Tunneling with Time Dependence

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("student 6")

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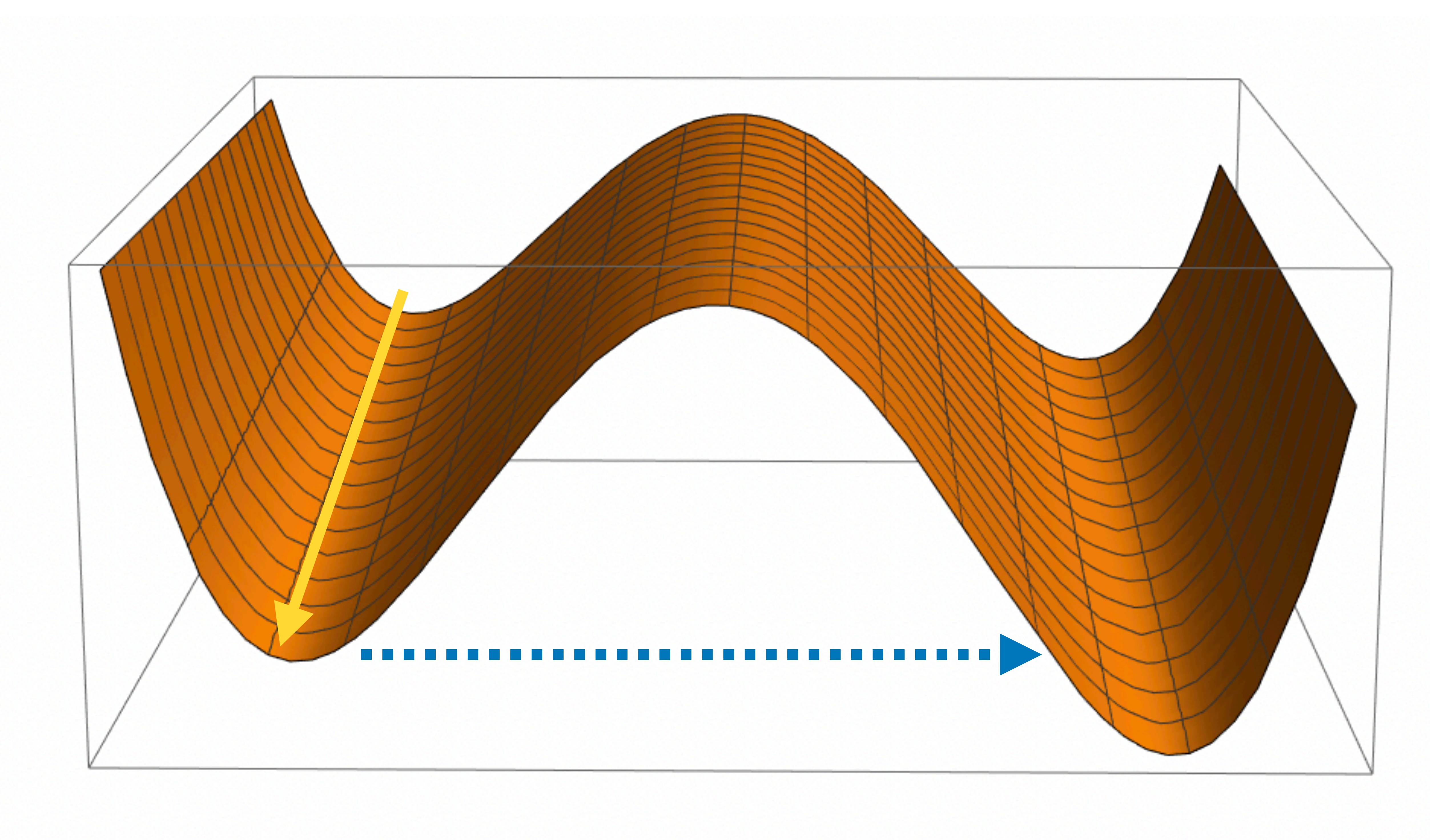
Semiclassics provides useful tools/intuition for many problems in QFT. Vacuum decay is a prototypical example.



How do we approach semiclassics of bubble nucleation on time-dependent backgrounds?

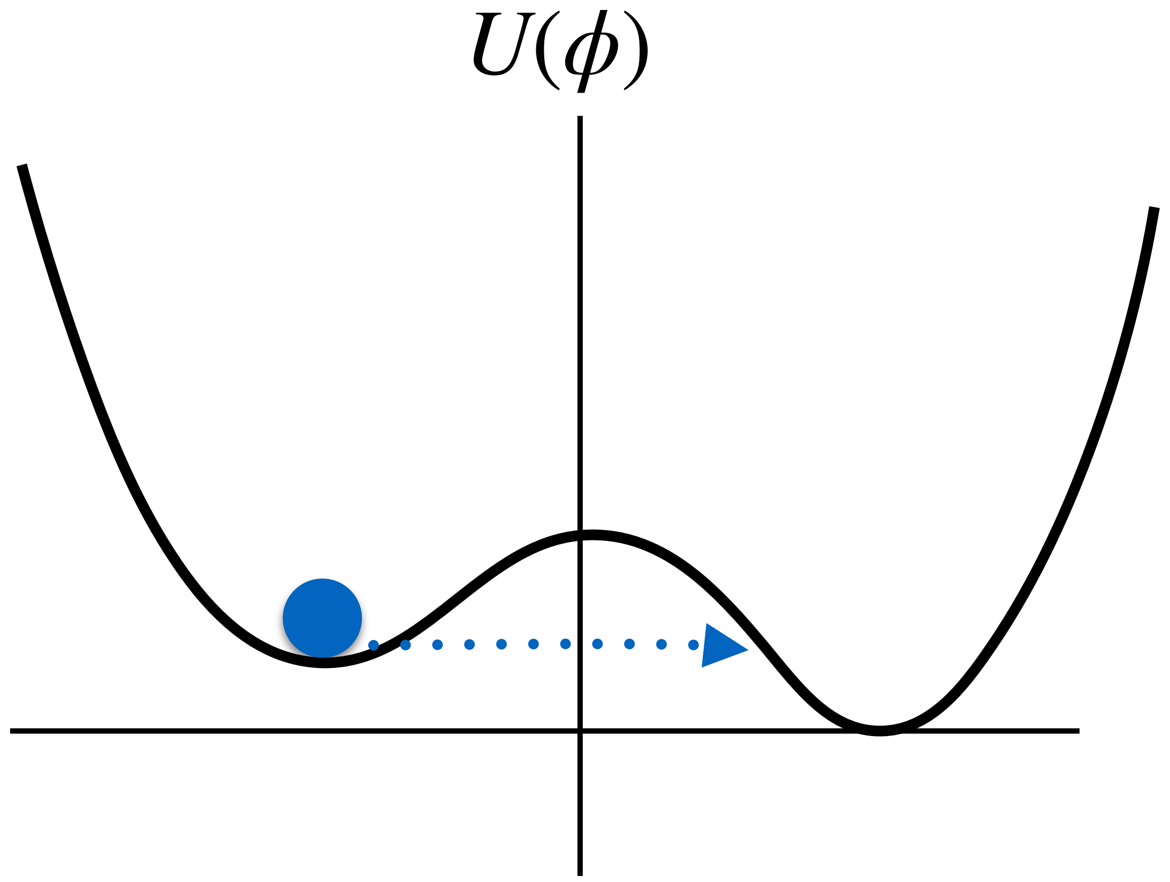
Euclidean continuation will result in a complex action

