3D Theory of the Ion Channel Laser

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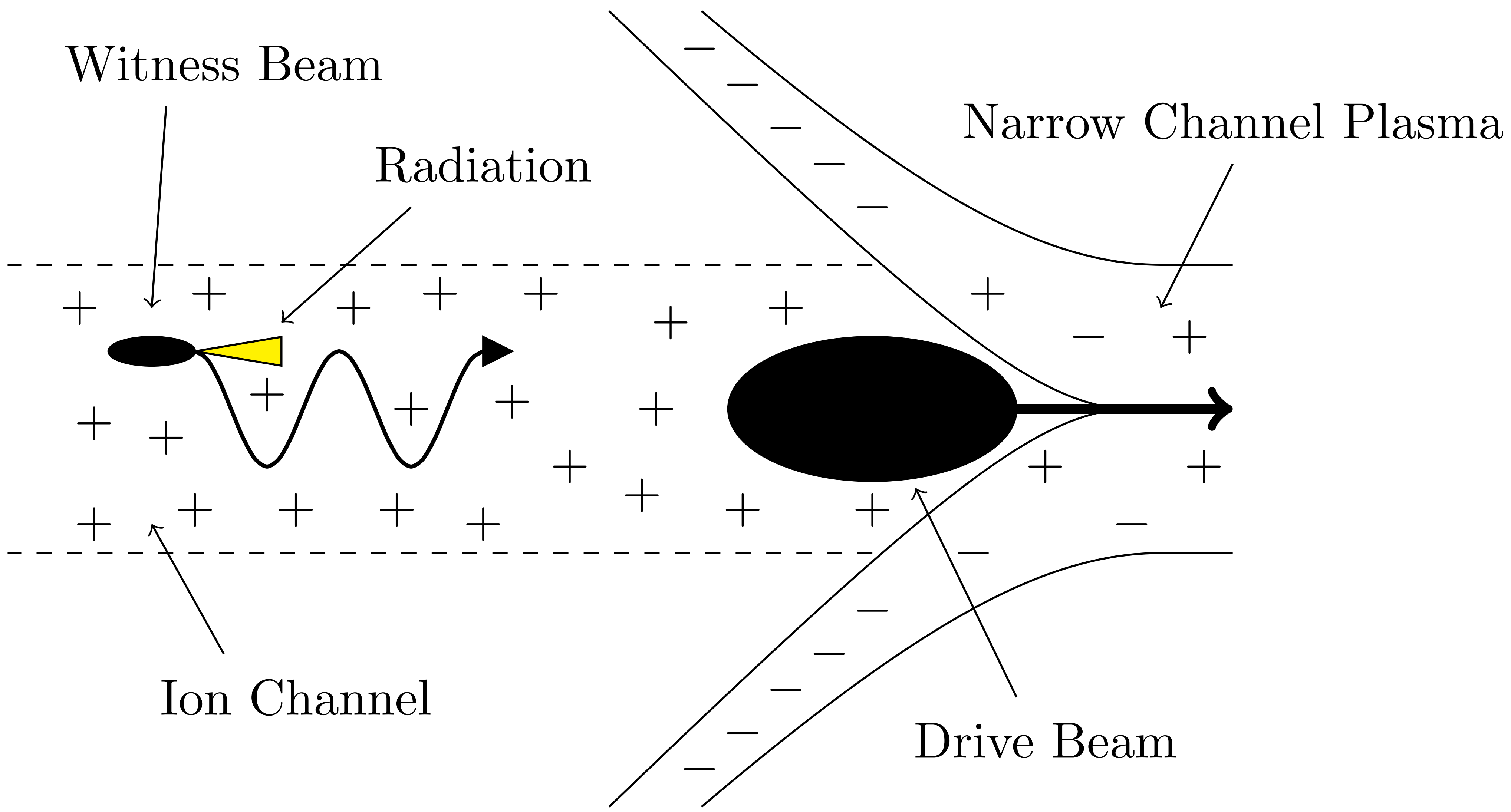
