# ...Nucleus-Nucleus Screened Coulomb Interactions

As treated in [Boschini et al. (2011)], at small distances from the nucleus, the potential energy is a Coulomb potential, while - at distances larger than the Bohr radius - the nuclear field is screened by the fields of atomic electrons. The interaction between two nuclei is usually described in terms of an interatomic Coulomb potential (e.g., see Sections 2.2.1 and 2.2.2 of [Leroy and Rancoita (2016)] and Section 4.1 of [ICRU Report 49 (1993)]), which is a function of the radial distance r between the two nuclei

$$V(r) = \frac{zZe^2}{r}\ \Psi_I(r_r),(1)$$

where ez (projectile) and eZ (target) are the charges of the bare nuclei and $\Psi_I$ is the interatomic screening function. This latter function depends on the reduced radius $r_r$ given by

$$r_r = \frac{r}{a_I},(2)$$

where $a_I$ is the so-called screening length (also termed screening radius). In the framework of the Thomas–Fermi model of the atom (e.g., see Chapters 1 and 2 of [Torrens (1972)]) - thus, following the approach of [ICRU Report 49 (1993)] -, a commonly used screening length for z = 1 incoming particles is that from Thomas–Fermi (e.g., see [Thomas (1927), Fermi (1928)])

$$a_{TF} = \frac{C_{TF}\,a_0}{Z^{1/3}},(3)$$

and - for incoming particles with z ≥ 2 - that introduced by [Ziegler, Biersack and Littmark (1985)] (and termed universal screening length[1])

$$a_U = \frac{C_{TF}\,a_0}{z^{0.23} + Z^{0.23}},(4)$$

where

$$a_0 = \frac{\hbar^2}{me^2}$$

is the Bohr radius, m is the electron rest mass and

$$C_{TF} = \frac{1}{2}\left(\frac{3\,\pi}{4}\right)^{2/3} \simeq 0.88534$$

is a constant introduced in the Thomas–Fermi model. The simple scattering model due to [Wentzel (1926)] - with a single exponential screening-function $\Psi_I(r_r)$ {e.g., see [Wentzel (1926)] and Equation (21) in [Fernandez-Vera et al. (1993)]} - was repeatedly employed in treating single and multiple Coulomb-scattering with screened potentials (e.g., see [Fernandez-Vera et al. (1993)] - and references therein - for a survey of such a topic and also [Molière (1947, 1948), Bethe (1953), Butkevick et al. (2002), Boschini et al. (2010)]). The resulting elastic differential cross section differs from the Rutherford differential cross section by an additional term - the so-called screening parameter - which prevents the divergence of the cross section when the angle θ of scattered particles approaches 0˚. The screening parameter $A_{s,M}$ [e.g., see Equation (21) of [Bethe (1953)])] - as derived in [Molière (1947, 1948)] for the single Coulomb scattering using a Thomas–Fermi potential - is expressed[2] as

$$A_{s,M} = \left(\frac{\hbar}{2\,p\,a_I}\right)^2\left[1.13 + 3.76 \times \left(\frac{\alpha zZ}{\beta}\right)^2\right](5)$$

where $a_I$ is the screening length - from Eqs. (3, 4) for particles with z = 1 and z ≥ 2, respectively; α is the fine-structure constant; p (βc) is the momentum (velocity) of the incoming particle undergoing the scattering onto a target supposed to be initially at rest; c and ℏ are the speed of light and the reduced Planck constant, respectively. When the (relativistic) mass - with corresponding rest mass m - of the incoming particle is much lower than the rest mass (M) of the target nucleus, the differential cross section - obtained from the Wentzel–Molière treatment of the single scattering - is:

$$\frac{d\sigma^{WM}(\theta)}{d\Omega} = \left(\frac{zZe^2}{p\beta c}\right)^2 \frac{1}{(2A_{s,M}+1-\cos\theta)^2} \quad (6)$$

$$= \left(\frac{zZe^2}{2p\beta c}\right)^2 \frac{1}{\left[A_{s,M}+\sin^2(\theta/2)\right]^2} \quad (7)$$

