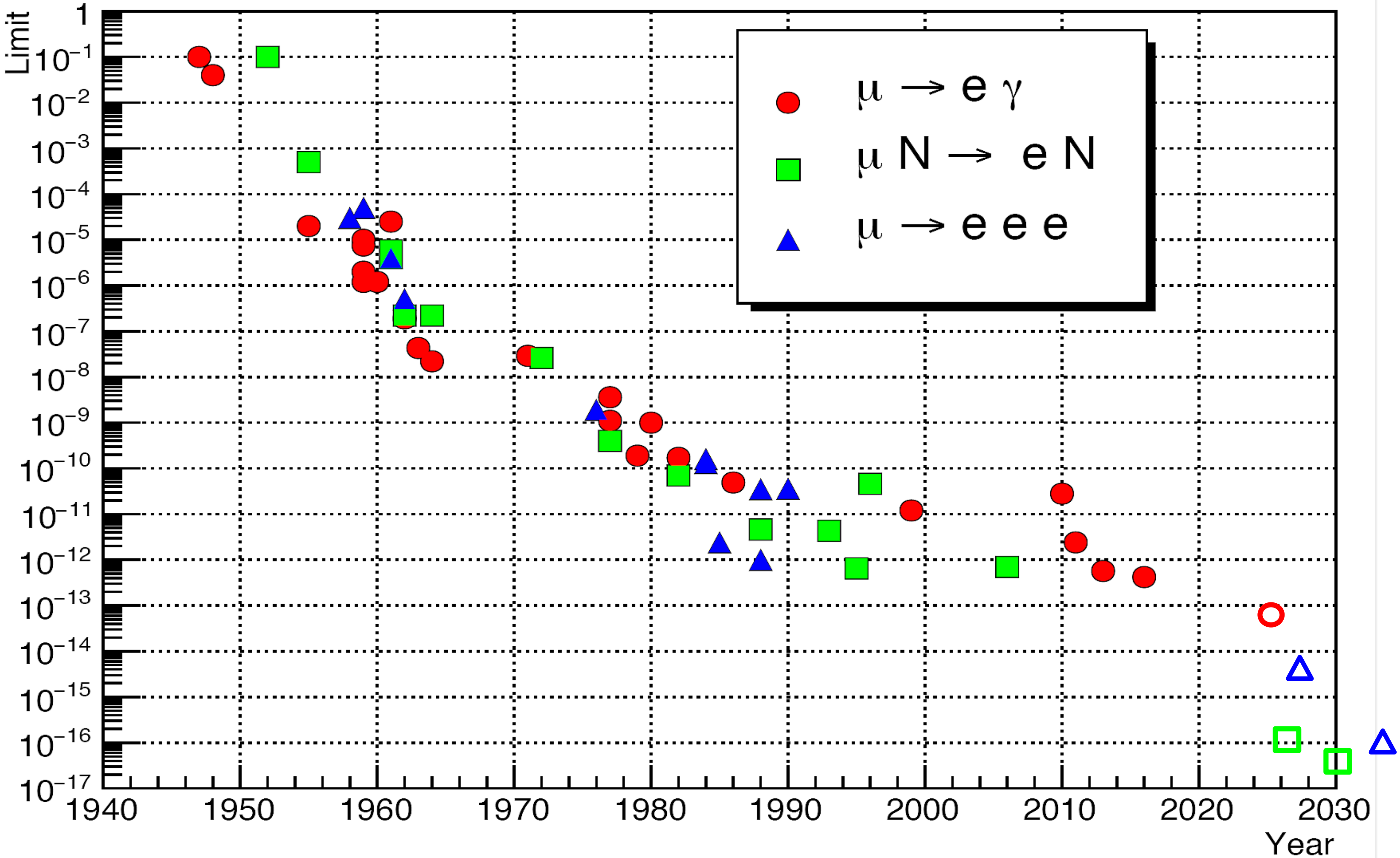
Normalization of $\mu \rightarrow e$ conversion measurements

Compilation of Calibbi and Signorelli +

We anticipate many exciting new CLFV results in the near future

Two μ^+ decay branching fractions plus μ^- to e^- nuclear conversion



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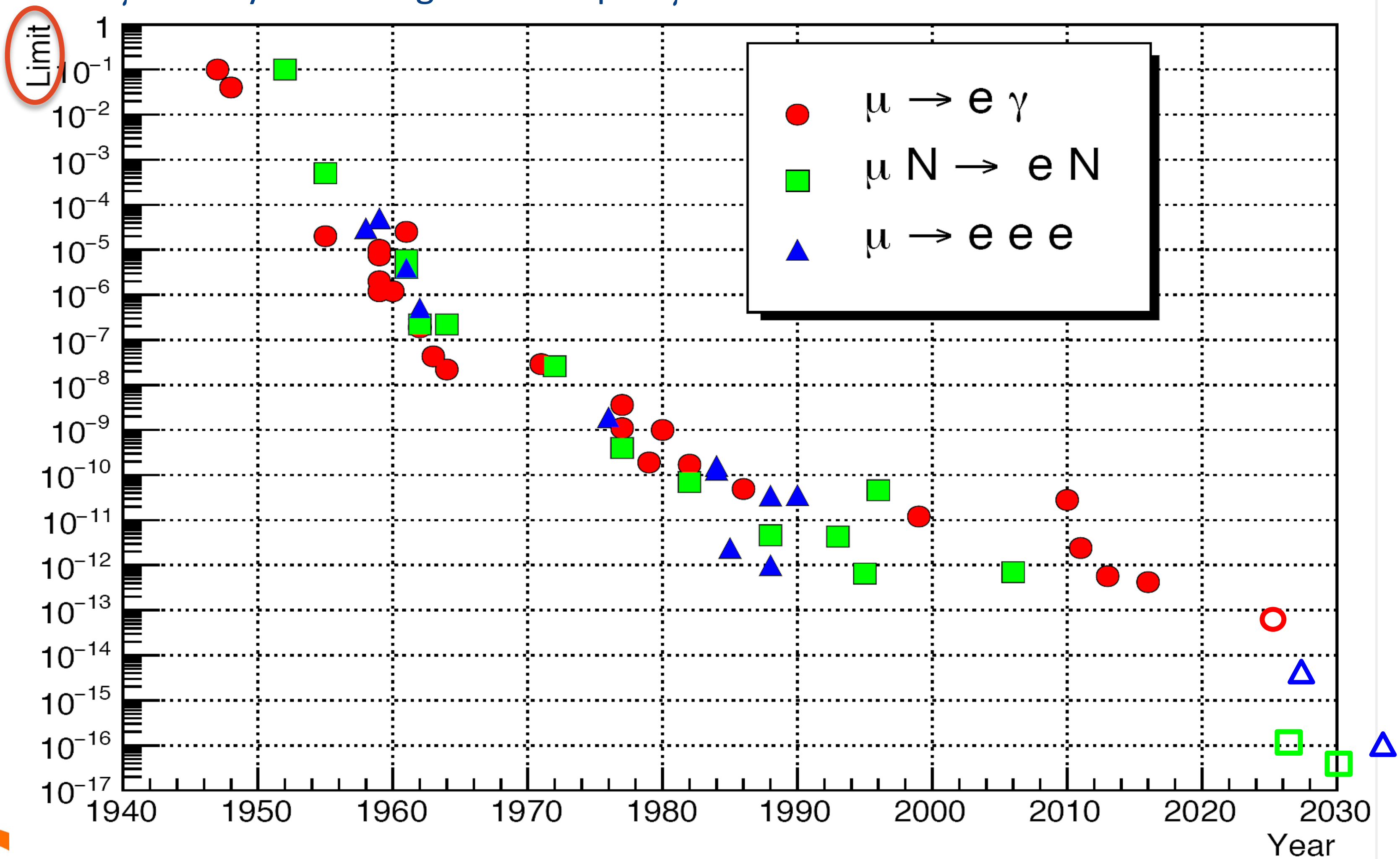


90% CL limits on CLFV processes

For particle decay rates, we quote a dimensionless "branching ratio" or "branching fraction"

$$\mu \to e\gamma$$

$$B(\mu^+ \to \mu^+\gamma) = \frac{\Gamma(\mu \to e\gamma)}{\Gamma(\mu^+ \to e^+\nu\bar{\nu})} \text{ where } \Gamma(\mu^+ \to e^+\nu\bar{\nu}) = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

$$\mu \to eee$$

$$B(\mu^+ \to e^+e^-e^+) = \frac{\Gamma(\mu^+ \to e^+e^-e^+)}{\Gamma(\mu^+ \to e^+\nu\bar{\nu})}$$

For $\mu \to e$ conversion, we quote the conversion rate (sec^{-1}) relative to muon capture into all final states to derive a dimensionless quantity

$$\mu^- N \to e^- N$$

$$R_{\mu e} \equiv \frac{\Gamma(\mu^- + N(A,Z) \to e^- + N(A,Z))}{\Gamma(\mu^- + N(A,Z) \to \text{ all captures})}$$

Normalization of $\mu \to e$ conversion results - history

- The original method of normalizing the $\mu \to e$ conversion rate to ordinary muon capture can be traced to S. Weinberg and G. Feinberg, *Phys. Rev. Lett.* **3**, 111 (1959)

$$R_{\mu e} \equiv \frac{\Gamma\left(\mu^- + N(A,Z) \to e^- + N(A,Z)\right)}{\Gamma\left(\mu^- + N(A,Z) \to \text{ all captures}\right)}$$

(sometimes $B_{\mu \to e}(Z)$)

- This choice involves a Standard Model process and a BSM process
  - *n.b.* in 1959, there was no such thing as a Standard Model
  - This approach mixes the nuclear physics into the BSM physics in an unfortunate way
- This choice was also motivated by consultation with an experimentalist:

“ 9. We are indebted to Dr. Juliet Lee-Franzini for a discussion of the relevant experimental problems.”