Based on work to appear with Manthos Karydas and Hao Zhang

Semiclassical theory of vacuum decay by bubble nucleation

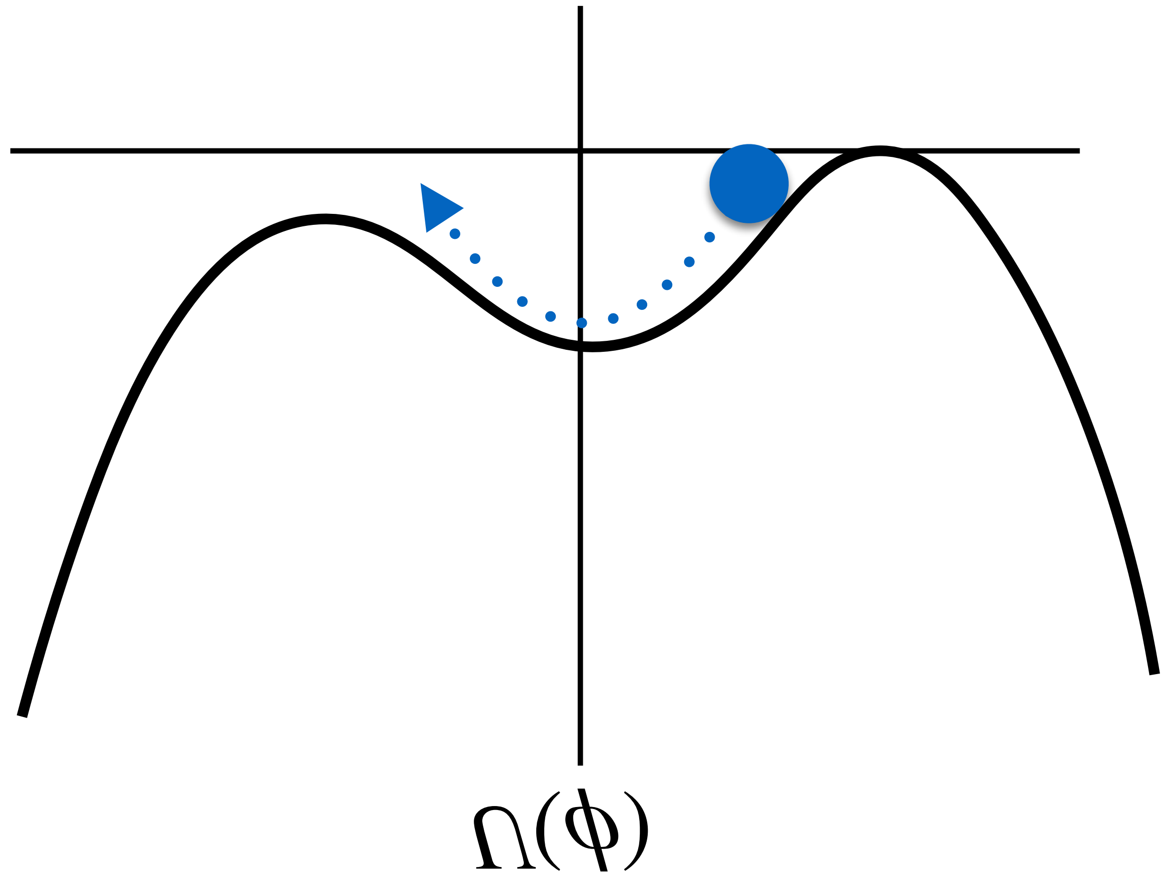
Typical tunneling potential

$$V(\phi) = \frac{1}{2}\lambda(\phi^2 - a^2)^2 - (\epsilon/2a)\phi$$



We take $t \to -i\tau$ and look for O(4) invariant saddle points. Solve a shooting problem:

$$\frac{d^2\phi}{d\rho^2} + \frac{3}{\rho}\frac{d\phi}{d\rho} = U'(\phi)$$



$$\Gamma \approx v^4 \exp(-S_E/\hbar)$$

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- real-valued fields remain real

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Two generalizations:

- $t \to e^{i\gamma}\tau$
- $V(\phi) \to V(\phi, \chi_0(t))$

O(4) invariance broken, DOFs complexified, final states may carry momentum

Loss of O(4) invariance is a technical complication because ODEs => PDEs.

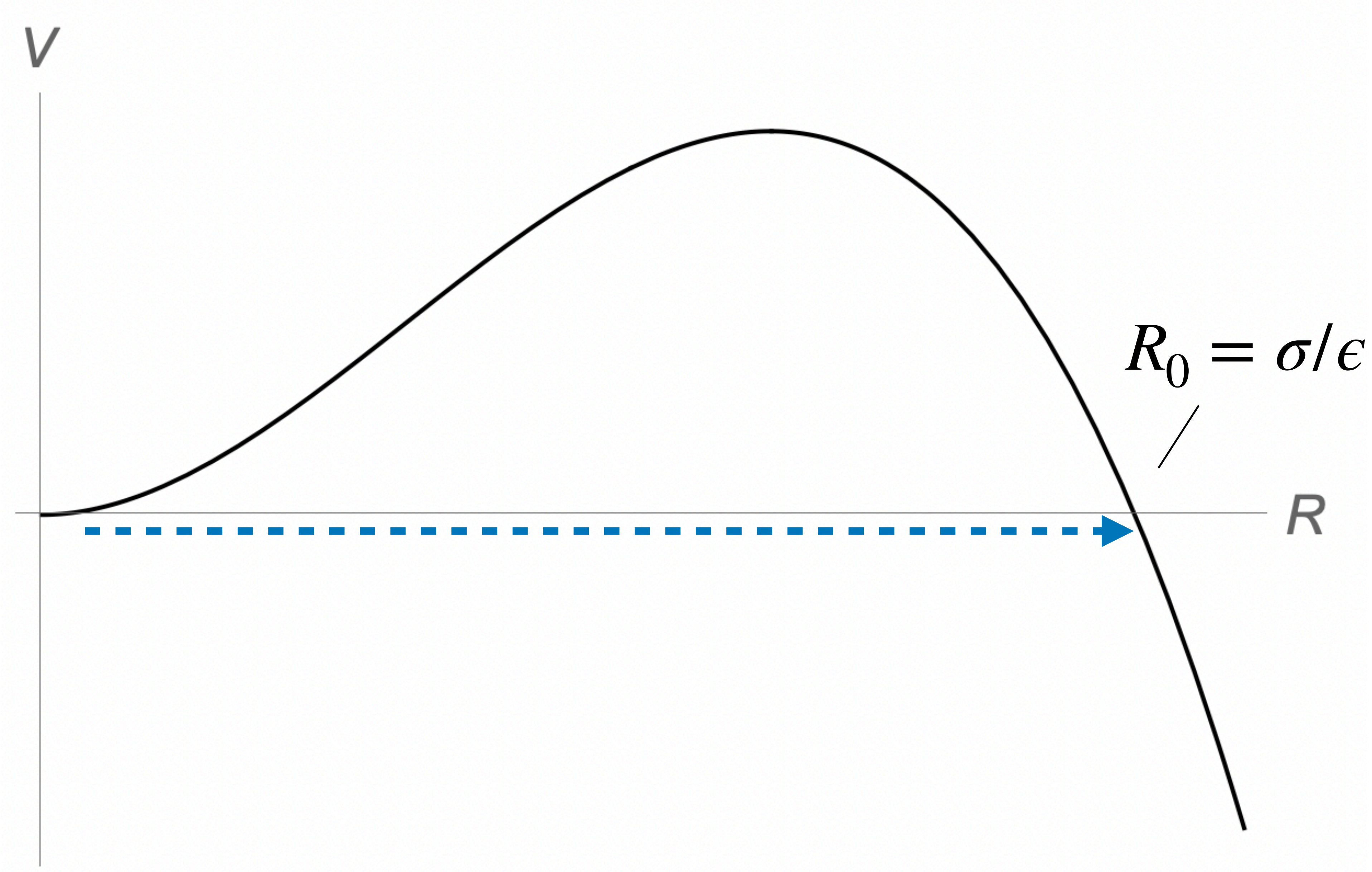
To avoid this complication we will work with effective QM models
for collective coordinate DOFs.   QM => back to ODEs.

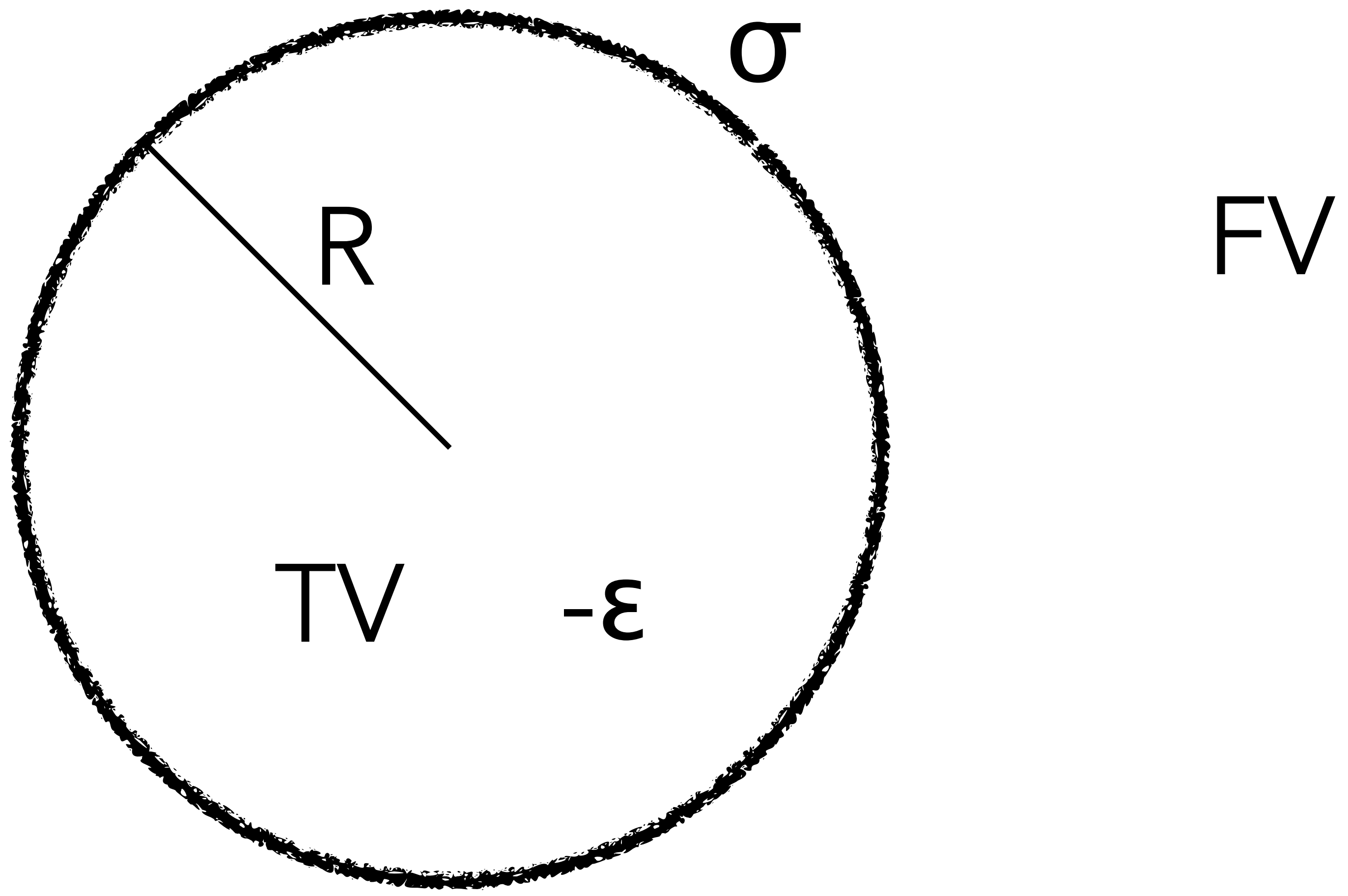
Start by reviewing how this works for the standard, time-indep, Euclidean case,
then extend to $t \to e^{i\gamma}\tau$,
then add time dependence

standard, time-independent Euclidean analysis:

thin-wall effective Lagrangian captures leading semiclassics.