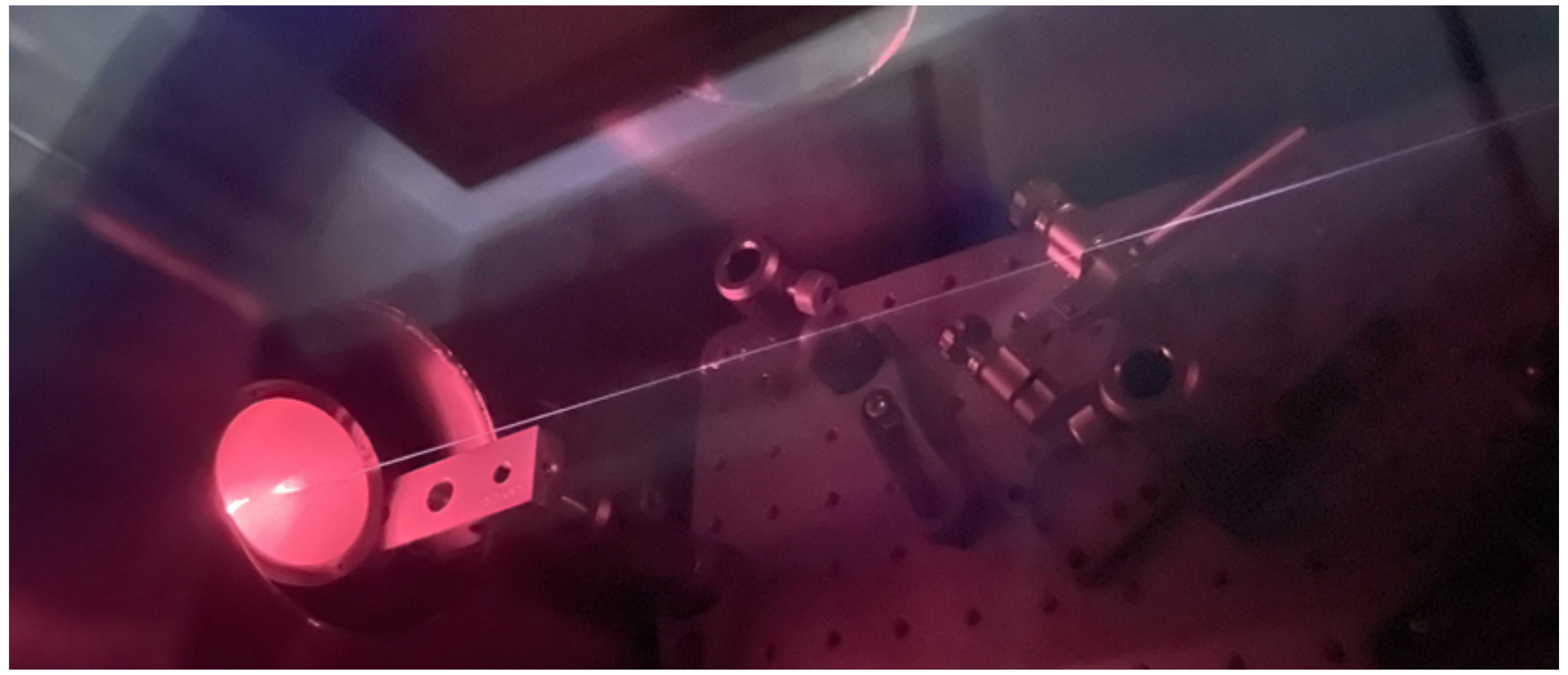
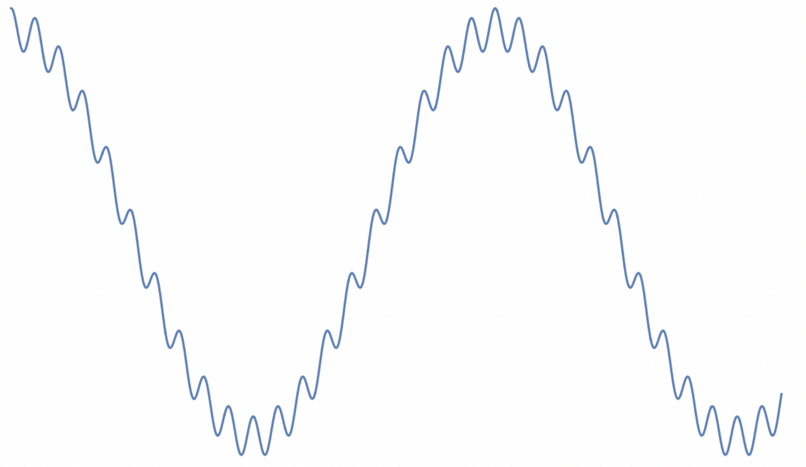
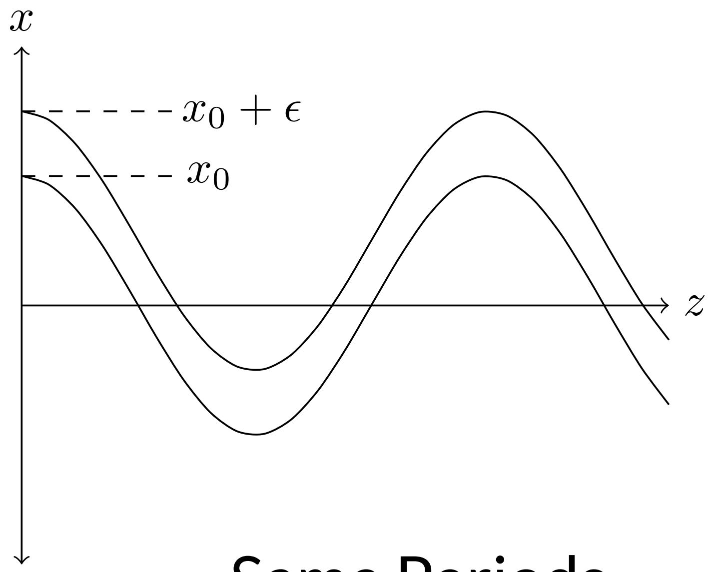