- Experimental limits have been reported as $R_{\mu e}$ ever since
- There are both theoretical and experimental reasons why **this is not optimal**
  - The actual BSM theoretical calculation is of the absolute **rate** of $\mu \to e$ conversion
  - The experimental measurement is of the **rate** of $\mu \to e$ conversion normalized to muon stops in the target
  - Normalizing the rate to muon capture to produce a “quasi-branching fraction” mixes a coherent BSM numerator with an incoherent SM denominator
    - this has unfortunate consequences for the (Z,A) dependence

Normalization

- How does one compare sensitivity of a rare branching fraction with a nuclear conversion?
- I am going to argue that the conventional approach to presenting $\mu \to e$ conversion results (limits, or some day, actual measurements), is less than optimal
- I am going to say many obvious things, but this is necessary to construct the argument, so I hope you won't be bored
- To date, in the absence of an observation, the conventional approach of normalizing $\mu \to e$ conversion to μ capture has been serviceable, but should Mu2e or COMET make an observation and then turn to a determination of the Lorentz structure of the New Physics via experiments on different elements, this may matter
  - Such a Z dependence comparison of different couplings requires discernment of 5 to 10% differences
- The classic approach of ascertaining the Z,A dependence of elastic conversion (*c.f.* Kitano *et al*. or Cirigliano *et al.*) can be sharpened a bit by a different approach to normalization
- This requires the best possible modeling of the nuclear physics aspect of the calculation of the conversion rate, which Léo Borrel has just discussed, and a revised take on how experimental results are presented

R. Kitano, M. Koike and Y. Okada, *Phys. Rev.* **D66**, 096002 (2002)
V. Cirigliano, R. Kitano, Y. Okada and P. Tuzon *Phys. Rev.* **D80**, 013002 (2009)

Calculating the measured conversion rate

- **Theory** calculates $\Gamma\left(\mu^- N(A,Z) \to e^- + N(A,Z)\right)$ and then presents the result as

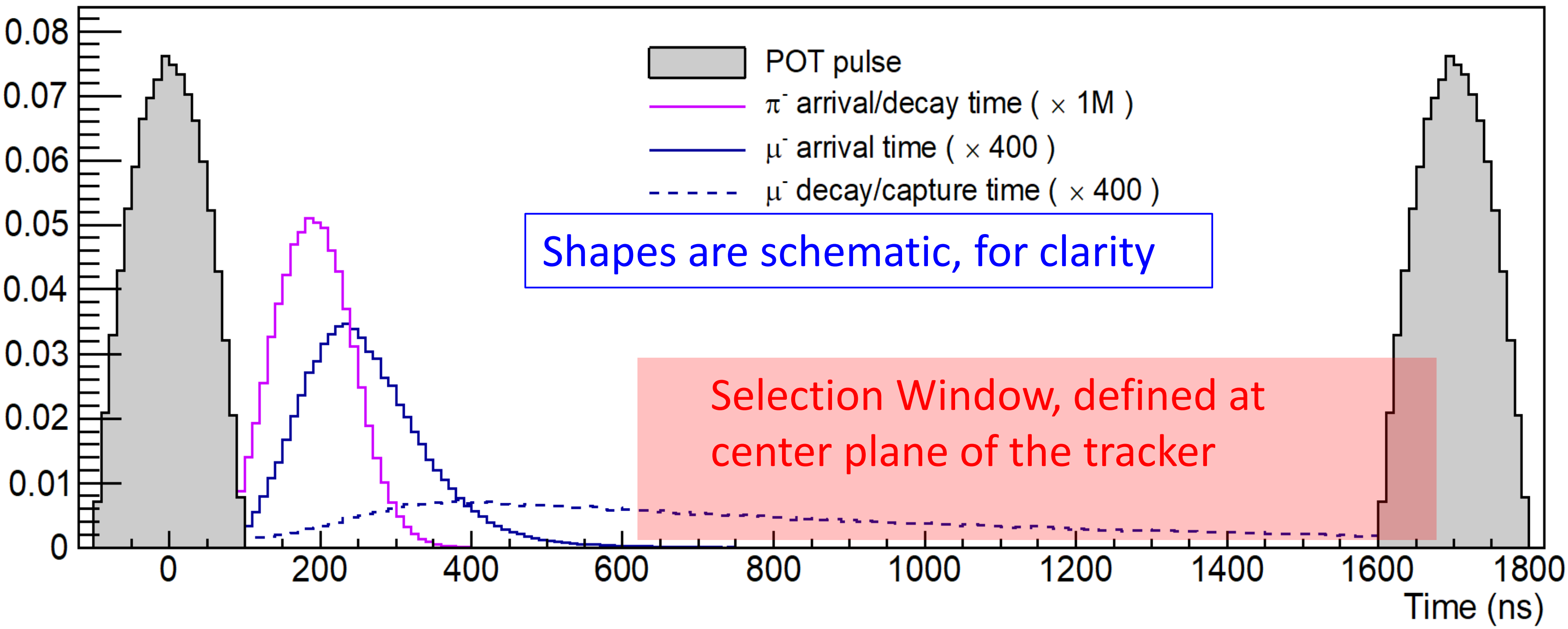
$$R_{\mu e} \equiv \frac{\Gamma\left(\mu^- N(A,Z) \to e^- + N(A,Z)\right)}{\Gamma\left(\mu^- N(A,Z) \to \text{ all captures}\right)} = \frac{\Gamma\left(\mu^- N(A,Z) \to e^- + N(A,Z)\right)}{\omega_{\text{capture}}}$$

  à la Weinberg and Feinberg

- **Experiments** use the muon lifetime (τ_{DIO} + ω_{capture}) (864 ns for aluminum) to count the number of muon stops in the target as the denominator
- Theory and experiment have different objectives
- The historical theory choice of dividing by ω_{capture} minimizes the uncertainty of the muon-nucleus overlap integrals with the proton and neutron distributions, but the measured capture rate involves both coherent and incoherent capture process
- The experimental choice calculates the effective live time in order to properly count muon stops. Then, knowing the net efficiency, we can calculate Γ.
- Let's look more closely at ω_{capture}

The pulsed beam

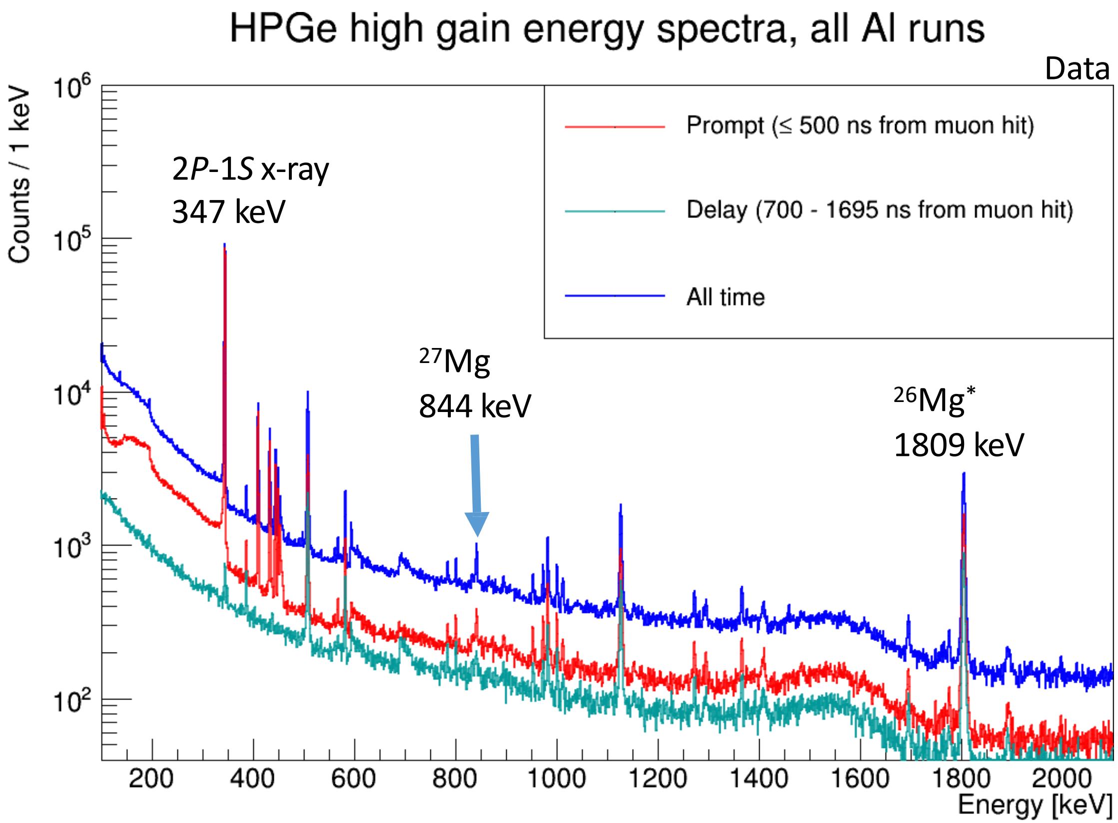
Pulsed proton beam based on muonic aluminum lifetime of 864 ns