$$L = \underbrace{-\sigma R^2 \sqrt{1-(\partial_t R)^2}}_{\text{tension+lorentz contraction}} + \underbrace{\epsilon R^3}_{\text{pressure}}$$





similar L can be used for Schwinger model, BON,...

$$A = \langle \mathrm{TV\ bubble}, \mathrm{T_f} = 0 | \mathrm{FV}, \mathrm{T_i} \rangle = ?$$

$$e^{iS} = \exp\left(i \int_{T_i}^{T_f} dt \left[-\sigma R^2 \sqrt{1 - (\partial_t R)^2} + \epsilon R^3 \right] \right)$$

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$$t \to -i\tau$$ continuation of $\mathrm{T_i}$

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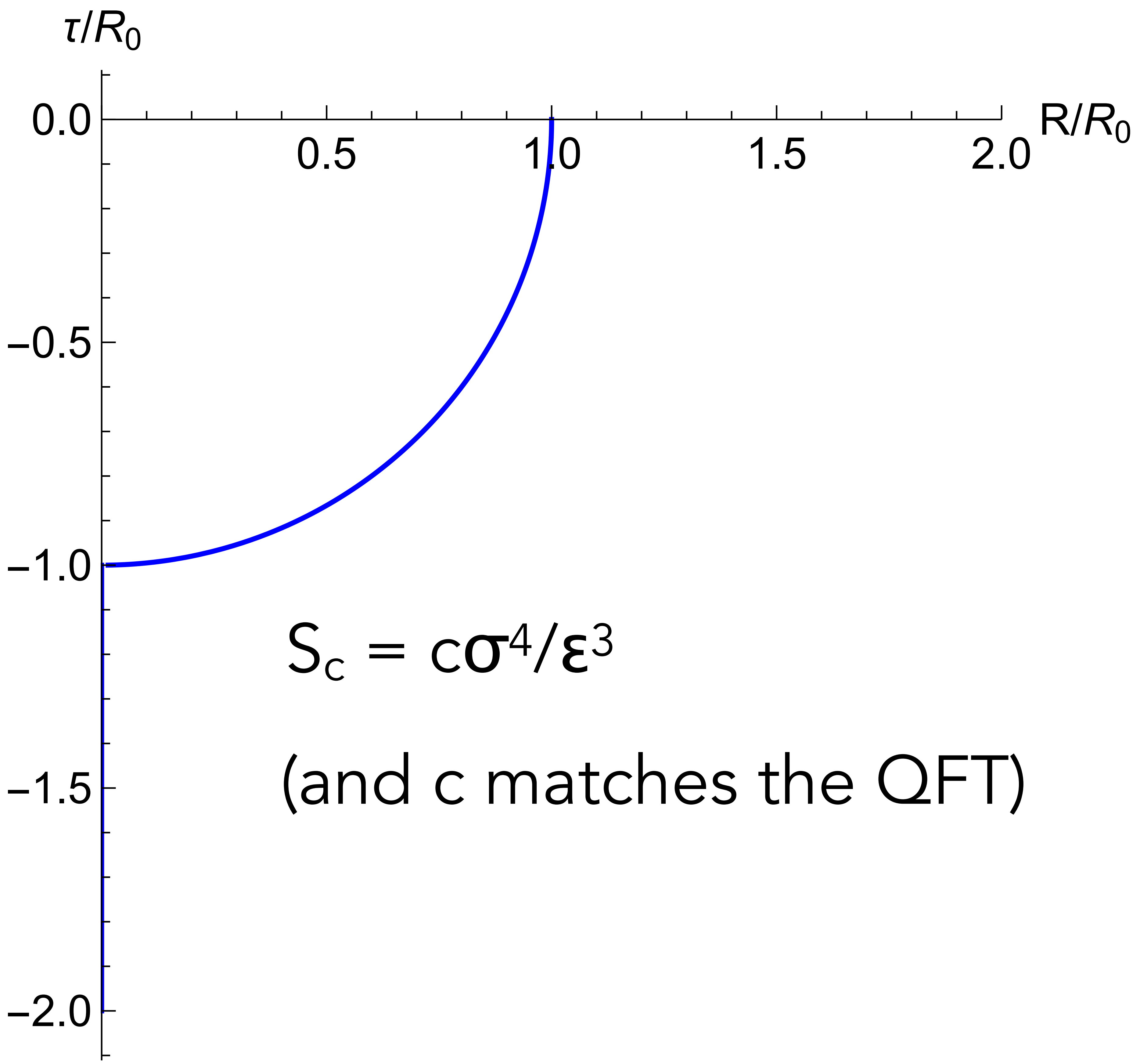
Zero-energy saddle point for which R=0 at some τ_0:

$$R = \Theta(\tau + R_0)\sqrt{R_0^2 - \tau^2}.$$

$$R_0 = \sigma/\epsilon \ , \qquad \tau_0 = -R_0$$

"Amplitude to nucleate a bubble at rest at t=0"



$S_c = c\sigma^4/\epsilon^3$

(and c matches the QFT)

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Technically this joins two zero-energy solutions piecewise.
Still a stationary point of the action: $\sigma R^2 \sqrt{1 + (\partial_\tau R)^2}$

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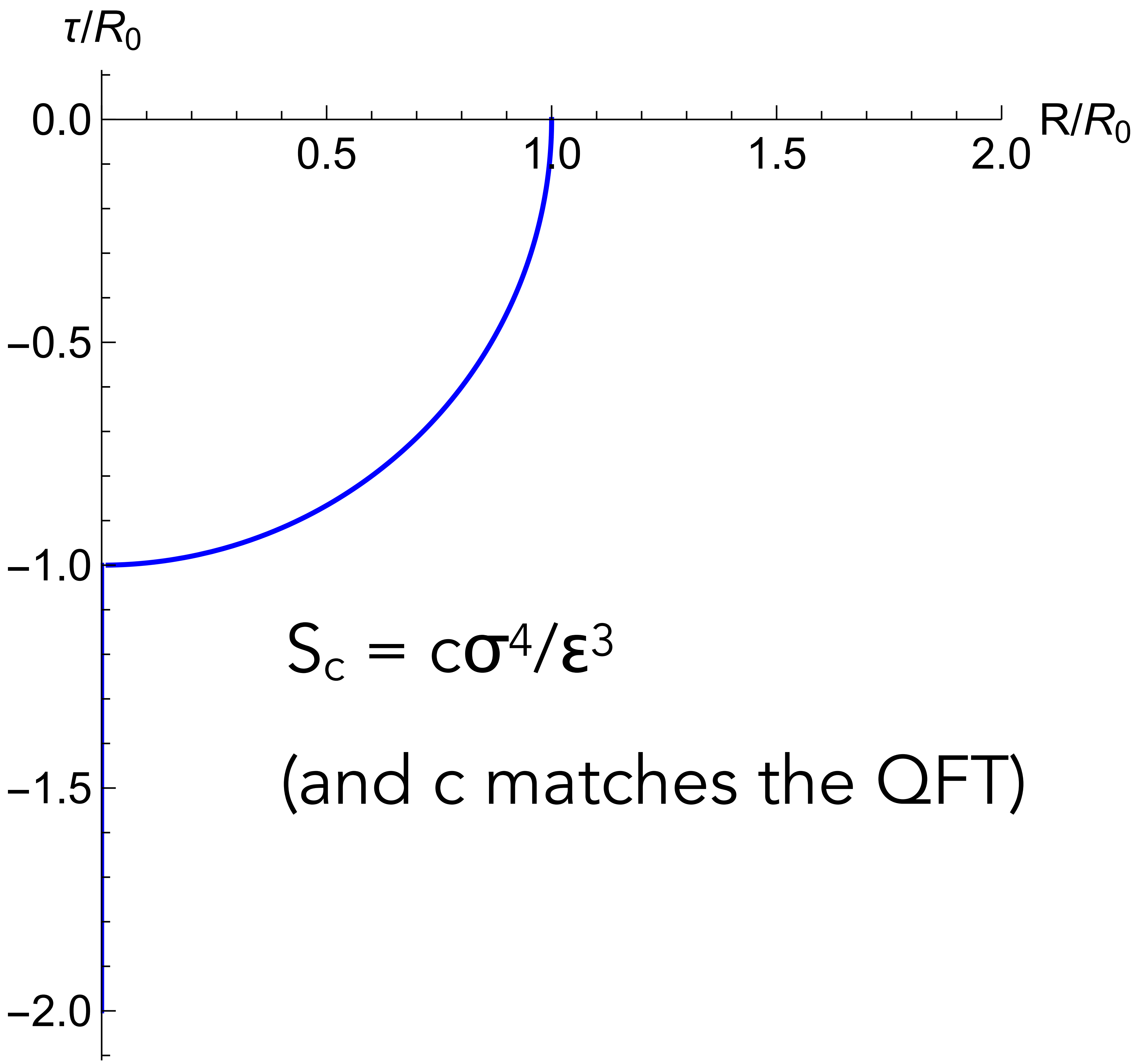

