{11}(v_1 + i_1 Z_0) + S_{12}(v_2 + i_2 Z_0) \quad (22)$$

$$(v_2 - i_2 Z_0) = S_{21}(v_1 + i_1 Z_0) + S_{22}(v_2 + i_2 Z_0) \quad (23)$$

where I have used lower case v and i with subscripts 1 and 2 to denote the voltage and current at ports 1 and 2 respectively. We will consider how the noise at port 1 propagates to port 2. Since we are assuming a small $S_{12} = S_{21}^*$, we can ignore the last term on the RHS of (22), which is of second order in the cross-coupling. If we wanted to include it, we would have to consider the contributions from all the other feeds. In order to solve equations (22) and (23), we need to apply the boundary conditions. At port 2 the boundary condition is

$$v_2 = -i_2 Z_2 \quad (24)$$

and at port 1 it is

$$v_1 = (i_t - i_1) Z_1 \quad (25)$$

where Z_1 and Z_2 are the input impedances of the amplifiers at port 1 and port 2, respectively. The solution of these equations is straight-forward, and the result is:

$$i_1 = \frac{(1 - S_{11}) Z_1}{(1 - S_{11}) Z_1 + (1 + S_{11}) Z_0} i_t \quad (26)$$

$$i_2 = \frac{-2 S_{21} Z_0 Z_1}{[(1 - S_{22}) Z_2 + (1 + S_{22}) Z_0][(1 - S_{11}) Z_1 + (1 + S_{11}) Z_0]} i_t \quad (27)$$

Of course, i_1 and i_2 are the currents flowing into the antennas at ports 1 and 2, respectively. We are interested in the amplifier input currents. The amplifier input current at port 1 is

$$i_t - i_1 = \frac{(1 + S_{11}) Z_0}{(1 - S_{11}) Z_1 + (1 + S_{11}) Z_0} i_t \quad (28)$$

The current at port 2 is simply $-i_2$

$$-i_2 = \frac{2 S_{21} Z_1}{(1 + S_{11})[(1 - S_{22}) Z_2 + (1 + S_{22}) Z_0]} (i_t - i_1) \quad (29)$$

If $Z_1 = Z_2 = Z_0$, (29) simplifies to

$$-i_2 = \frac{S_{21}}{1 + S_{11}} (i_1 - i_1) \quad (30)$$

The currents can be converted into temperatures or, more precisely, the temperature of the noise induced into port2 can be given relative to the noise temperature in port 1.

$$\frac{T_2}{T_1} = \frac{|i_2|^2 \mathrm{Re}(Z_2)}{|i_t - i_1|^2 \mathrm{Re}(Z_1)} \tag{31}$$

$$= \left| \frac{2S_{21}Z_1}{(1+S_{11})[(1-S_{22})Z_2 + (1+S_{22})Z_0]} \right|^2 \frac{\mathrm{Re}(Z_2)}{\mathrm{Re}(Z_1)} \tag{32}$$

4 Conclusions

It is explicitly shown that the cross-coupled noise power can be described in terms of a single noise source, and a explicit formula for the coupling has been derived.