Normalization of conversion in Al

- The physics quantity we seek is $R_{\mu e} = \frac{\Gamma(\mu^- + N \rightarrow e^- + N)}{\Gamma(\mu^- + N \rightarrow \text{all captures})}$
- The numerator is our electron signal
- We do not generally directly measure the muon capture rate in a conversion search
- The denominator is measured indirectly
  - Lifetime of the muon – decay or capture on N $\quad \frac{1}{\Gamma} = \frac{1}{\Gamma_{\text{decay}}} + \frac{1}{\Gamma\text{capture}}$
- The lifetime of the muonic atom and the muon capture rate on many nuclei are well-known Review by D. Measday, Phys. Rep. 35, 243 (2001)
- The stopping target for both Mu2e and COMET Phase I is aluminum: ${}^{27}_{13}Al$ which is essentially 100% of stable isotopes (foil or screen targets may contain small amounts of other elements)
- There are three clear γ signals produced by μ^- stopping in Al
  - Measure the rate of x-rays from muonic atoms (prompt after a muon stop)
    - **347 keV** 2*P*-1*S* transition muonic atom in Al, 79.8(8)% per muon stop
    - Need good timing to estimate number remaining in the live window
  - Measure a γ resulting from muon capture to an excited nuclear state
    - **1809 keV** γ produced immediately in 51(5)% of captures, 31.1% of stops
      $\mu^- + {}^{27}_{13}Al \rightarrow {}^{26}_{12}Mg^* + n + \nu_\mu \qquad {}^{26}_{12}Mg^* \rightarrow {}^{26}_{12}Mg + \gamma(1809)$ confirmed in the AlCap experiment)
  - Measure γ from decay of longer-lived isotopes produced in muon capture
    $\mu^- + {}^{27}_{13}Al \rightarrow {}^{27}_{12}Mg + \nu_\mu \qquad {}^{27}_{12}Mg \rightarrow {}^{27}_{13}Al + \gamma(844) + e^- + \bar{\nu}_e$ (9.5 minute half-life)
    - **844 keV** γ 9.2(1.5)% of captures, 5.7% of stops

AlCap HPGe Photon Data



The experimental goal is to ascertain the number of muons stopped in the Al target
Mu2e also uses a HPGe detector (STM) at the rear of the detector hall

Normalization of $\mu \to e$ conversion results

- The issue comes down to the method of normalization
  - There is an approach to presenting the results that clarifies the physics and minimizes the nuclear physics complications (Z, A) dependence, coherent *vs* incoherent
  - This also facilitates the comparison to $\mu \to e\gamma$ and $\mu \to eee$ measurements that are manifestly reported as decay branching fractions
- We actually measure the number of conversion electron candidates for a given number of muons stopped in the Al target in our live window
  - This requires knowledge of the muon lifetime in the Al atom

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TABLE IV. Compendium of total muon capture results for medium and heavy nuclei. (Z_{eff} is taken from Ref. 77. When it is underlined, it is an estimate. Entries in parentheses in column 4 are not given in the original reference.)

Z (Z_{eff})	Element	Mean life (ns)	Total capture rate (10^6/s)	Huff factor	Refs.
13 (11.48)	Al	880 $\pm$10	0.691 $\pm$0.020	0.993	40
		864 $\pm$2	0.662 $\pm$0.003		42
		905 $\pm$12	0.650 $\pm$0.015		45
		864.0$\pm$1.0	0.7054$\pm$0.0013		a

  - The muon mean life and the total capture rate are, of course, related, but it is the muon lifetime that is relevant for our measurement

Coherent *vs* incoherent processes

- The "Conversion Rate" à la Weinberg and Feinberg $CR = \frac{\Gamma(\mu \to e \text{ conversion})}{\Gamma(\text{nuclear capture})}$

  yields the fraction of all nuclear encounters that result in conversion

- However, exclusive $\mu \to e$ conversion is a coherent process over the nucleus while μ capture $\mu^- + p \to \nu_\mu + n$ in a nuclear environment is an incoherent process involving excitation of the residual nucleus via giant dipole excitation, multi-neutron production, fission, …

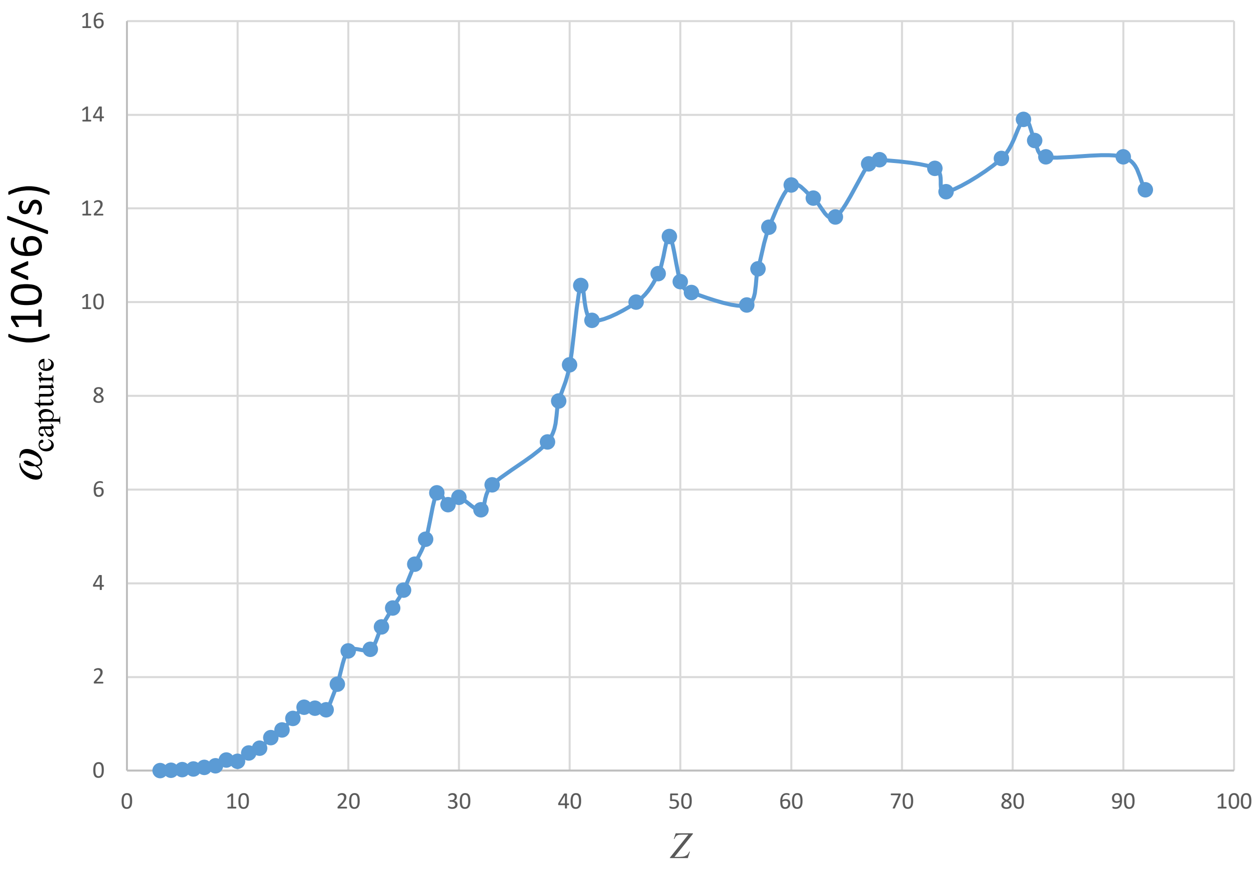
Reaction	Observed γ-ray yield	Estimated ground-state transition	Missing yields	Total yield
$^{27}\mathrm{Al}(\mu^-, \nu)^{27}\mathrm{Mg}$	10(1)	0	3	13
$^{27}\mathrm{Al}(\mu^-, \nu n)^{26}\mathrm{Mg}$	53(5)	4	4	61
$^{27}\mathrm{Al}(\mu^-, \nu 2n)^{25}\mathrm{Mg}$	7(1)	3	2	12
$^{27}\mathrm{Al}(\mu^-, \nu 3n)^{24}\mathrm{Mg}$	2	3	1	6
$^{27}\mathrm{Al}(\mu^-, \nu p x n)^{26\text{-}23}\mathrm{Na}$	2	2	1	5
$^{27}\mathrm{Al}(\mu^-, \nu \alpha x n)^{23\text{-}21}\mathrm{Ne}$	1	2	0	3
Total	75(5)	14	11	100