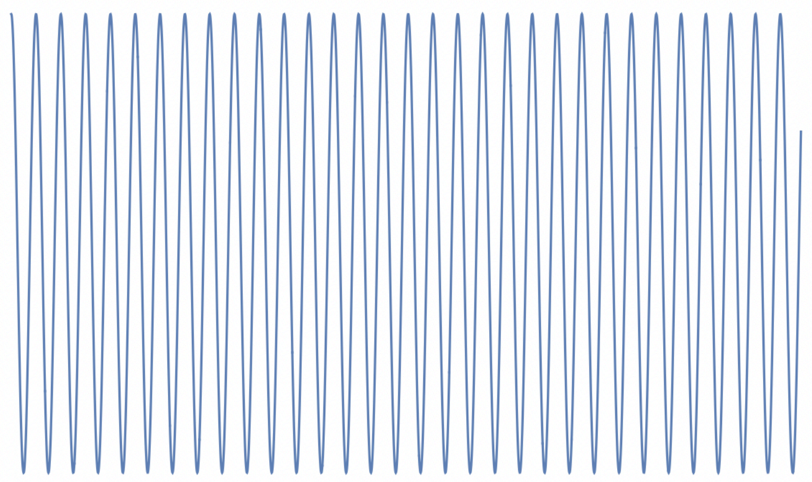
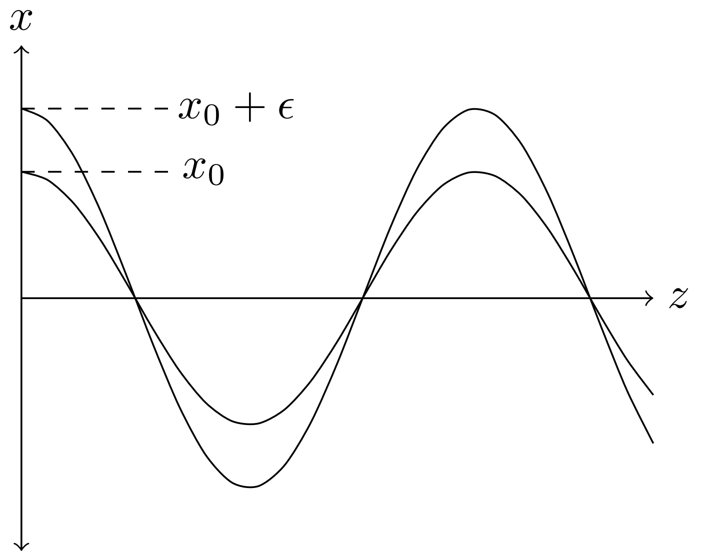
Ion Channel Laser