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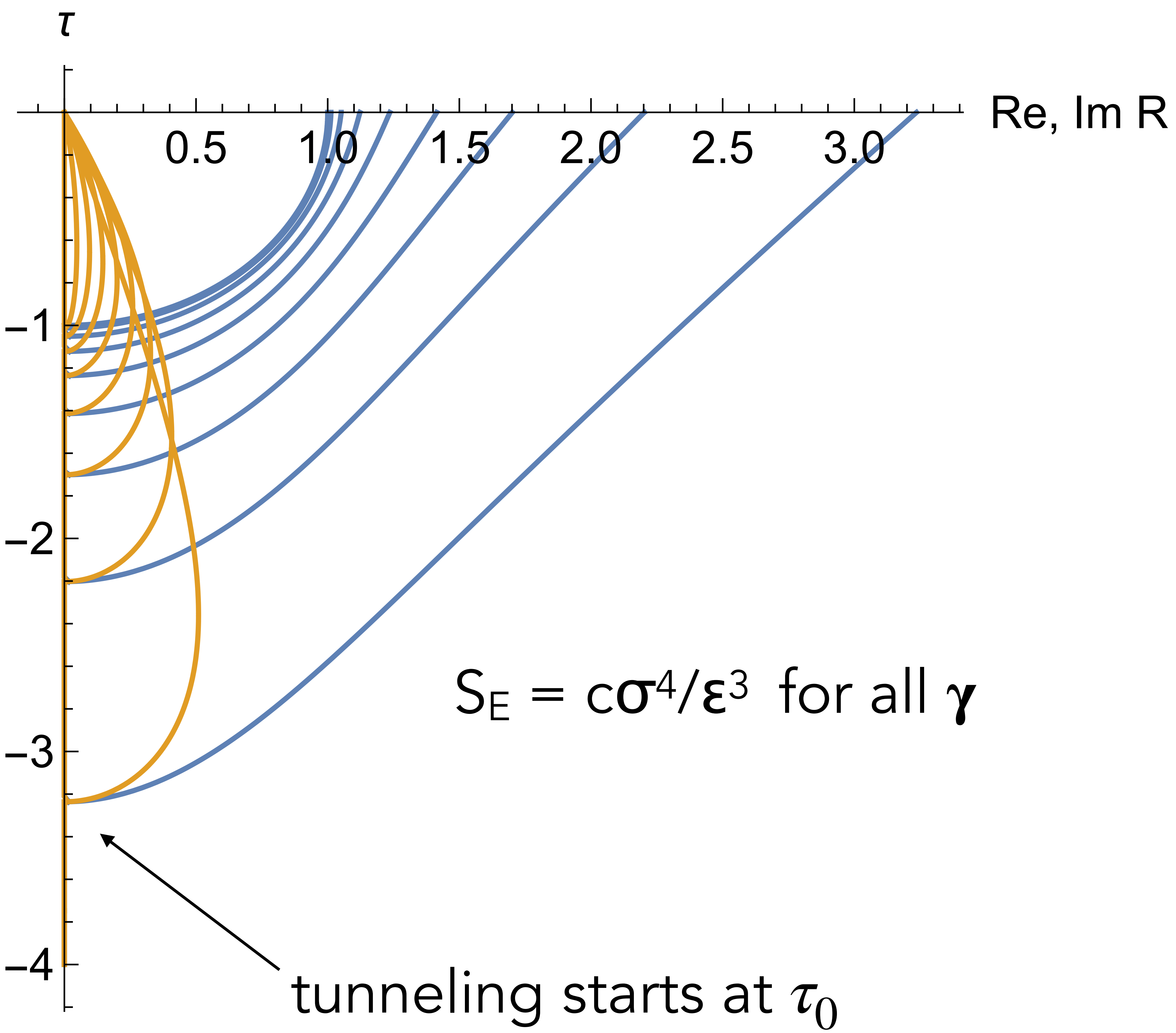

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tunneling starts at τ_0

goes to $-\infty$ in the real-time limit

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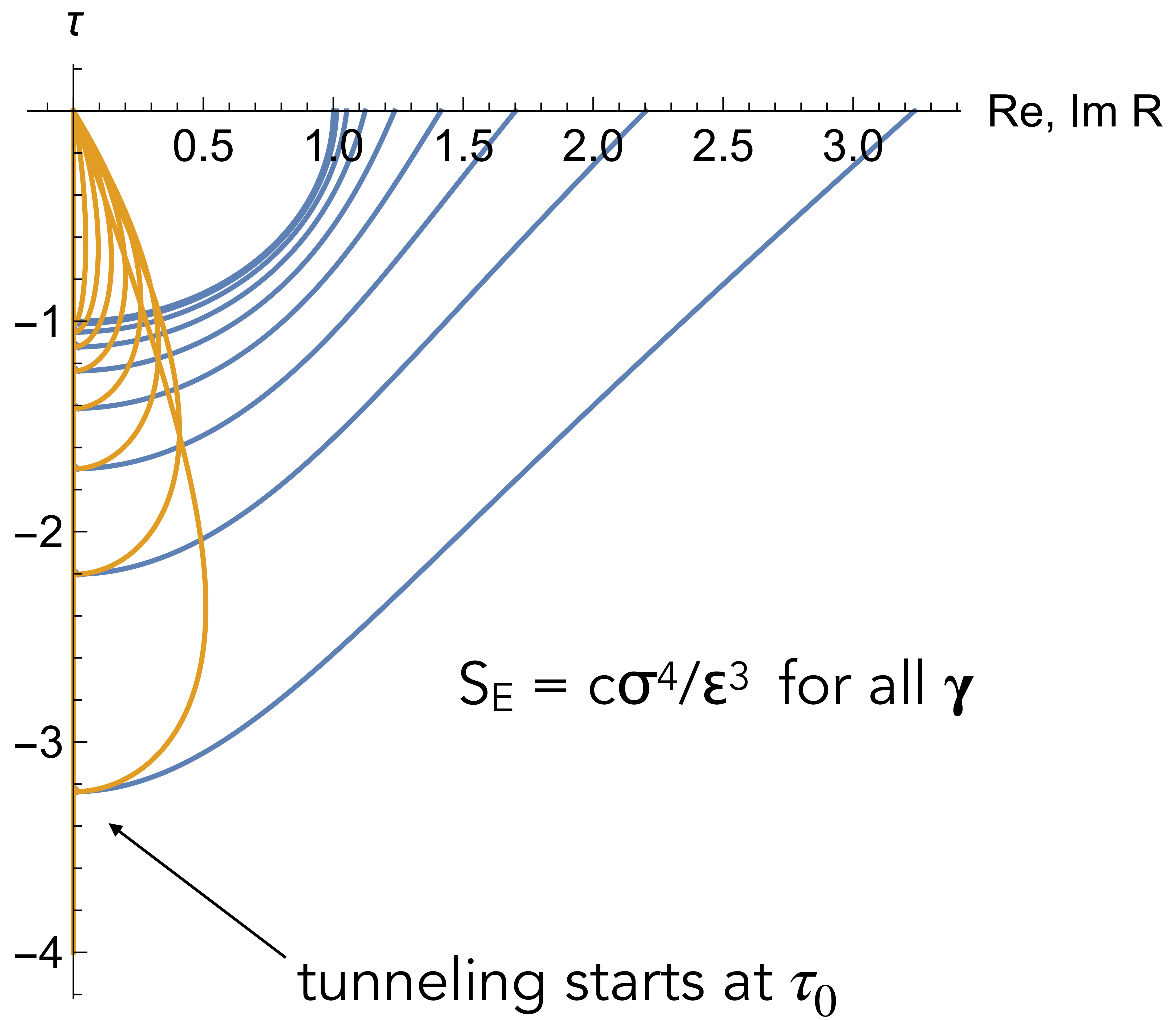
Return to real R at $\tau = 0$,
but beyond the classical turning point R_0,
with nonzero momentum

What do they mean?



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amplitudes with semiclassical wavepackets:

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$$S_c = -iS_i(R_i) + iS_f(R_f) - i\int_{T_i}^{T_f} d\tau\, L_c(\partial_\tau R, R, \tau)$$

$$L_c(\partial_\tau R, R, \tau) = e^{i\gamma} L(e^{-i\gamma}\partial_\tau R, R, e^{i\gamma}\tau).$$

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same bulk EOM,
boundary variations relate
initial and final semiclassical
R,p to features of the states:

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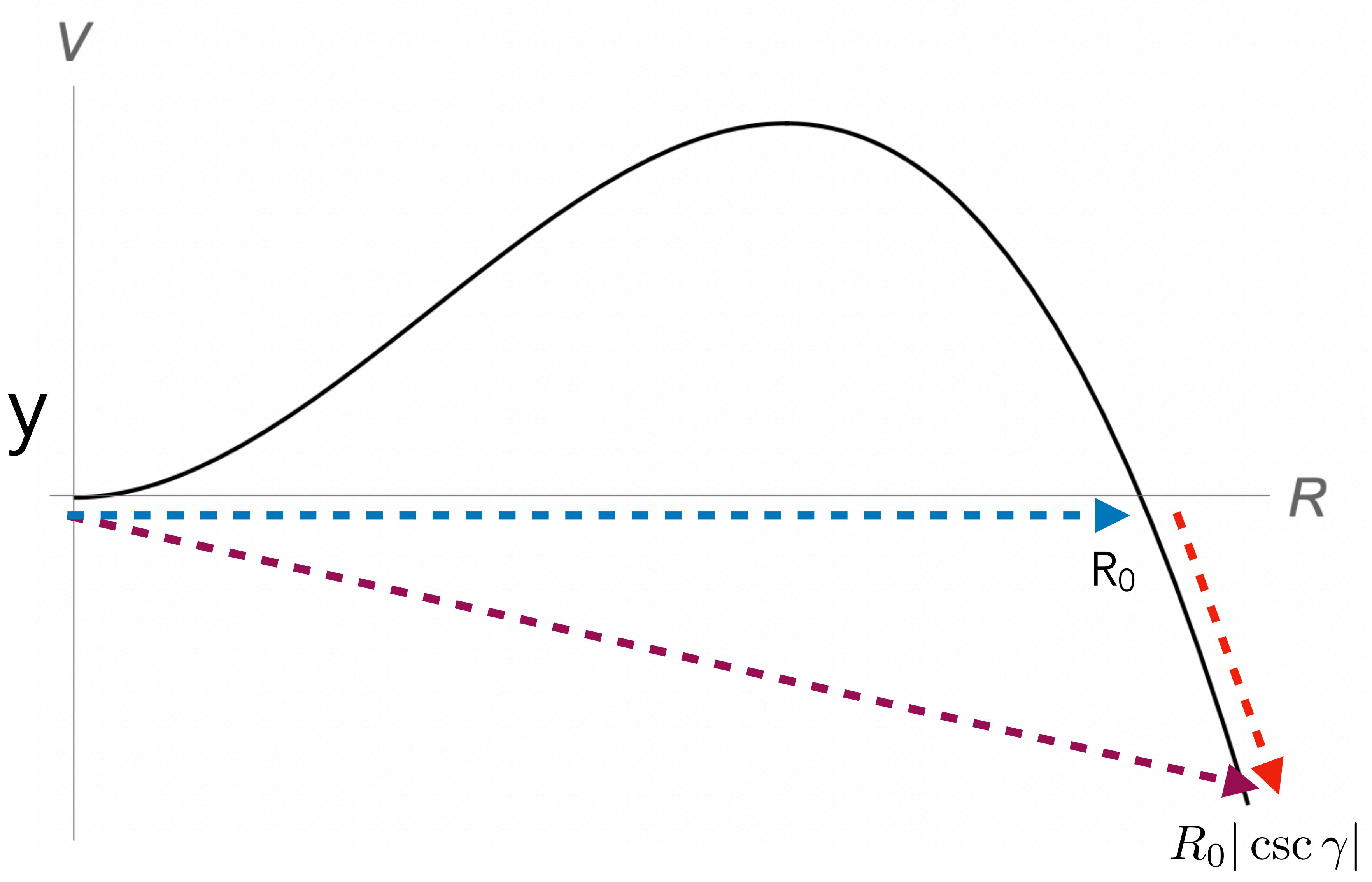
semiclassical trajectories also appx certain packet amplitudes!

e.g. $\mathrm{Re}(p_f)$ = <p> for packets: $\psi_f = Ne^{-(R_f-R_0)^2/2\sigma^2+ipR_f} \Rightarrow \mathrm{Re}\,S_f'(R_f) = p$ = $\mathrm{Re}(p_f)$

Interpretation:

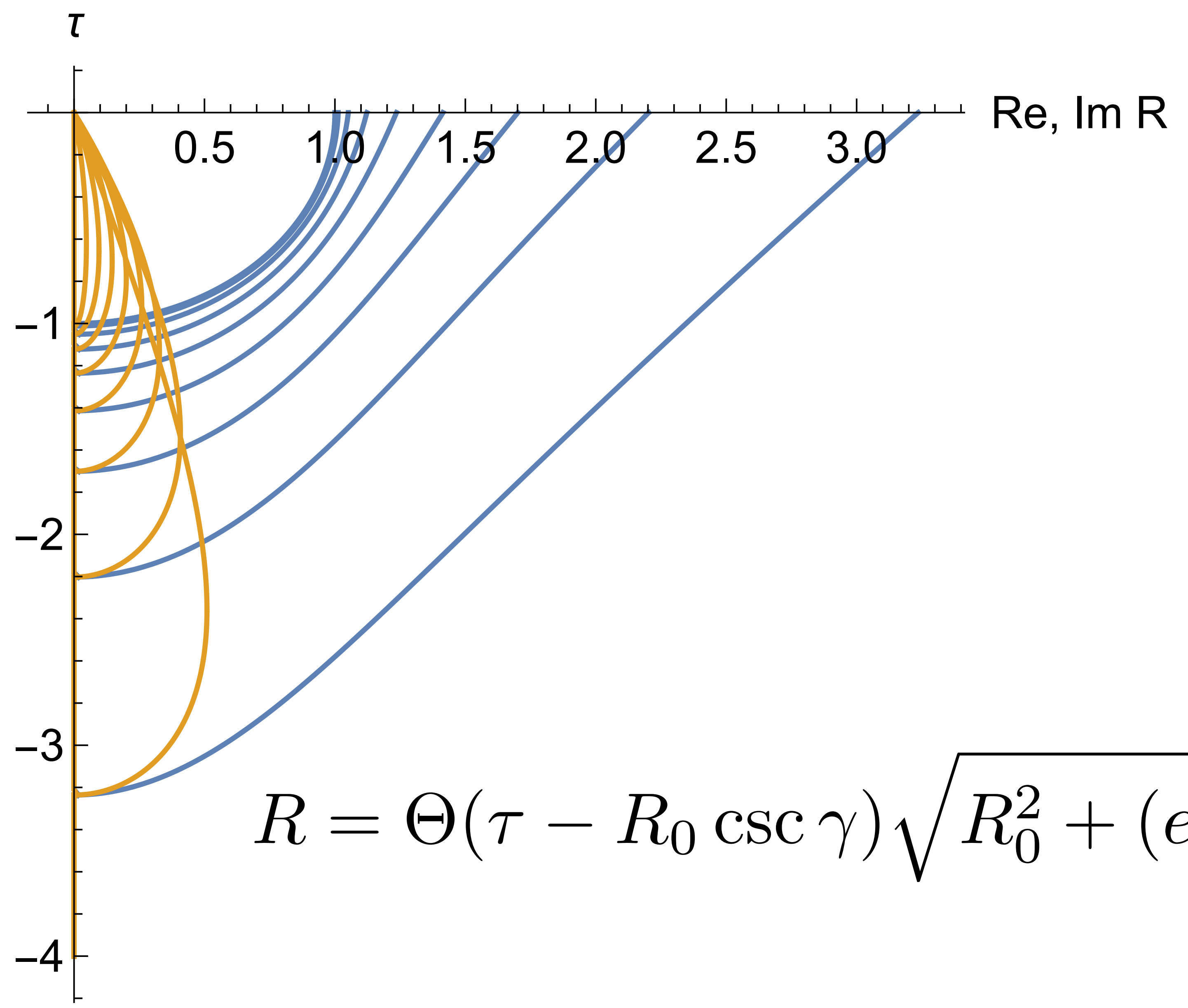
- The solutions for general $\boldsymbol{\gamma}$ connect the false vacuum to wavepackets outside the barrier with zero classical energy

  connected by classical, real time evolution to the nucleation point of the critical bubble ($R=R_0$, at rest)



- tunneling starts earlier (only makes sense if $T_i < R_0 \csc \boldsymbol{\gamma}$)