Measday, Stocki, Moftah and Tam
Phys. Rev. C**76**, 035504 (2007)

- Thus the calculation of the μ capture rate involves matrix elements involving transitions to Mg, Na and Ne which have nothing to do with $\mu \to e$ conversion

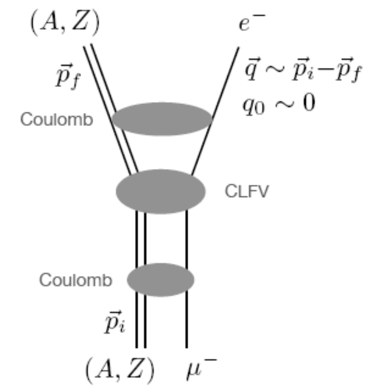
- It is true that normalizing to ω_{capture} yields the fraction of the New Physics over all the things that the μ does in interacting with the nucleus, but this has the effect of mixing complex, and irrelevant, nuclear processes into a study of Z dependence

Measured μ capture rates



An EFT for $\mu \rightarrow e$ conversion

- Haxton, Rule, McElvain and Ramsey-Musolf *et al.* (e-print: 2208.07945[nucl-th]) have formulated a detailed EFT for a variety of potential conversion targets with a nucleon level description of CLFV
- They employ the term **elastic $\mu \rightarrow e$ conversion**, which is what experiments are sensitive to, since the resulting monoenergetic electron is the experimental signature
- The elastic channel picks out particular CLFV operators via *P* and *CP* selection rules
  - Energy transfer to the nucleus is negligible and the three-momentum transfer scale $q \sim m_\mu$ is comparable to the inverse nuclear size
  - This is comparable to the situation in direct detection WIMP dark matter

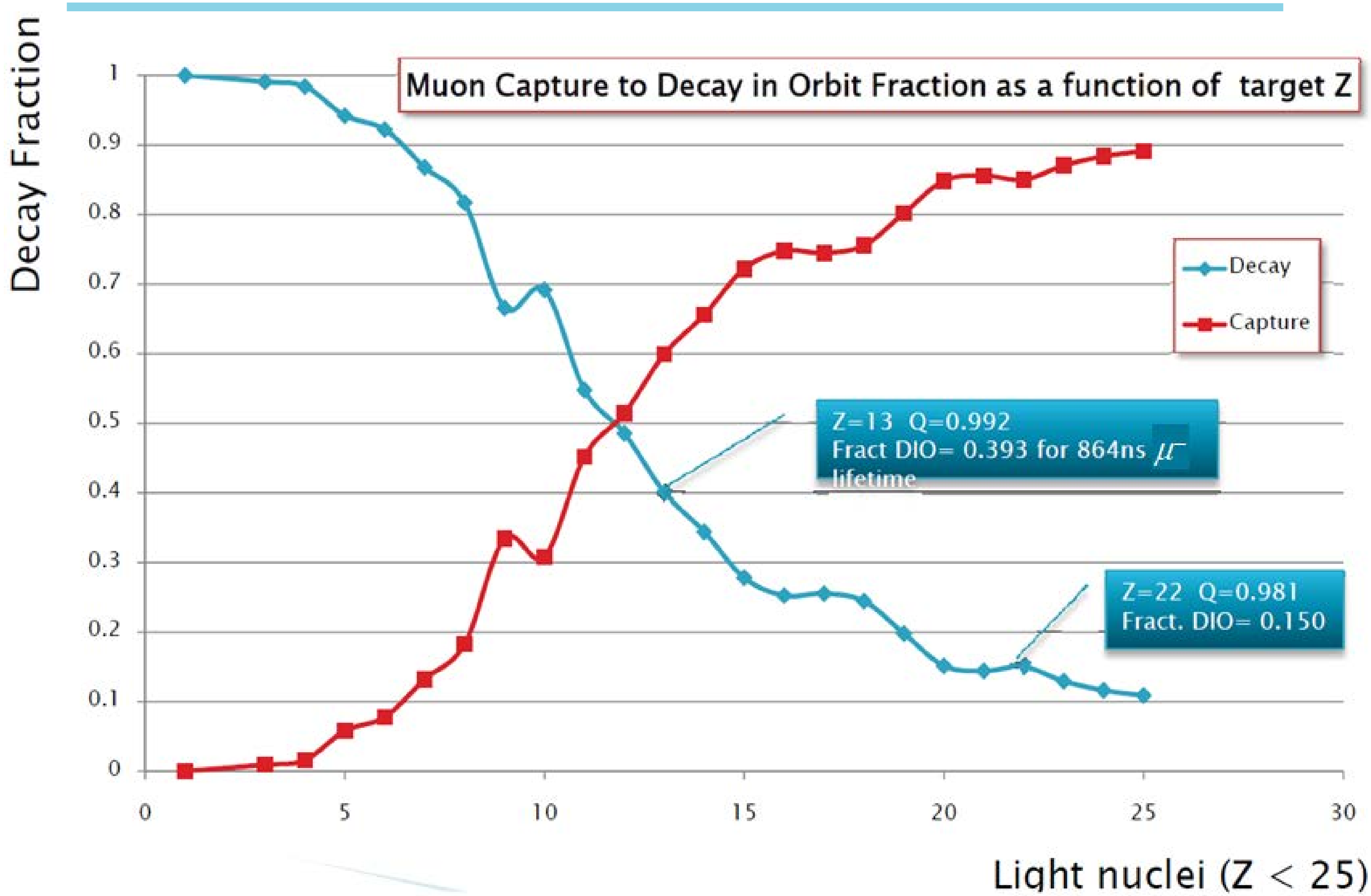


- This is emphatically not the case in nuclear muon capture
  - In my opinion the classical approach to normalization is thus calculating apples over oranges
- It makes more sense to quote as an experimental result the measured conversion rate, which is in fact what we measure and what theorists actually calculate
- This convention also helps to clarify the Z,A dependence of the conversion rate by removing as much as possible of the incoherent nuclear physics

What's happenin'

- Note that the leptonic vertex in $\mu \rightarrow e$ conversion is **inclusive** (a sum over all partial waves), while the nucleon vertex is **exclusive**, selecting particular operators (intermediate and final-state) for particular final states
- The spin of the candidate nucleus acts as a filter for CLFV operators
  - ***n.b.*** Spin independent and spin dependent couplings have been studied for several years
- To calculate the conversion rate we need to know how many muons are candidates for conversion
  - That is, at any given time in our live window, how many muons remain
    - This is given by the **mean lifetime** $\Lambda_{\mathrm{T}} = \Lambda_{\mathrm{C}} + Q\Lambda_{\mathrm{D}}$ not by the muon capture rate
      - The stopped muon population is depleted as a function of time by both DIO and nuclear capture
      - The use of the mean muon lifetime in an Al muonic atom (864 ns) just allow to count surviving muons. It does not deal with the fate of the muons

DIO vs nuclear capture fraction $\Lambda_T = \Lambda_C + Q\Lambda_D$



A revised normalization convention

- **A new convention**
  Present both experimental and theoretical results as the "branching fraction" or "conversion fraction" of $\mu \to e$ conversion relative to the free muon decay rate, as with $\mu \to e\gamma$ and $\mu \to eee$

$$
\begin{aligned}
\Gamma(\mu^- + N(Z,A) \to e^- + N(Z,A)) = \; & 2G_{\mathrm{F}}^2 \left| A_R^* D + \tilde{g}_{LS}^{(p)} S^{(p)} + \tilde{g}_{LS}^{(n)} S^{(n)} \right. \\
& \left. + \tilde{g}_{LV}^{(p)} V^{(p)} + \tilde{g}_{LV}^{(n)} V^{(n)} \right|^2 + 2G_{\mathrm{F}}^2 \left| A_L^* D + \tilde{g}_{RS}^{(p)} S^{(p)} \right. \\
& \left. + \tilde{g}_{RS}^{(n)} S^{(n)} + \tilde{g}_{RV}^{(p)} V^{(p)} + \tilde{g}_{RV}^{(n)} V^{(n)} \right|^2
\end{aligned}
$$

$$
\Gamma(\mu^- \to e^- \nu \overline{\nu}) = \frac{G_F^2 m_\mu^5}{192\pi^3}
$$

$$
BR \text{ (or } CR)(\mu \to e(A,Z)) = \frac{\Gamma(\mu^- + N(Z,A) \to e^- + N(Z,A))}{\Gamma(\mu^- \to e^- \nu \overline{\nu})}
$$