- The ICL is similar to the FEL, but uses a uniform ion channel instead of a magnetic undulator to transversely oscillate particles
- Narrow channel plasma eliminates accelerating field
- Strong ion channel focusing increases gain (ρ) and allowable energy spread by an order of magnitude, decreases gain length by the same amount.
- More stringent emittance requirements than the FEL
- **ICL physics has subtle but important differences from FEL physics**



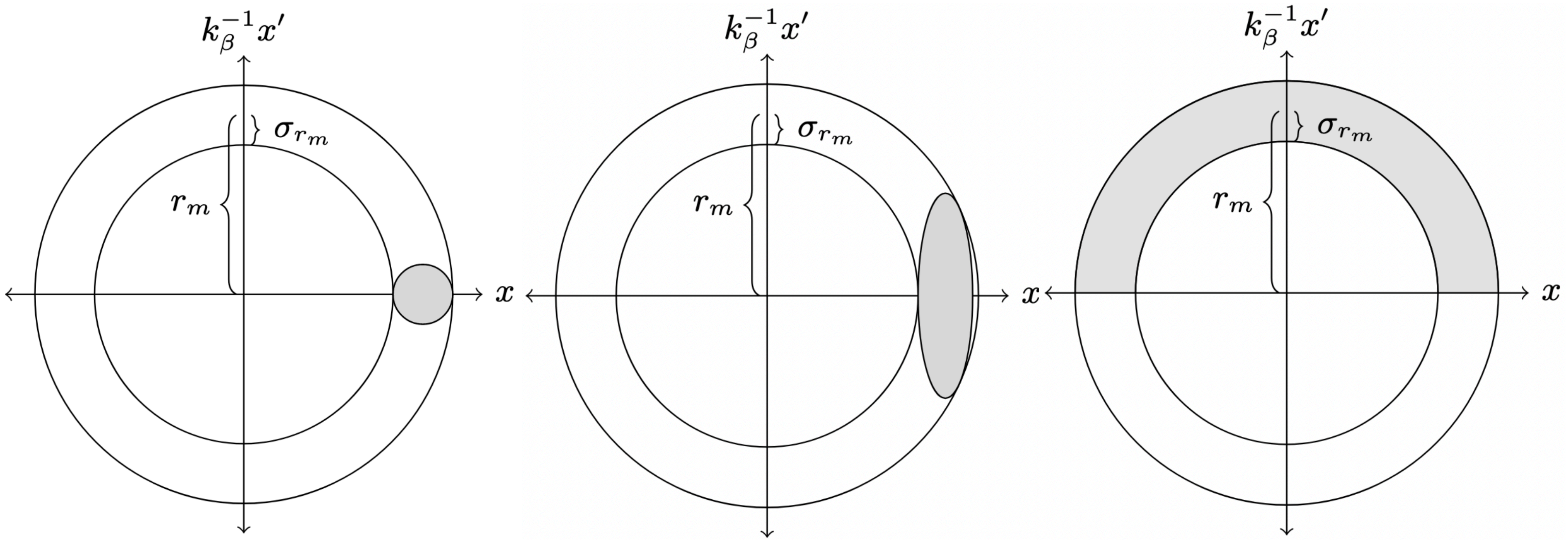


ICL vs FEL

	Field	Oscillation Period	Undulator Parameter	Emittance Constraint	Resonant Condition	Particle Motion	Effect of Offset
FEL	$\mathbf{B} = B_0 \sin(k_u z)\hat{\mathbf{y}}$	λ_u	$K = \dfrac{eB_0}{mck_u}$	$\epsilon_n \lesssim \dfrac{\gamma \lambda_1}{4\pi} \dfrac{\overline{\beta}}{L_{G0}}$	$\lambda = \dfrac{\lambda_u}{2\gamma^2}\left(1 + \dfrac{K^2}{2} + \gamma^2 \theta_\beta^2\right)$	 $\mathbf{x}(z) = \mathbf{x}_u(z) + \mathbf{x}_\beta(z)$	 Same Periods Same Amplitudes Same Radiation Wavelengths
ICL	$\mathbf{E} = \dfrac{en_0}{2\epsilon_0}\mathbf{x}_\perp$	$\lambda_\beta = \lambda_p \sqrt{2\gamma}$	$K = \gamma k_\beta r_m$ Betatron Oscillation Amplitude	$\dfrac{\sigma_{r_m}}{r_m} \lesssim \dfrac{1 + \frac{K^2}{2}}{K^2}\rho$	$\lambda = \dfrac{\lambda_\beta}{2\gamma^2}\left(1 + \dfrac{K^2}{2}\right)$	 $\mathbf{x}(z) = \mathbf{x}_\beta(z)$	 Same Periods Different Amplitudes Different Radiation Wavelengths