$$\langle \psi_f(R,T_f)|\psi_i(R,T_i)\rangle$$

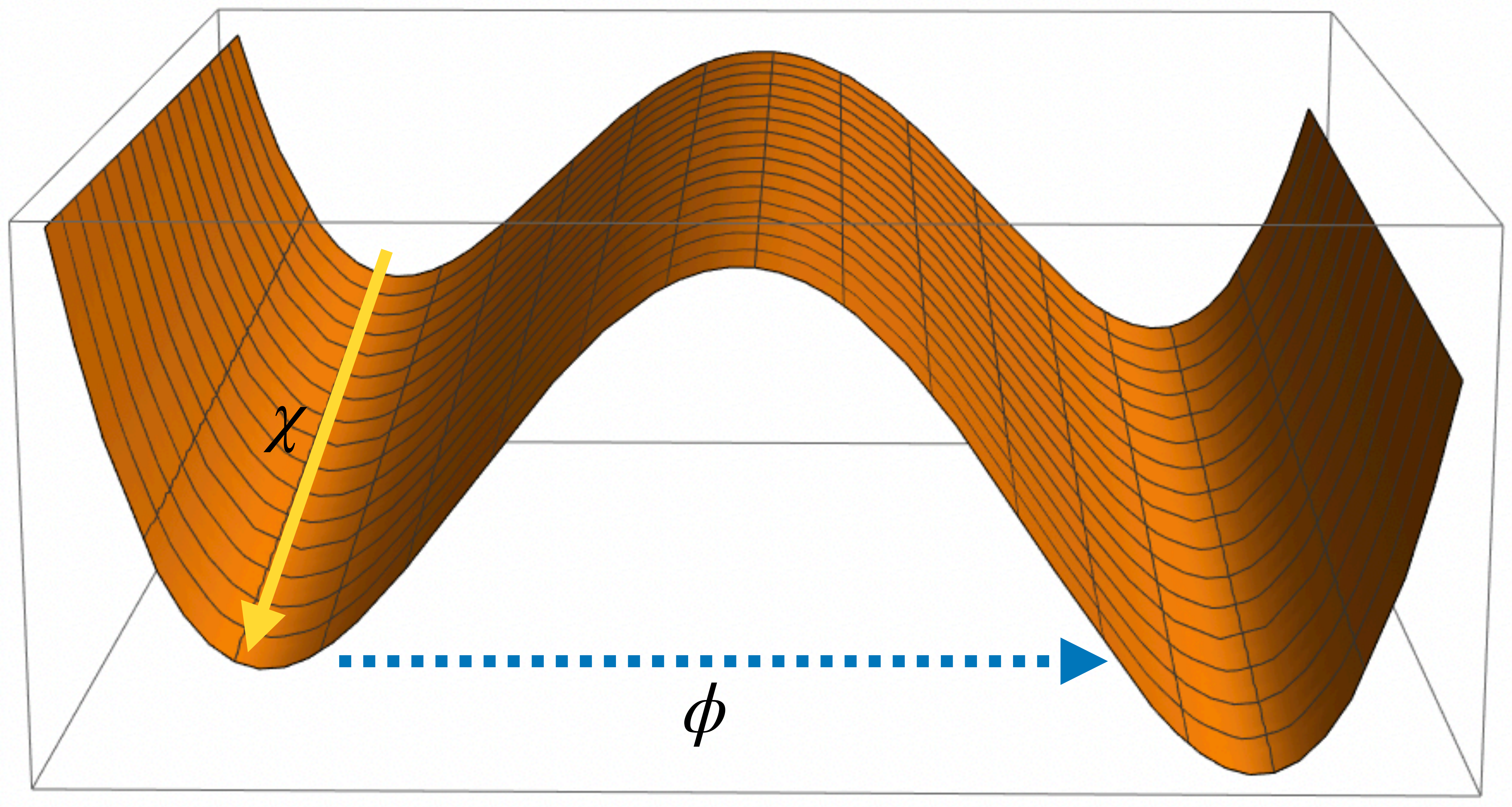


$$R = \Theta(\tau - R_0 \csc\gamma)\sqrt{R_0^2 + (e^{i\gamma}\tau - R_0\cot\gamma)^2}$$

$\boldsymbol{\gamma}$ is accounting for the time translation collective coordinate

Now let us add time dependence.

This can arise, for example, if the tunneling field ϕ is weakly coupled to another field χ slowly rolling in a shallow potential.



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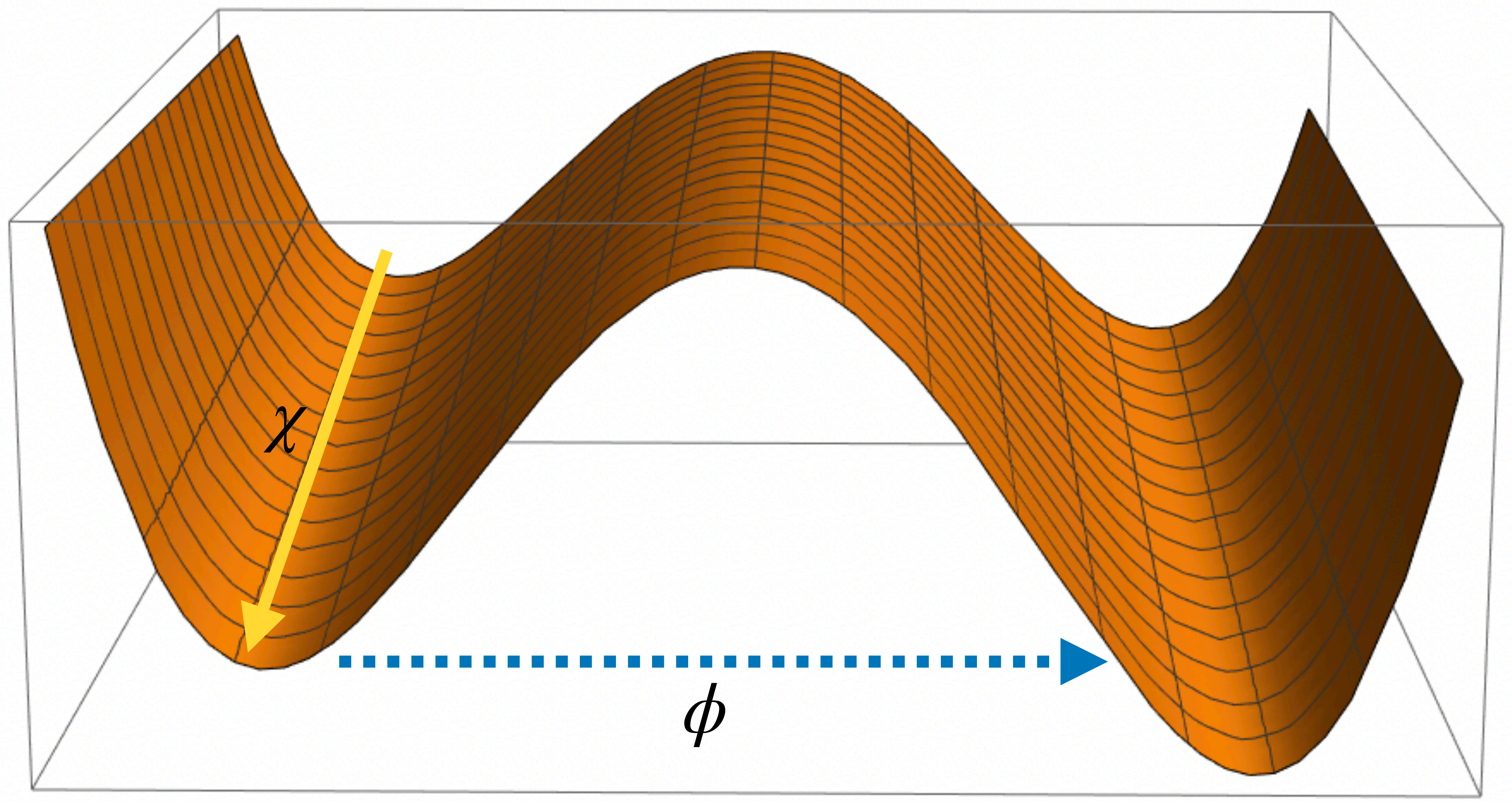


at zeroth order in the coupling, same thin-wall profile for ϕ + rolling solution for χ

integrate out space to get $L = -\sigma(t) R^2 \sqrt{1 - (\partial_t R)^2} + \epsilon(t) R^3$ $\qquad \chi(t) \to \sigma(t), \epsilon(t)$

could also include backreaction of ϕ profile on χ,
but higher order and may be subleading to corrections to the thin-wall limit

$$L = -\sigma(t) R^2 \sqrt{1-(\partial_t R)^2} + \epsilon(t) R^3 \quad + \quad t \to e^{i\gamma}\tau$$

Look for nontrivial small-R solutions that can connect to the R=0 “vacuum”

Relevant solutions behave as

$$\boxed{\begin{aligned} R &= c\sqrt{\tau-\tau_0}\,(1+\mathcal{O}(\tau-\tau_0)) \ , \\ c &= ie^{i\gamma/2}\sqrt{\frac{6(\sigma_0+e^{i\gamma}\tau_0 v_\sigma)}{3i(\epsilon_0+e^{i\gamma}\tau_0 v_\epsilon)+v_\sigma}} \end{aligned}}$$

where

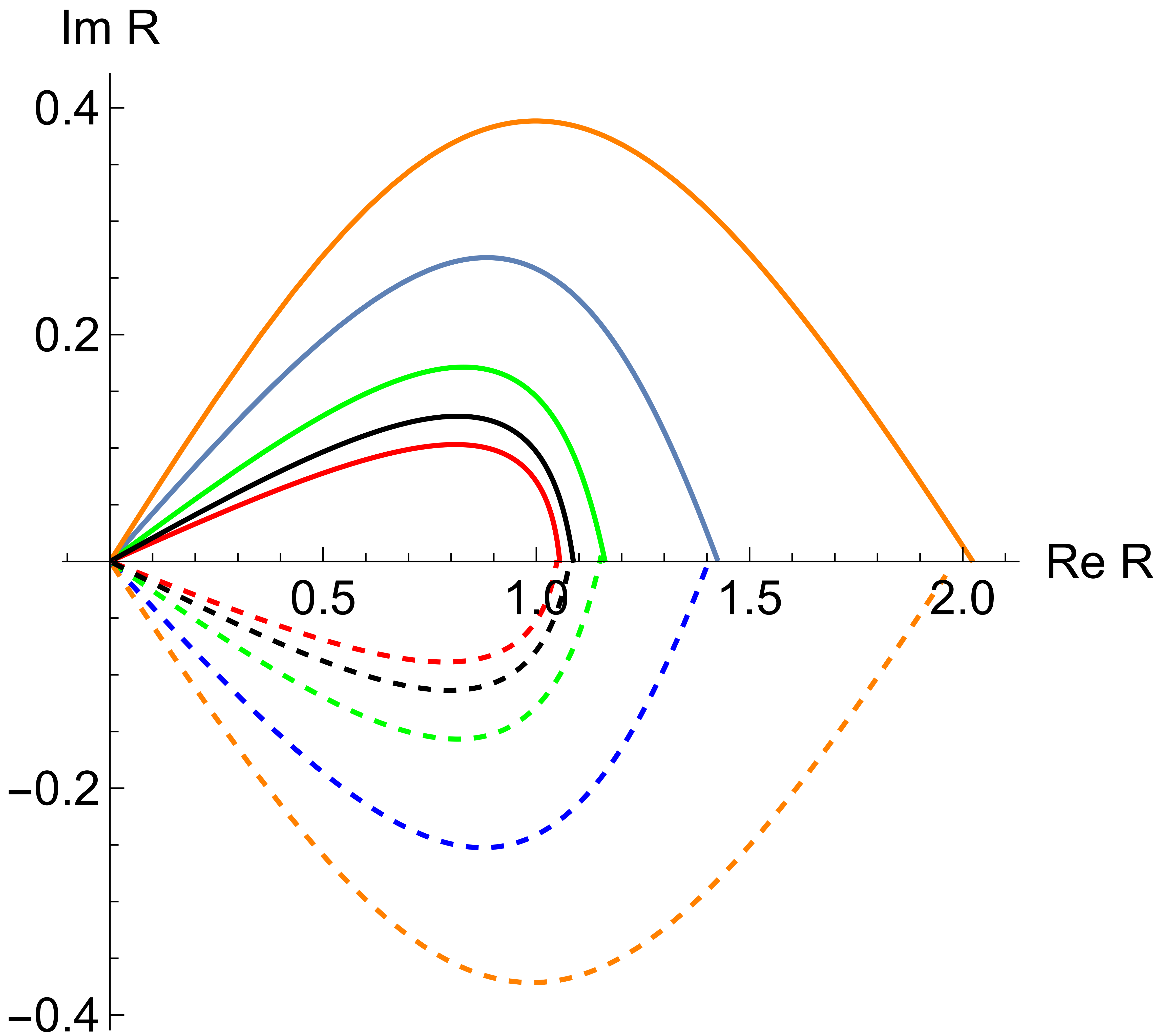
$$\sigma(e^{i\gamma}\tau) = \sigma_0 + e^{i\gamma} v_\sigma(\tau-\tau_0) + \ldots ,$$
$$\epsilon(e^{i\gamma}\tau) = \epsilon_0 + e^{i\gamma} v_\epsilon(\tau-\tau_0) + \ldots$$