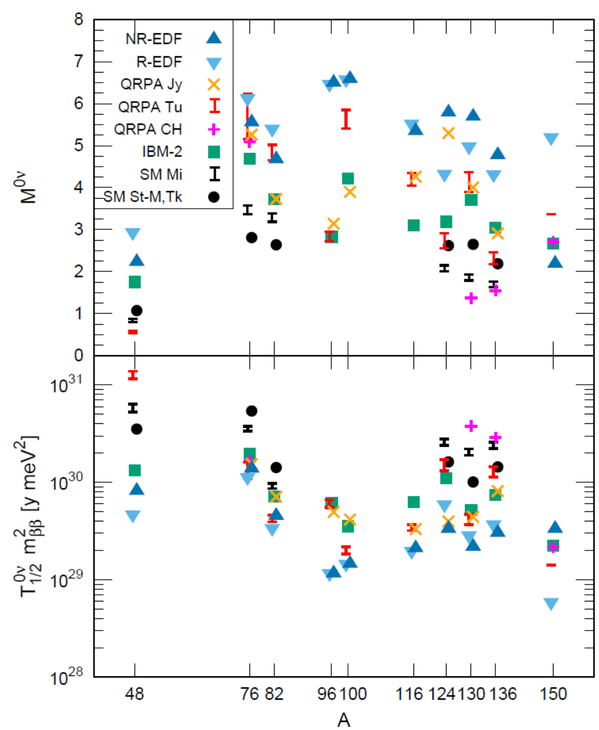
- In practice, use most comprehensive EFT for $\Gamma(\mu^- + N(Z,A) \to e^- + N(Z,A))$

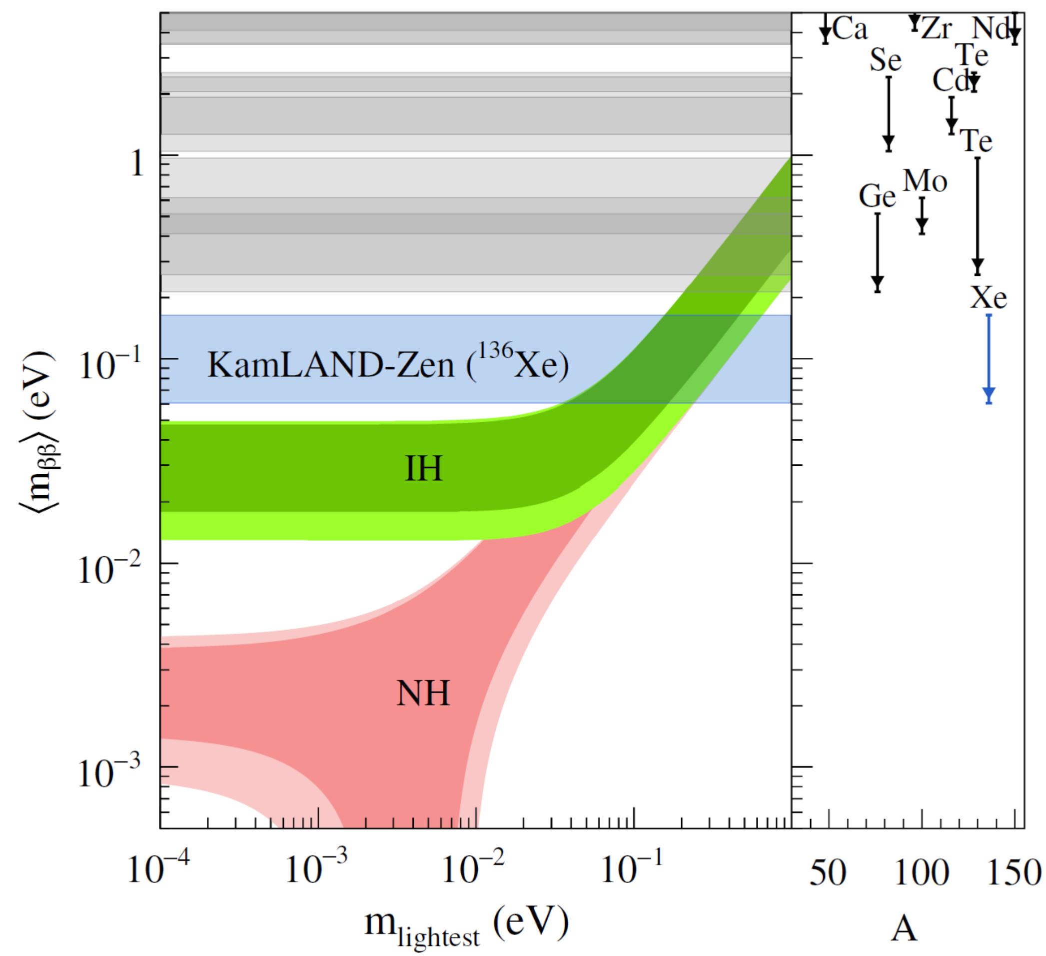
Presentation of $\beta\beta$ results

- $\beta\beta$ compilations separate the New Physics from the nuclear physics

$$\frac{1}{T_{1/2}^{0\nu}} = G^{0\nu}(Q,Z)\left|M^{0\nu}\right|^2 \left|\langle m_{\beta\beta}\rangle\right|^2$$

$$\langle m_{\beta\beta}\rangle = \sum_i^N \left|U_{ei}\right|^2 e^{i\alpha_i} m_i$$





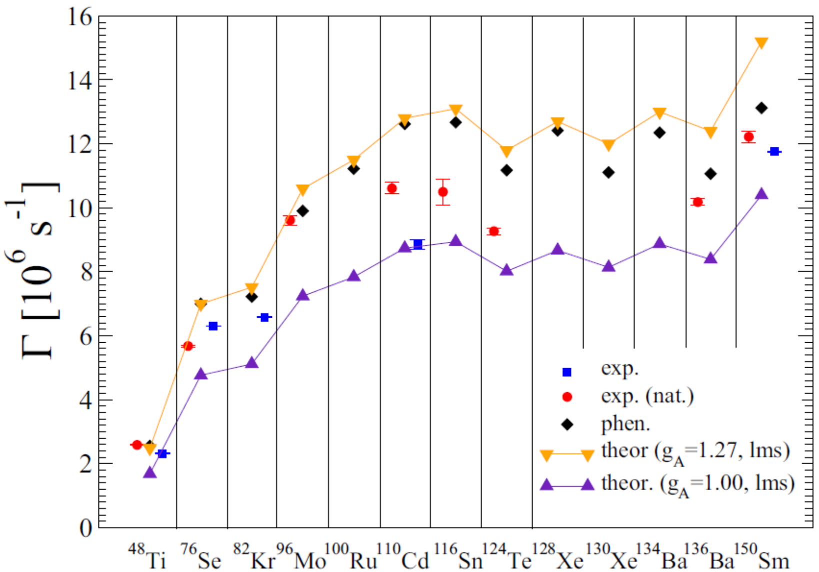
Calculating the muon capture rate

- Interesting in itself for shedding light on a variety of nuclear physics questions, as well as overlap of the calculation with nuclear physics of $\beta\beta$ decay
  - The traditional approach is to factorize the muon and nuclear elements (*c.f. F.* Šimovic, R. Dvornický and P. Vogel, *Phys. Rev*, **C102** 034301 (2020))

$$\Gamma = m_\mu \frac{(G_\beta m_\mu^2)^2}{2\pi}(C_V B_{\Phi V} + C_A B_{\phi A} + C_P B_{\phi P}) \qquad B_{\Phi K} = \sum_k \frac{E_{\nu_k}^2}{m_\mu^2} B_{\Phi K}^k(p_{\nu_k}) \quad K{=}V,\, A,\, P$$

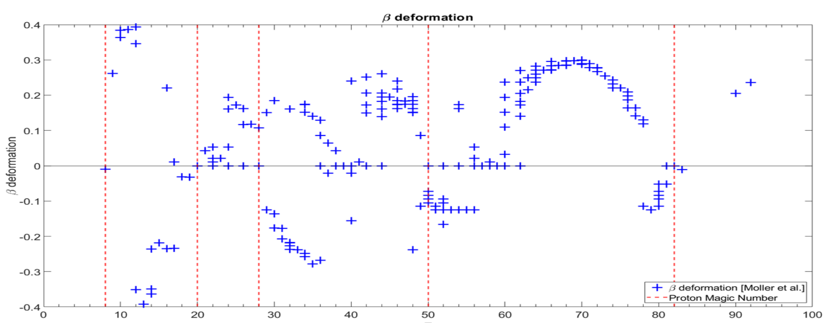
$$B_{\Phi K}^k(p_{\nu_k}) = \frac{1}{\hat{J}_i}\sum_{M_i M_k}\int \frac{d\Omega_\nu}{4\pi}\left|\langle J_k M_k|\sum_{j=1}^{A}\tau_j^- e^{i\mathbf{p}_{\nu_k}\cdot\mathbf{r}_i} O_K \frac{\Phi_g(r_i)}{m_\mu^{3/2}}|J_i M_i\rangle\right|^2$$

Must evaluate all matrix elements connecting the nuclear ground state to allowed excited states. Matrix elements are usually evaluated using the (heavily model-dependent) Random Phase Approximation