ICL Emittance Constraints

- $\dfrac{\sigma_{r_m}}{r_m} \lesssim \dfrac{1+\frac{K^2}{2}}{K^2}\rho$
- Not an *emittance* constraint but a *transverse oscillation amplitude spread* constraint
- Phase space manipulation can somewhat soften the emittance requirement
- Still, **ICLs have extremely stringent emittance requirements**



$$\frac{\epsilon_n}{\gamma\lambda} \lesssim \frac{1}{\pi}\frac{1+\frac{K^2}{2}}{K^2}\rho^2$$

$$\frac{\epsilon_n}{\gamma\lambda} \lesssim \frac{1}{\pi}\left(\frac{2}{5}\right)^{\frac{5}{8}}\left(\frac{1+\frac{K^2}{2}}{K^2}\right)^{\frac{1}{4}}\rho^{\frac{5}{4}}$$

$$\frac{\epsilon_n}{\gamma\lambda} \lesssim \frac{2}{\pi}\rho^{*}$$

FEL

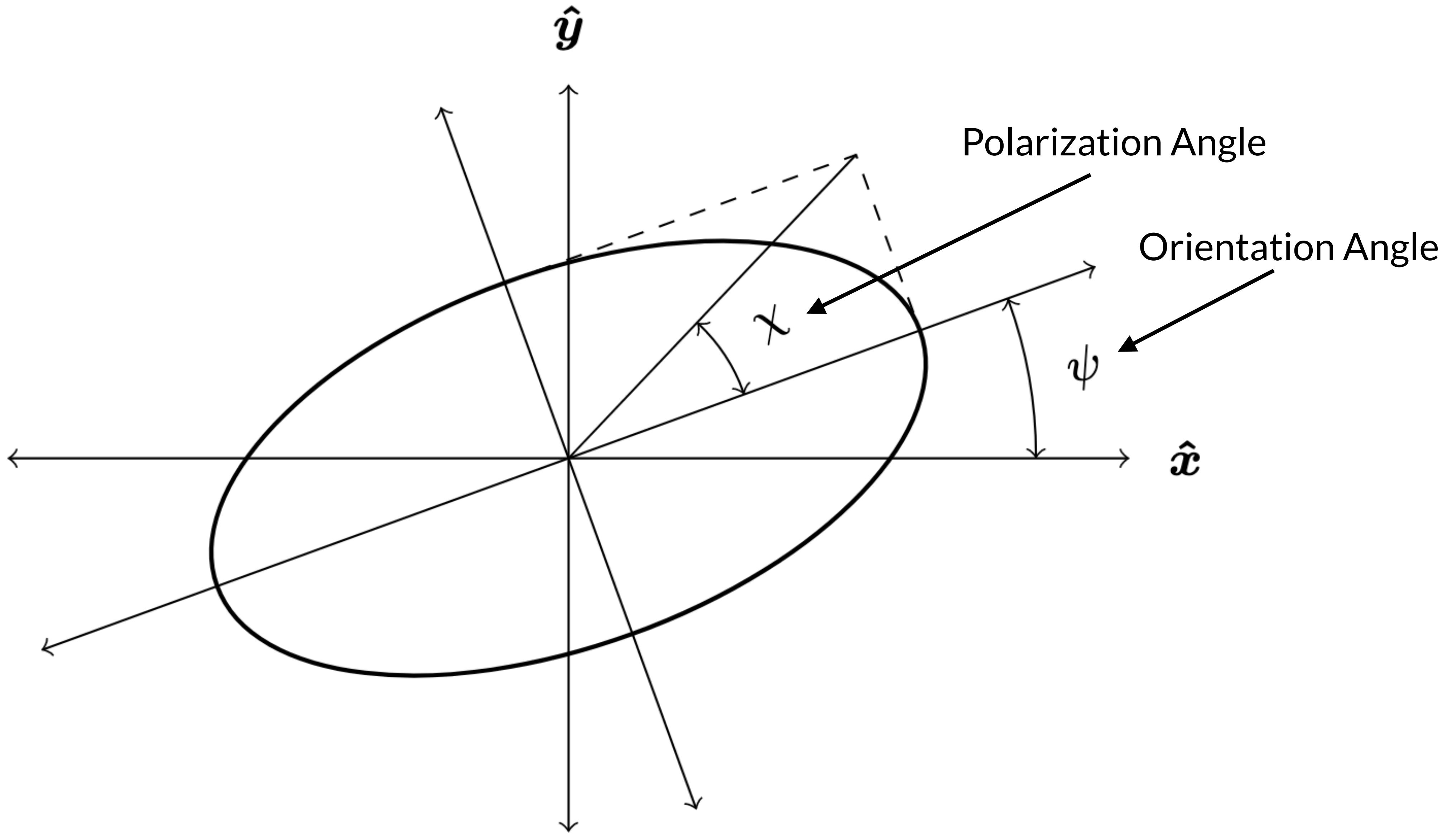
$$\frac{\epsilon_n}{\gamma\lambda} \lesssim \frac{1}{4\pi}\frac{\bar{\beta}}{L_{G,0}}$$