To determine τ_0, shoot for solutions that return to Im R=0 at τ=0 examples with slow linear evolution of **σ**, **ε** and a range of **γ**

final states generally acquire small Im p_f — slightly off-peak

$$\psi_f = N e^{-(R_f - R_0)^2/2\sigma^2 + ipR_f}$$

$$\mathrm{Re}\, p_f = -\sigma R_0^2 \csc^2\gamma \cot\gamma + \mathcal{O}(v_{\epsilon,\sigma})$$



v=0 on-shell action $S \sim R_0^3\sigma_0, R_0^4\epsilon_0$

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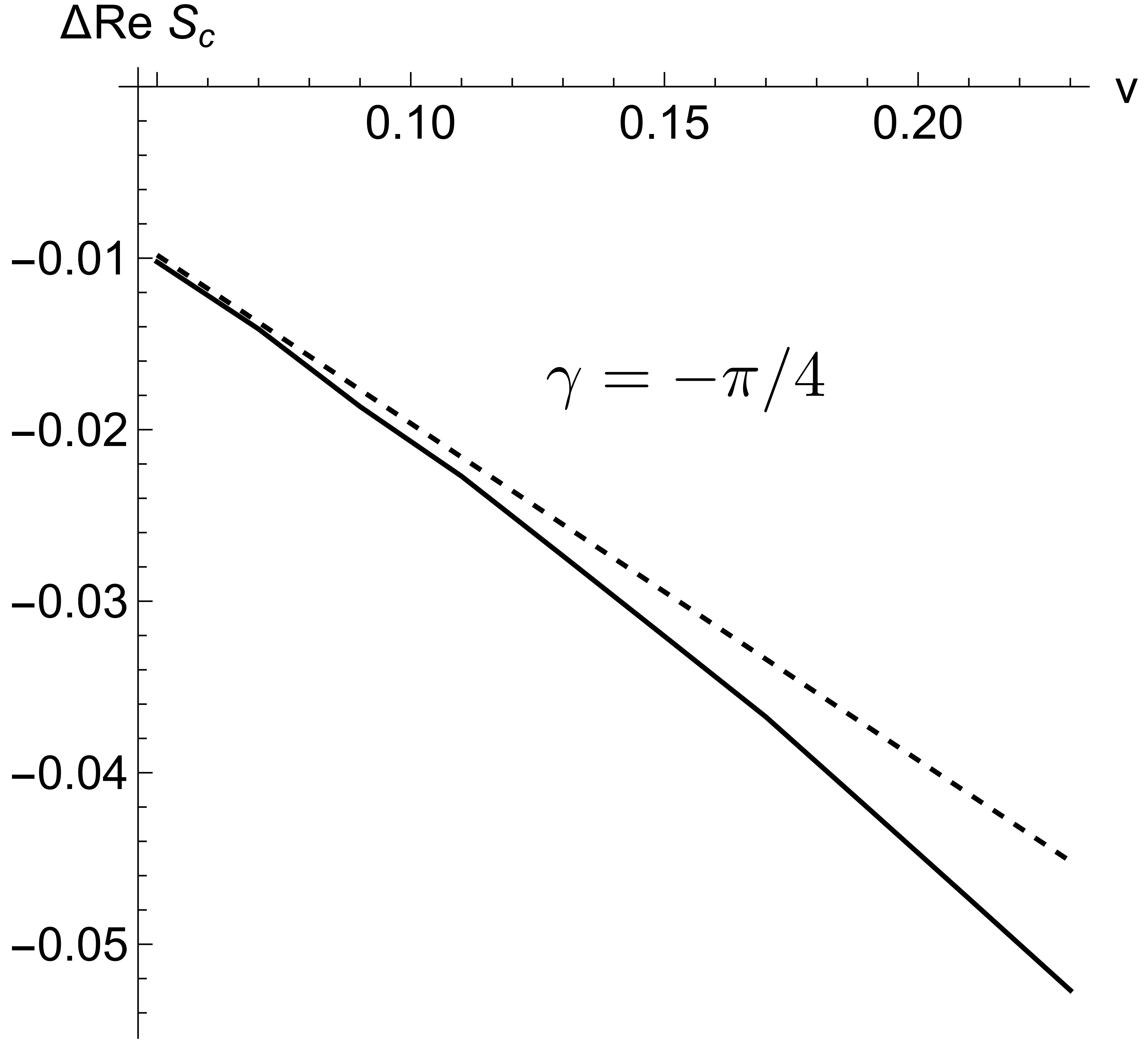
$$\Delta\mathrm{Re}\ S_c|_{\mathcal{O}(v)} = \frac{\pi}{4}\cot(\gamma)R_0^4\left(v_\sigma - \frac{3}{4}R_0 v_\epsilon\right)$$



For fixed **γ**, small cf. leading order
if variations are slow.

But unbounded as **γ** approaches 0, pi…
what **γ** should we use?

Recall tunneling starts at $\tau_0 = R_0 \csc \gamma$ — require > T_i

If e.g. barrier is growing monotonically, starting earlier = larger amplitudes

$$\gamma = \csc^{-1}(T_i/R_0) \quad \Rightarrow \quad \boxed{\Delta \mathrm{Re}\ S_c|_{\mathcal{O}(v),\mathrm{max}} = -\frac{\pi}{4}\sqrt{T_i^2 - R_0^2}R_0^3\left(v_\sigma - \frac{3}{4}R_0 v_\epsilon\right)}$$

if +ve, favors earlier decay

if +ve, favors later decay

Otherwise later is better, $\gamma = -\pi/2$ (Euclidean) and leading O(v) correction vanishes

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What precise question does exp(-S_c) address?

T_i is the latest time when we're sure there was no bubble
maximizing over $\boldsymbol{\gamma}$ computes the leading decay probability over times up to T_f=0

Coupling to gravity

Three approaches to keep the problem tractable (not exhaustive):

- Rigid Minkowski => rigid something time-dep e.g. FRW
- Effective action for a membrane coupled to gravity
- Exact solutions from analytic continuation

Rigid Minkowski => rigid FRW:

$$ds^2 = -dt^2 + a(t)^2 dx_i^2$$

$$a \approx 1 + Ht \qquad (HR_0 \ll 1)$$

$$S \approx \int d^4x \left[\mathcal{L}_{flat} + Ht(3\mathcal{L}_{flat} + (\partial_i \phi)^2)\right]$$

background spacetime provides the time dependence

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$$\Rightarrow L_{eff} = \underbrace{-(1+3Ht)\sigma R^2 \sqrt{1-(\partial_t R)^2} + (1+3Ht)\epsilon R^3}_{\text{same as previous}} + \underbrace{\frac{\sigma H t R^2}{\sqrt{1-(\partial_t R)^2}}}_{\text{new term}}$$

O(H) contribution to the on-shell action vanishes

Membrane effective action:

in the static case, use a small generalization of an effective action found by Visser (1992)

EH + spherical brane:

$$\text{(1)} \quad S = \frac{1}{16\pi G_N} \int_{\mathcal{M}_{1,2}} \sqrt{|g|} d^4x \, (R_{1,2} - 2\Lambda_{1,2}) + \frac{1}{8\pi G_N} \int_{\mathcal{T}} \sqrt{|h|} d^3y K_{1,2} - \mu \int_{\mathcal{T}} \sqrt{|h|} d^3y$$

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Effective action for R_{brane}:

(2) $$S^{\mathrm{eff}}_{\mathrm{brane}} = \frac{1}{2G_N}\int d\lambda\Bigg[-2R\dot{R}\sinh^{-1}\left(\frac{\dot{R}}{\sqrt{N^2 f_{\mathrm{dS}_T}}}\right) + 2R\sqrt{f_{\mathrm{dS}_T}N^2 + \dot{R}^2} + 2R\dot{R}\sinh^{-1}\left(\frac{\dot{R}}{\sqrt{N^2 f_{\mathrm{dS}_F}}}\right) - 2R\sqrt{f_{\mathrm{dS}_F}N^2 + \dot{R}^2} - 8\pi G_N \mu N R^2\Bigg]$$

$$\boxed{f_{T,F} = 1 - R^2/L^2_{1,2}}$$

- λ is an arbitrary parametrization and N is the lapse on the world"line" $-N^2 d\lambda^2 = -f_{T,F}dt^2_{T,F} + f^{-1}_{T,F}dR^2$
- The arcsinh is Visser's trick to rewrite in 1st order form
- Looks rather different, but actually equivalent to previous for $G_N \to 0$

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dS/dN=0 in (2) = junction condition from (1):

$$R\sqrt{f_{dS_T} + \dot{R}^2} - R\sqrt{f_{dS_F} + \dot{R}^2} - 4\pi\mu G_N R^2 = 0$$

gauge fix N=1, λ=proper time

N provides reparam invariance => equiv to H=0

This is the only independent equation (dS/dR=0 gives the proper time derivative of it)

Now we continue $\lambda \to e^{i\gamma}\tau$

The continued energy is

$$E = e^{i\gamma}\frac{\pi}{G_N}\left[R\sqrt{f_{dS_F} + e^{-2i\gamma}\dot{R}^2} - R\sqrt{f_{dS_T} + e^{-2i\gamma}\dot{R}^2} + 4\pi G_N \mu R^2\right] = 0$$

One solution is R=0.