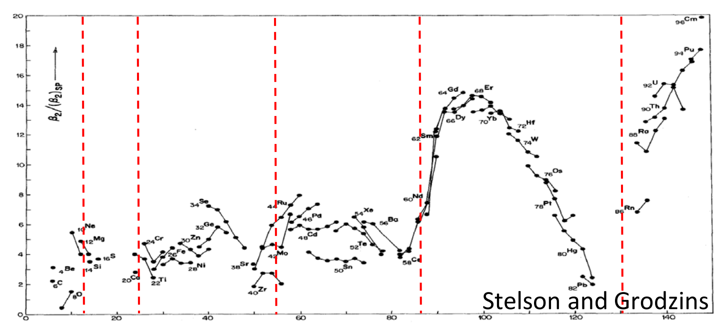


Recapping Léo Borrel's presentation

- In an attempt to better separate the CLFV New Physics from the nuclear physics, we revisited older calculations of the Z,A dependence of $\mu \rightarrow e$ conversion
    - Added charge distributions of a substantial number of new nuclei, many having large quadrupole deformations, measured using muonic x-ray spectra. We combined the muonic x-ray and electron scattering data using Barrett moments and accounting for quadrupole deformation effects on *rms* radii
    - Rather than use neutron distributions obtained by scaling charge distributions by N/Z, we used the Zhang *et al.** compilation that employs deformed relativistic Hartee-Bogoliubov theory
        - Only even-even nuclei (odd-even and even-odd nuclei show deviant isotope behavior due to unpaired spins)



protons

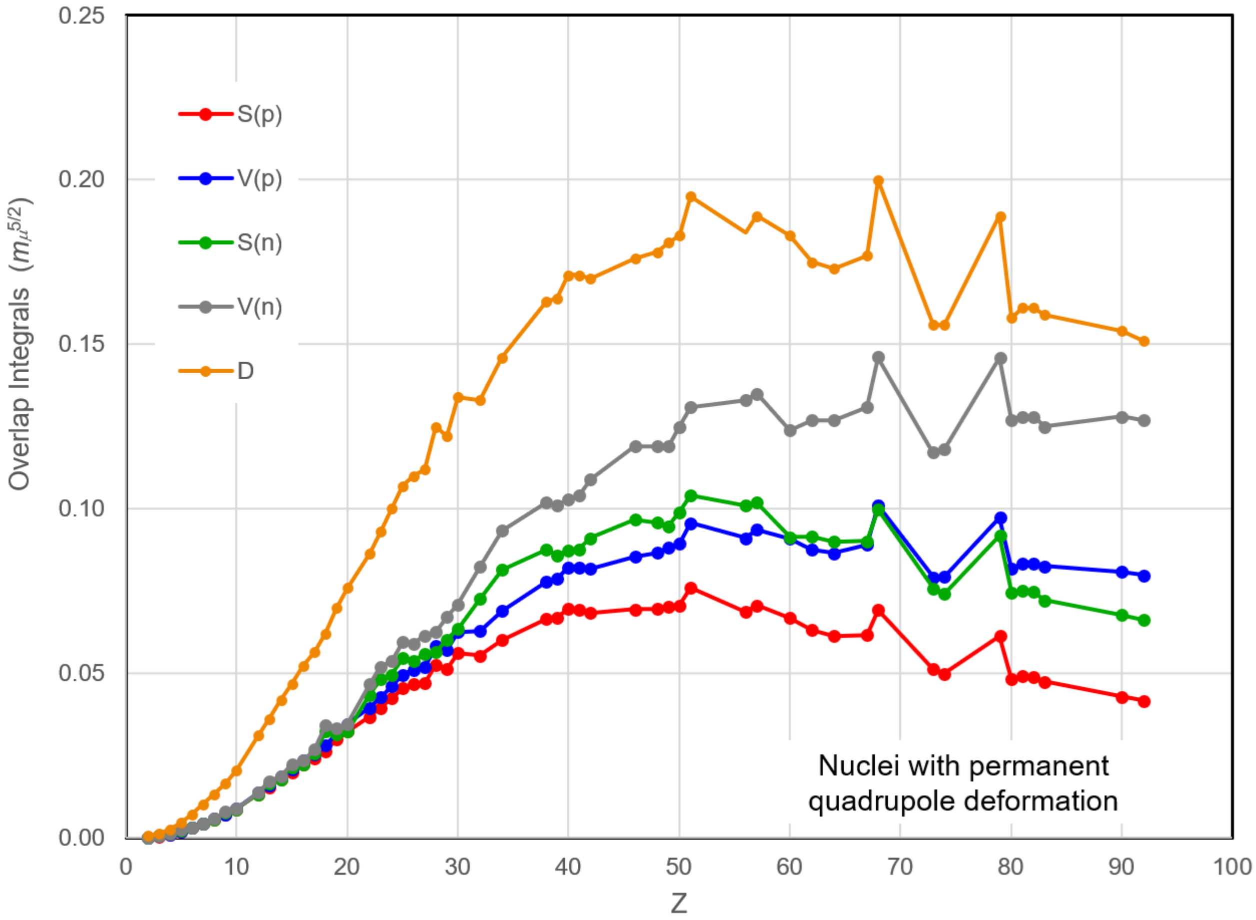


neutrons

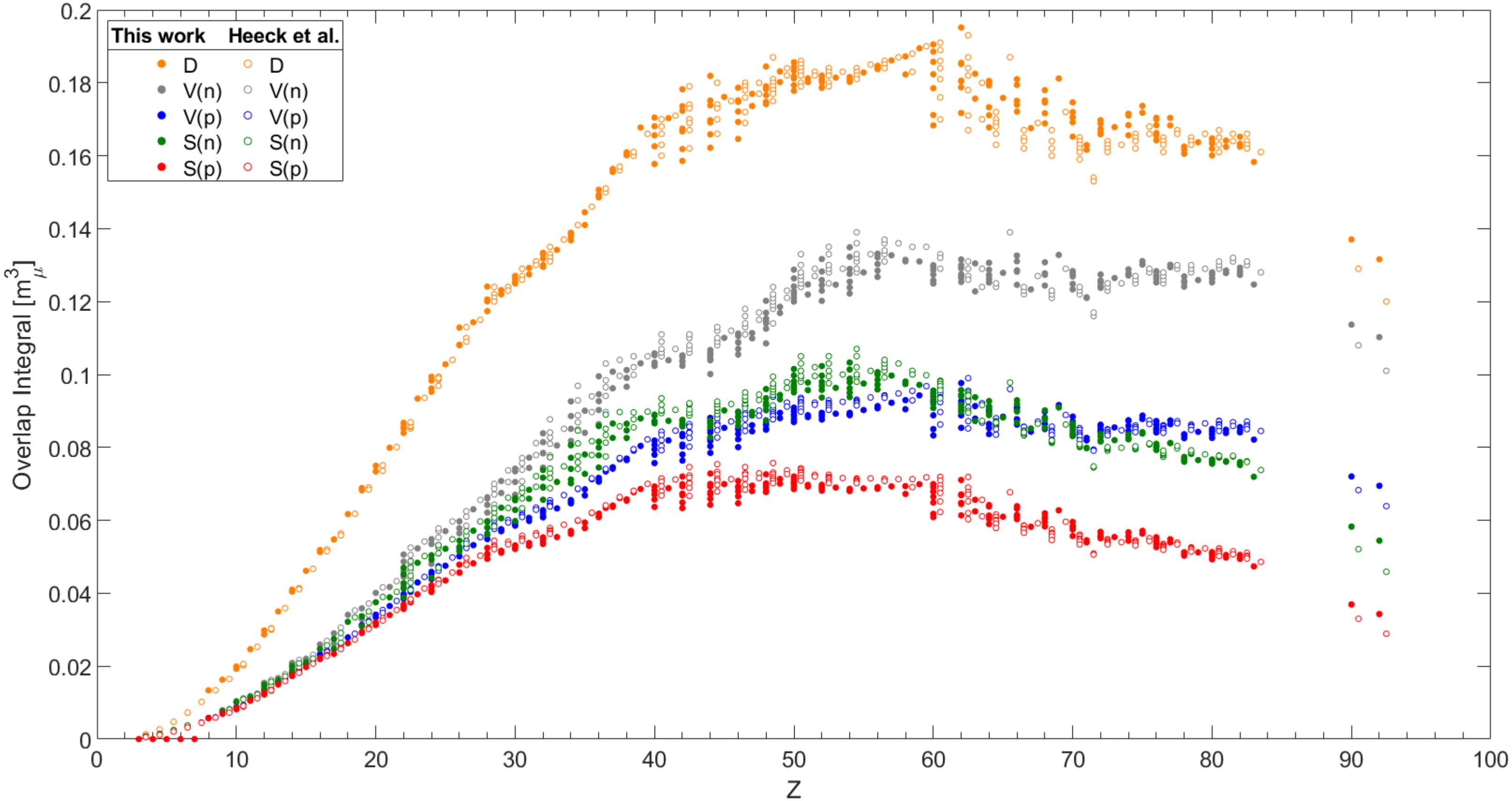
- We then compute the Z, A dependence of conversion in a new way