*Depends on the precise way emittance is defined

Further Peculiarities

- Multiple possible ICL "configurations"; this work focuses on the off-axis configuration (see Ersfeld *et al.* (2014))
- Unlike the undulator period, the betatron period depends on γ and thus is slightly different for each particle
- Particle oscillation phase is not fixed by the field but can be different for each particle, which changes microbunching physics
- Particles transversely oscillate across the radiation mode every oscillation

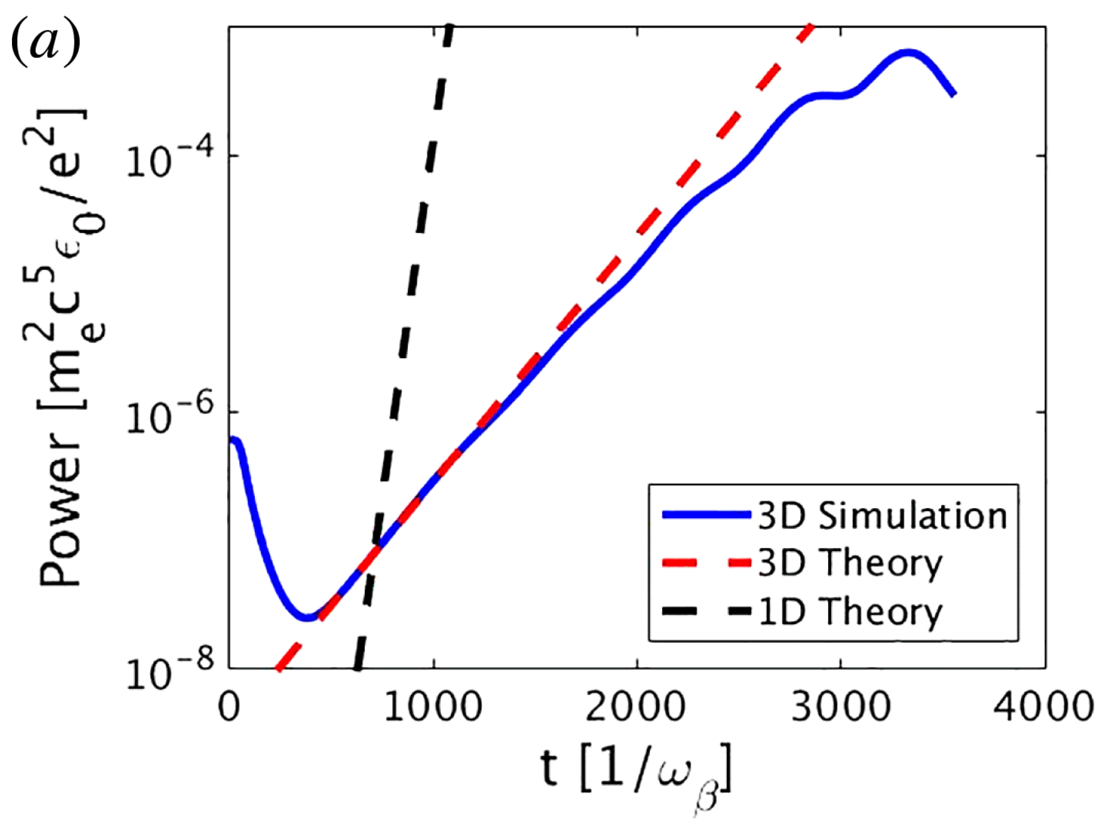
Transverse Oscillation Ellipse