(e.g., see Section 2.3 of [Fernandez-Vera et al. (1993)] and references therein). Equation (7) differs from Rutherford's formula - as already mentioned - for the additional term $A_{s,M}$ to sin $^2(\theta/2)$. The corresponding total cross section {e.g., see Equation (25) in [Fernandez-Vera et al. (1993)] } per nucleus is

$$\sigma^{WM} = \left(\frac{zZe^2}{p\beta c}\right)^2 \frac{\pi}{A_{s,M}(1+A_{s,M})}. \quad (8)$$

Thus, for $\beta \simeq 1$ (i.e., at very large p) and with $A_{s,M} \ll 1$, from Eqs. (5, 8) one finds that the cross section approaches a constant:

$$\sigma_c^{WM} \simeq \left(\frac{2zZe^2a_I}{\hbar c}\right)^2 \frac{\pi}{1.13+3.76\times(\alpha zZ)^2}. \quad (9)$$

In case of a scattering under the action of a central potential (for instance that due to a screened Coulomb field), when the rest mass of the target particle is no longer much larger than the relativistic mass of the incoming particle, the expression of the differential cross section must properly be re-written - in the center of mass system - in terms of an "effective particle" with momentum ($p_r'$) equal to that of the incoming particle ($p'_{in}$) and rest mass equal to the relativistic reduced mass

$$\mu_{rel} = \frac{mM}{M_{1,2}},$$

where $M_{1,2}$ is the invariant mass; m and M are the rest masses of the incoming and target particles, respectively (e.g., see [Boschini et al. (2010), Staruszkiewicz and Zalewski (1977), Fiziev and Todorov (2001)] and references therein). The "effective particle" velocity is given by:

$$\beta_r c = c\sqrt{\left[1+\left(\frac{\mu_{rel}c}{p'_{in}}\right)^2\right]^{-1}}.$$

Thus, the differential cross section[3] per unit solid angle of the incoming particle results to be given by

$$\frac{d\sigma^{WM}(\theta')}{d\Omega'} = \left(\frac{zZe^2}{2p'_{in}\beta_r c}\right)^2 \frac{1}{\left[A_s+\sin^2(\theta'/2)\right]^2}, \quad (10)$$

with

$$A_s = \left(\frac{\hbar}{2p'_{in}a_I}\right)^2 \left[1.13+3.76\times\left(\frac{\alpha zZ}{\beta_r}\right)^2\right] \quad (11)$$

and $\theta'$ the scattering angle in the center of mass system.

Furthermore (e.g., see Section 2.2.2 of [Leroy and Rancoita (2016)]), assuming an isotropic azimuthal distribution one can re-write Eq. (10) in terms of the kinetic energy transferred from the projectile to the recoil target as:

$$\frac{d\sigma^{WM}(T)}{dT} = \pi \left(\frac{zZe^2}{p'_{in}\,\beta_r c}\right)^2 \frac{T_{max}}{[T_{max}\,A_s + T]^2}. \quad (12)$$

Furthermore, since

$$\begin{aligned} \beta_r c &= \frac{p\,c^2}{E} \\ p'_{in} &= \frac{pM}{M_{1,2}} \\ T_{max} &= \frac{2p^2 M}{M_{1,2}^2} \end{aligned} \quad (13)$$

with p and E the momentum and total energy of the incoming particle in the laboratory, then one finds

$$\frac{T_{max}}{(p'_{in}\,\beta_r c)^2} = \frac{2E^2}{p^2 M c^4}.$$

Therefore, Eq. (12) can be re-written as

$$\frac{d\sigma^{WM}(T)}{dT} = 2\,\pi\,(zZe^2)^2 \frac{E^2}{p^2 M c^4} \frac{1}{[T_{max}\,A_s + T]^2}. \quad (14)$$

Equation (14) expresses - as already mentioned - the differential cross section as a function of the (kinetic) energy T achieved by the recoil target.

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