*K. Zhang et al., DRHBc Mass Table Collaboration, Atomic Data and Nuclear Data Tables 144 101488 (2022)

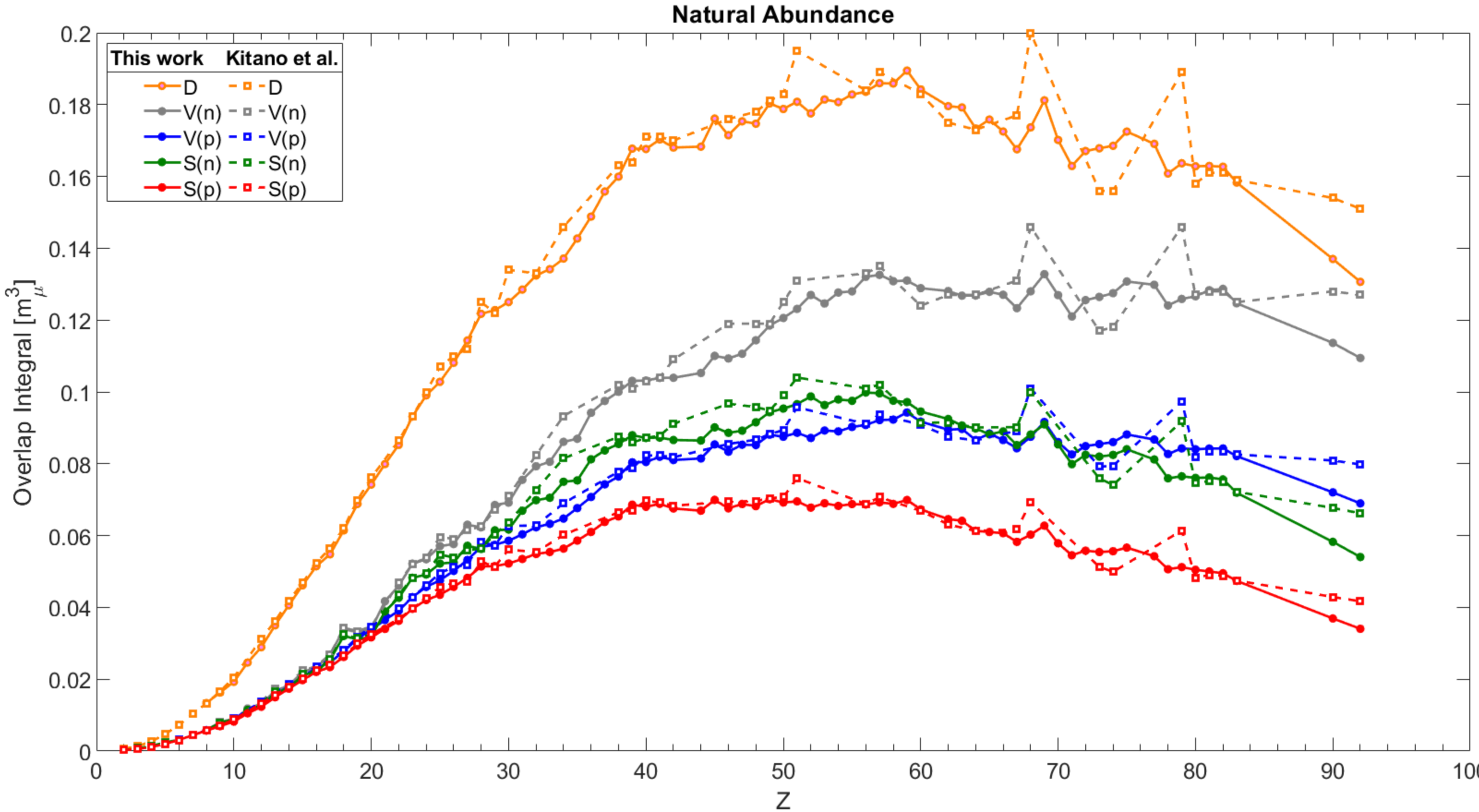
Overlap integrals of Kitano *et al.*



Overlap integrals by isotope (>1% abundance)



Overlap integrals weighted by natural abundance



Z,A dependence comparisons

The conventional normalization

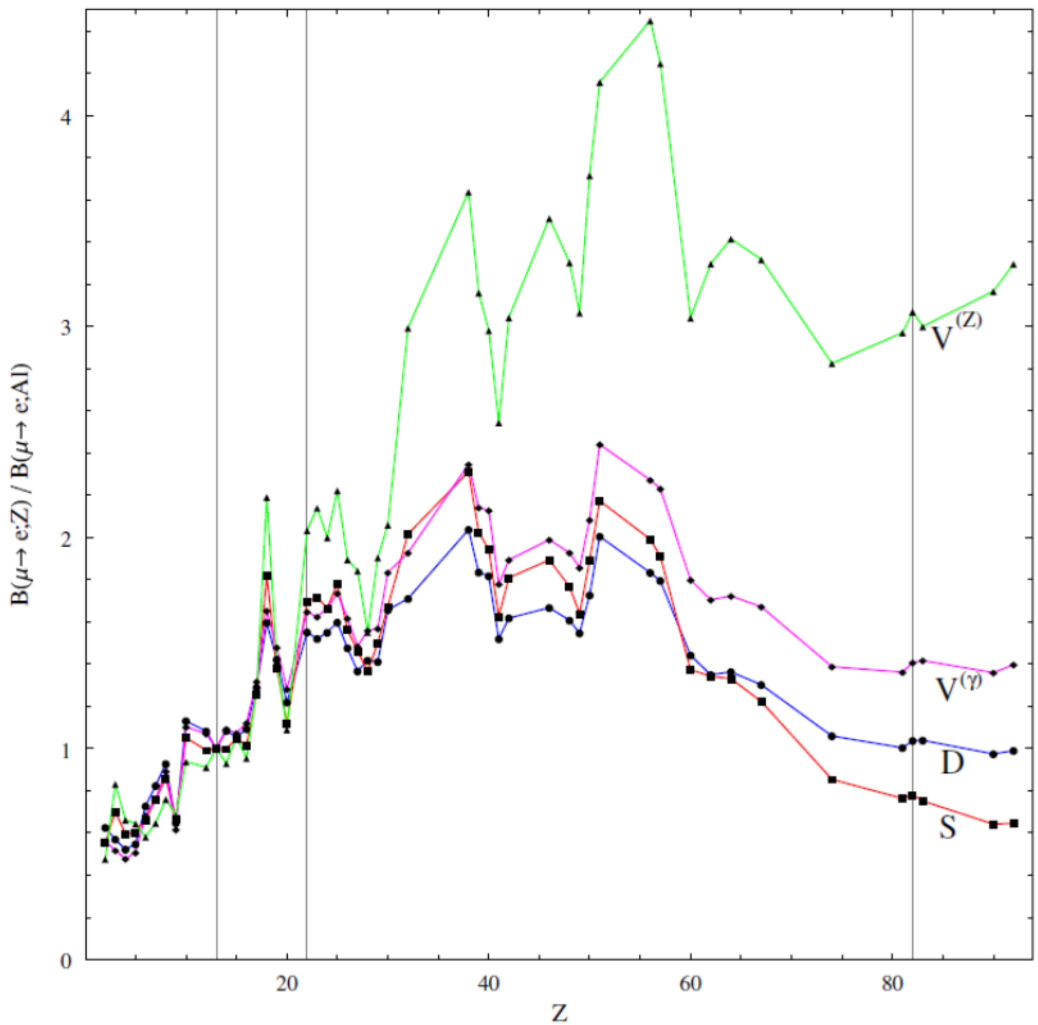
$$\frac{B_{\mu\to e}(Z) \equiv \dfrac{\Gamma_{conversion}(Z,A)}{\Gamma_{capture}(Z,A)}}{B_{\mu\to e}(\mathrm{Al}) \equiv \dfrac{\Gamma_{conversion}(13,27)}{\Gamma_{capture}(13,27)}}$$

The new normalization

$$\frac{B_{\mu\to e}(\mathrm{Z}) \equiv \dfrac{\Gamma_{conversion}(A,Z)}{\Gamma(\mu \to e\nu\overline{\nu})}}{B_{\mu\to e}(\mathrm{Al}) \equiv \dfrac{\Gamma_{conversion}(13,27)}{\Gamma(\mu \to e\nu\overline{\nu})}} = \frac{B_{\mu\to e}(Z) = \Gamma_{conversion}(Z,A)}{B_{\mu\to e}(\mathrm{Al}) \equiv \Gamma_{conversion}(13,27)}$$