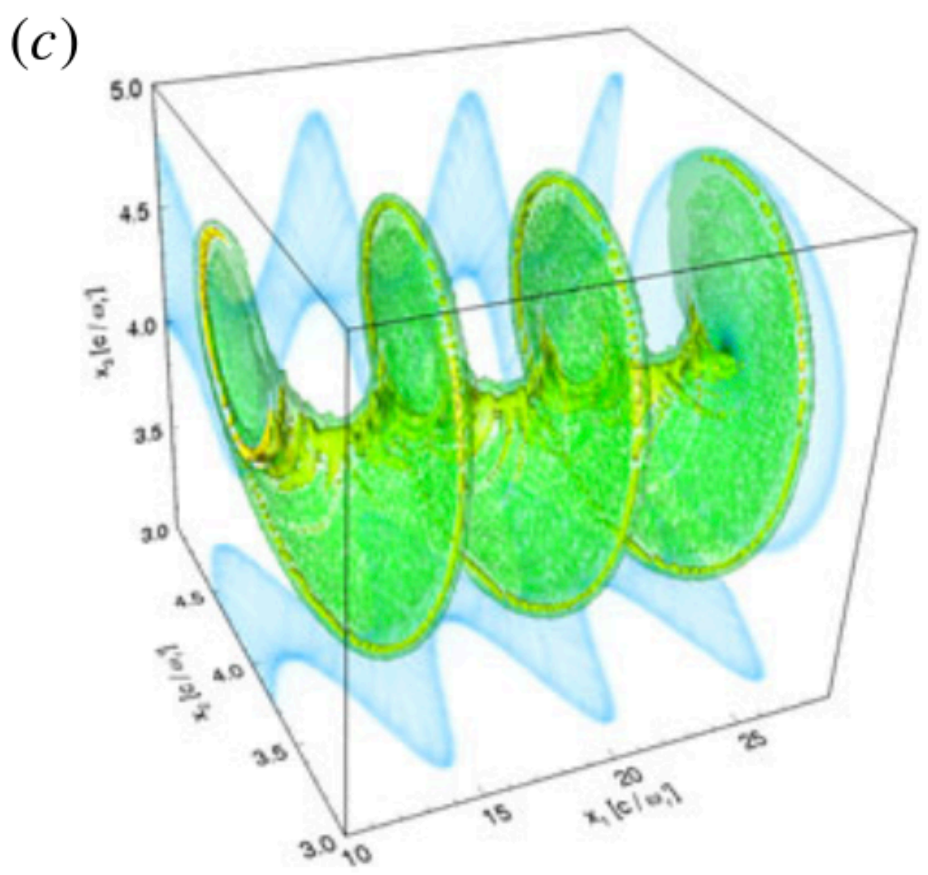


	Field	Particle Motion	Oscillation Phase ϑ	Oscillation Amplitude	Beam Size σ_{beam}
FEL	$\mathbf{B} = B_0 \sin(k_u z + \vartheta)\hat{\mathbf{y}}$	$x(z) = r_m \sin(k_u z + \vartheta)$	Fixed by field phase, same for each particle	$r_m \ll \sigma_{\text{radiation mode}}$	$\sigma_{\text{beam}} \sim \sigma_{\text{radiation mode}}$
ICL	$\mathbf{E} = \dfrac{en_0}{2\epsilon_0}\mathbf{x}_\perp$	$x(z) = r_m \sin(k_\beta z + \vartheta)$	Independent for each particle	$r_m \sim \sigma_{\text{radiation mode}}$	$\sigma_{\text{beam}} \ll \sigma_{\text{radiation mode}}$