To find the other, rearrange: $\boxed{e^{-2i\gamma}\left(\frac{dR}{d\tau}\right)^2 + 1 - \alpha^2 R^2 = 0}$

$$\alpha^2 = L_F^{-2} + \frac{\left(L_F^{-2} - L_T^{-2} - 16\pi^2 G_N^2 \mu^2\right)^2}{64\pi^2 G_N^2 \mu^2}$$

$$\approx R_0^{-2} + (2L_F)^{-2} + (2L_T)^{-2} + \mathcal{O}(G_N^2)$$

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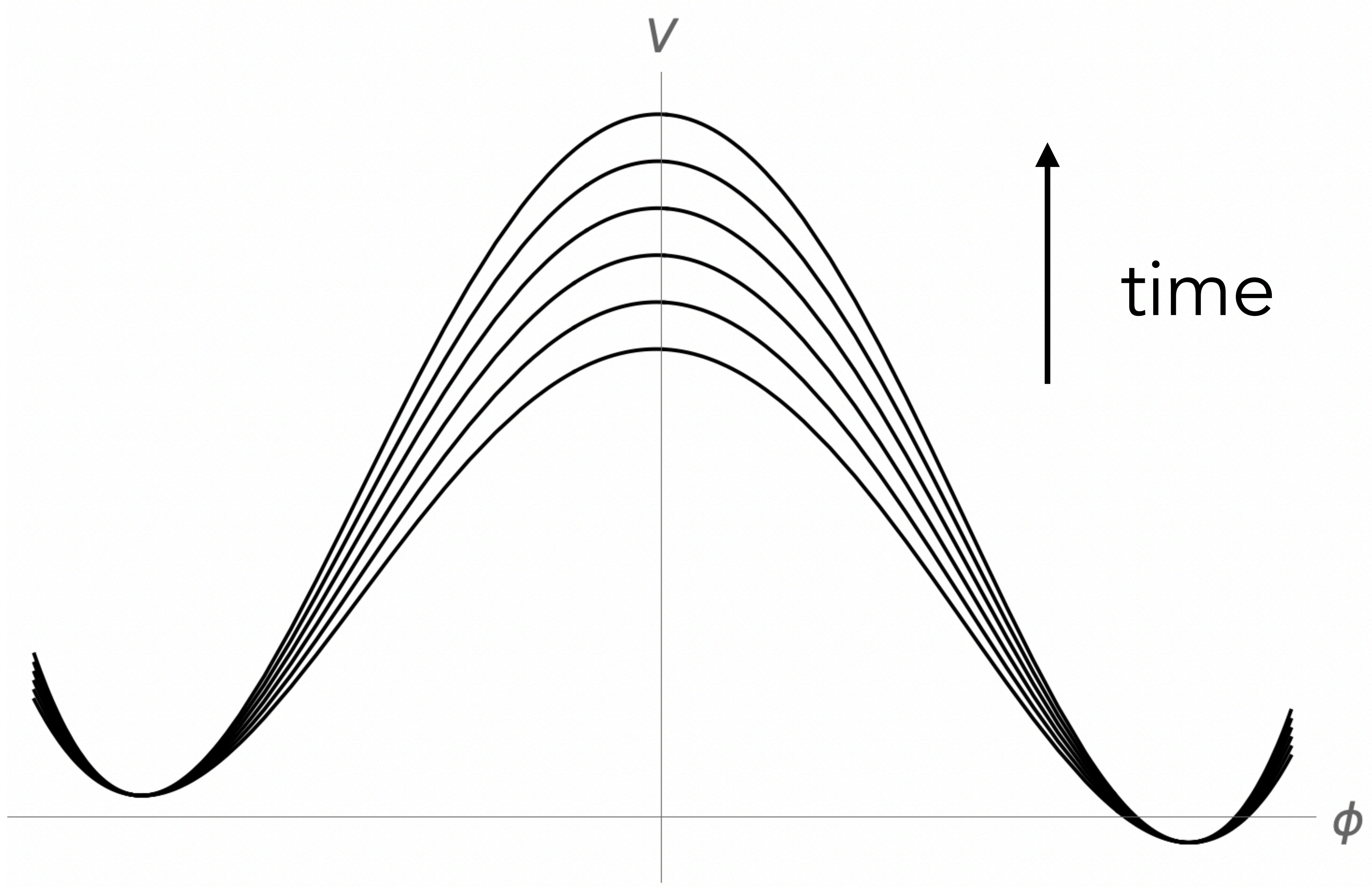
Relevant solutions: $R(\tau) = \alpha^{-1}\cosh\left[\alpha\, e^{i\gamma}(\tau - \tau_0) + i(\pi/2)\right]$ $\qquad \tau_0 = \frac{\pi}{2}\alpha^{-1}\csc(\gamma)$

Nucleation point: $R(0) = \alpha^{-1}\cosh((\pi/2)\cot\gamma)$ real

Similar to previous, real-time <E> of the final wavepacket vanishes

Simplest (space)time-dependent modification:
μ=>μ(t,r)

other modifications quite nontrivial…



This can again arise from certain (rather artificial) weak couplings to slowly-evolving spectators.

EOM:

$$\boxed{\frac{d}{d\lambda}\left[\frac{2R}{\left(\frac{\partial q}{\partial N}\right)}\left(\sqrt{f_{dS_T}+N^{-2}\dot{R}^2}-\sqrt{f_{dS_F}+N^{-2}\dot{R}^2}-4\pi G_N \mu R\right)\right]=-8\pi G_N N R^2\frac{\partial\mu}{\partial t_{dS_F}}}$$

$q \equiv f_{dS_F}^{-1}\sqrt{f_{dS_F}N^2+\dot{R}^2}$ gauge fix N=1 at the end

Can follow same procedure: continue, solve dμ/dt=0 case, compute dμ/dt correction to S

An exact solution

Lorentzian KK cosmology: $$ds^2 = -dy^2 + y^2 d\phi^2 + dt^2 + dx^2 + x^2 d\psi^2$$

If $t \sim t + 2\pi$ this is ordinary KK theory in a funny "Milne"-type coordinate system.

If $\phi \sim \phi + 2\pi$ it is a KK cosmology with the circle shrinking/growing in time y

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The singularity at y=0 is an annoyance which we can regulate by twisting the periodicity:

take $(t, \phi) \sim (t + 2\pi n \frac{\mu}{\sqrt{\mu - a^2}}, \phi + 2\pi n \frac{a}{\sqrt{\mu - a^2}})$ with $0 \leq a^2 \leq \mu$

The minimum circle radius in Milne patch is now $\mu/\sqrt{\mu - a^2}$. Also extend spacetime Milne->Mink (not completely innocuous)

An exact solution

This "vacuum" cosmology is unstable. There is a candidate "decay product":

$$ds^2 = \frac{r^2 + a^2\cosh^2\theta}{r^2 + a^2 - \mu} dr^2 + dt^2 - \frac{\mu}{r^2 + a^2\cosh^2\theta}(dt - a\sinh^2\theta d\phi)^2$$
$$+r^2\left(-\left[1 + \frac{a^2\cosh^2\theta}{r^2}\right] d\theta^2 + \sinh^2\theta\left[1 + \frac{a^2}{r^2}\right] d\phi^2 + \cosh^2\theta d\psi^2\right)$$

This is a real Lorentzian manifold given by the continuation $t \to it, \theta \to i\theta, \phi \to i\phi$ of the 5D Myers-Perry black hole with one angular momentum ($a = J/M, \mu = r_S$)

Geometry caps off smoothly at r = $r_H = \sqrt{\mu - a^2}$: a bubble of nothing

Asymptotics match the vacuum spacetime+periodicities ($\theta \to \tanh^{-1}(y/x),\ r \to \sqrt{x^2 - y^2}$)

The induced metric on the bubble wall is that of a spheroid which expands in time

The “instanton” (MP: $dt \to idt$) is quasi-Euclidean because MP is stationary rather than static,

$$dtd\phi \to idtd\phi$$

It can be joined smoothly to the nucleated bubble on a hypersurface of zero momentum

$S = \dfrac{\pi^2\mu^2}{G_5\sqrt{\mu - a^2}}$ recovers Witten’s result for a=0, diverges in the extremal limit where the minimum KK circle size diverges

Since the background is time dependent, there is no time translation symmetry and the instanton computes a probability rather than a rate.

Questions:

application to time-dependent Schwinger process?

generalization to multiple collective coordinate QM, full QFT?

membrane approach to other cases with dynamical gravity?

other exact instantons from other MP/black ring continuations?