Conventional normalization

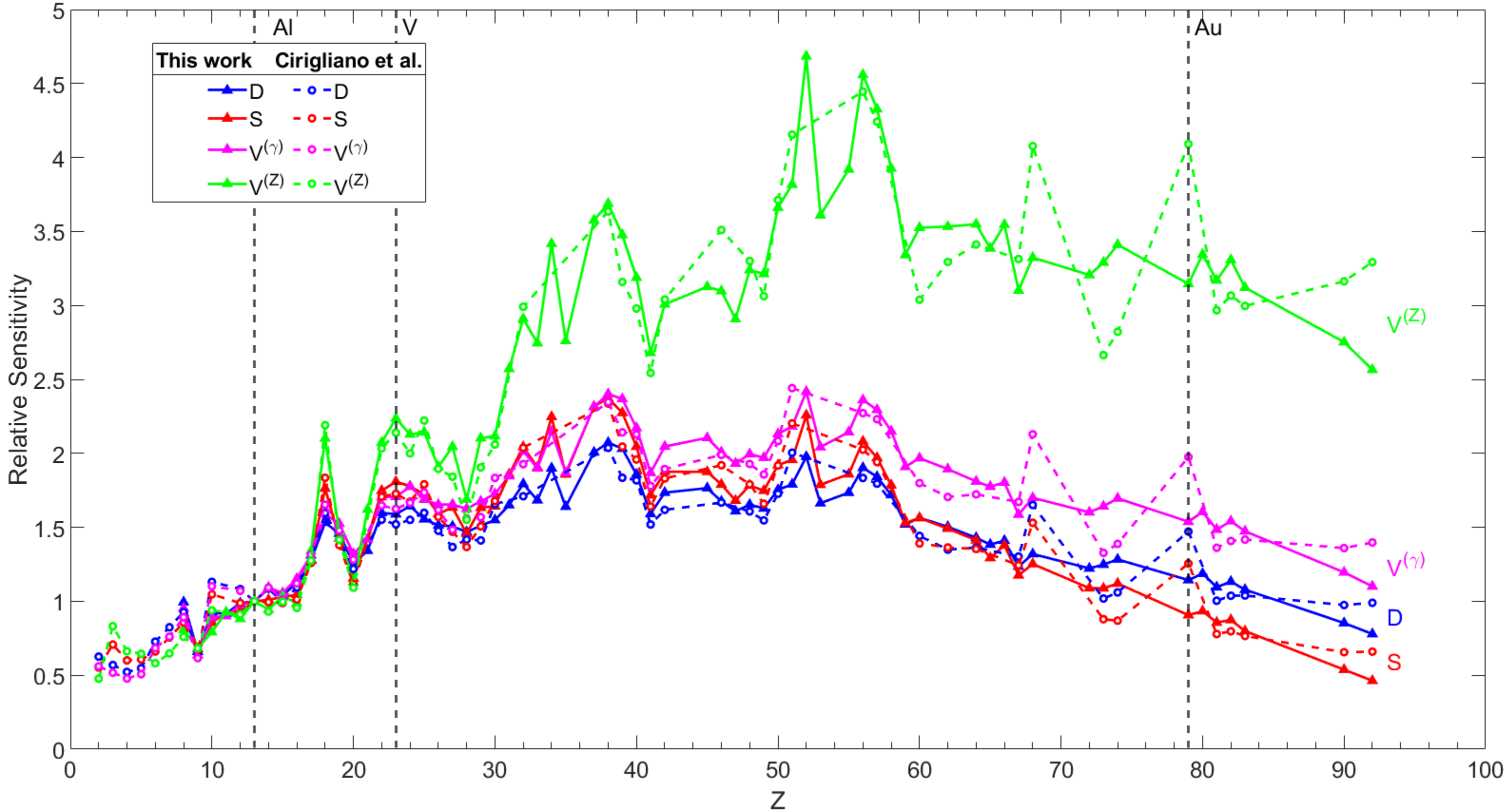
$$\frac{B_{\mu\to e}(Z) \equiv \dfrac{\Gamma_{conversion}(Z,A)}{\Gamma_{capture}(Z,A)}}{B_{\mu\to e}(Al) \equiv \dfrac{\Gamma_{conversion}(13,27)}{\Gamma_{capture}(13,27)}}$$



Cirigliano et *al.*

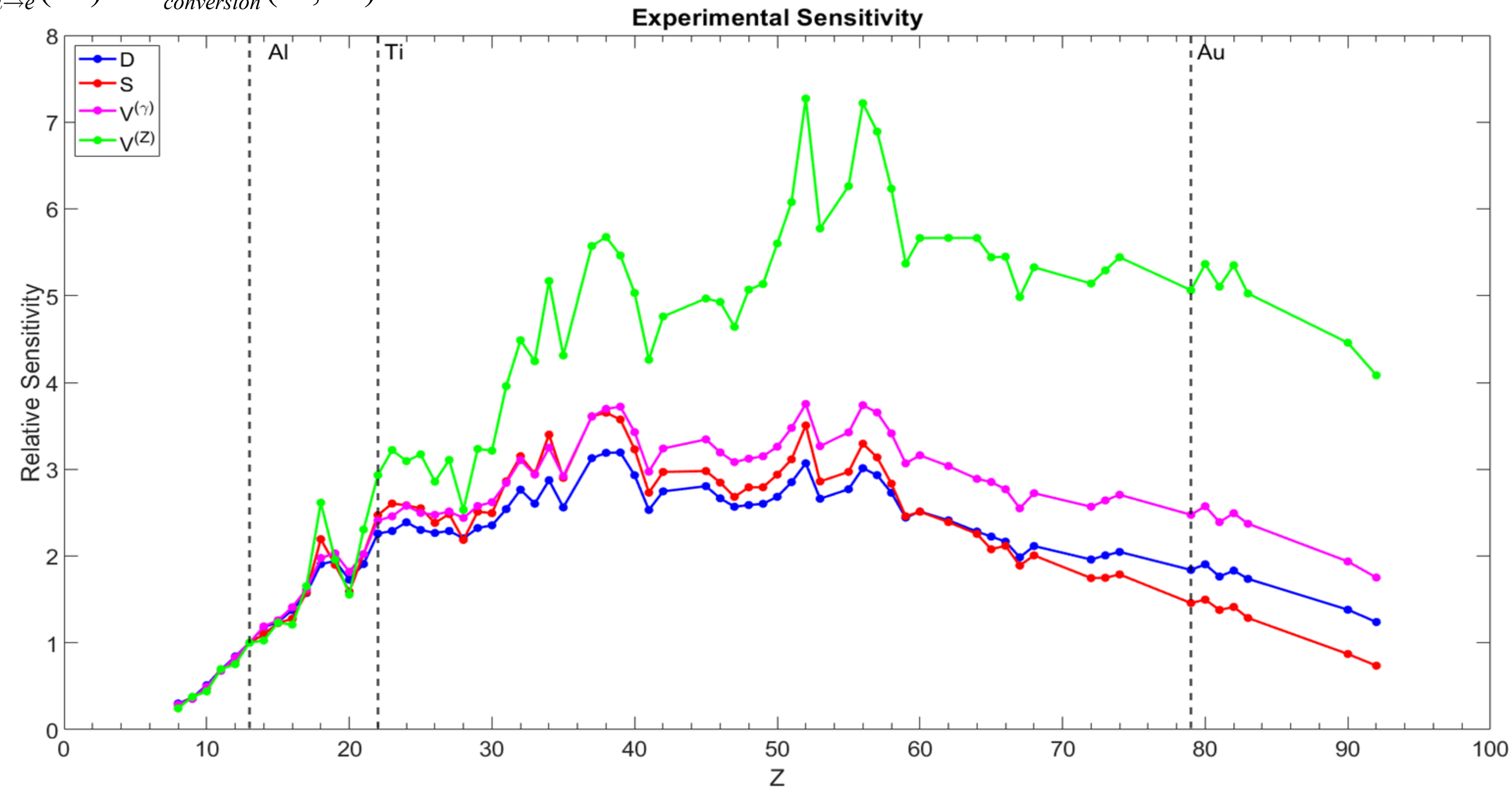
Comparison

$$\frac{B_{\mu\to e}(Z) = \Gamma_{conversion}(Z,A)}{B_{\mu\to e}(\mathrm{Al}) \equiv \Gamma_{conversion}(13,27)}$$



New normalization

$$\frac{B_{\mu\to e}(Z) = \Gamma_{conversion}(Z,A)}{B_{\mu\to e}(\mathrm{Al}) \equiv \Gamma_{conversion}(13,27)}$$



Conclusions

New normalization proposal

Theory ramifications

- Present the results that you actually calculate
- Present either calculated rate (Γ) or normalize to muon decay rate (BR) or (CR)
- Eliminates artificial normalization of a calculated coherent process to a measured incoherent process
- Facilitates comparison of conversion rates to decay rates in various models

Experimental ramifications

- Present the conversion fraction (or limit) normalized to the free muon decay rate, or just the conversion rate
- Normalize the conversion rate by what is actually measured:
  - Determine the number of muon stops in the target using the 2*P*-1*S* muonic x-ray (and/or muon capture γs)
  - Avoid presentation of experimental results divided by a looked-up muon capture rate, which results in extraneous Z,A structure dependence
- Corollary: NP limit comparisons such as the DeGouvea-Vogel or Davidson-Echenard plots should be revised to remove division by μ capture rate (.61 for Al)