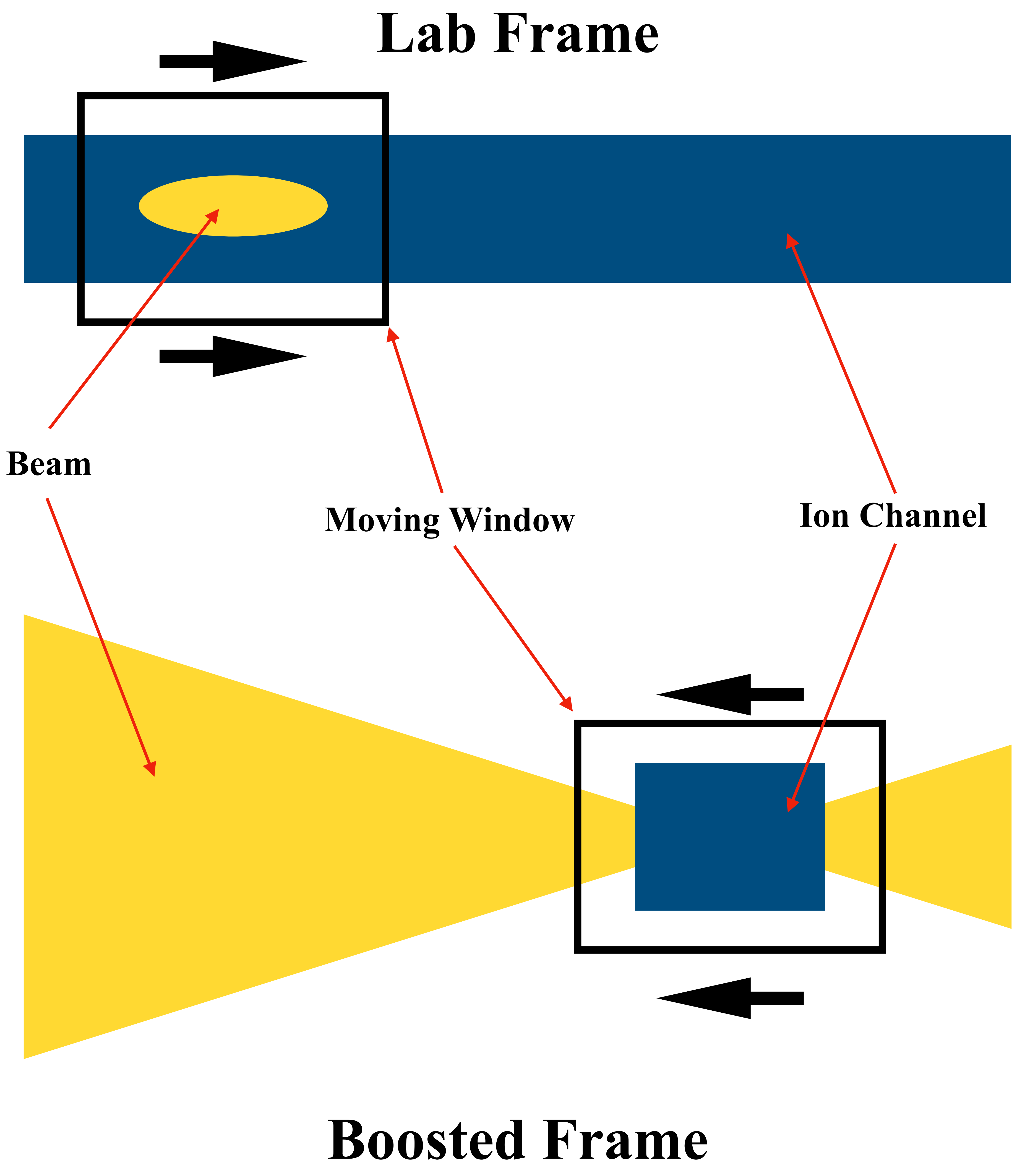
Simulation Challenges

- ICLs have extreme parameters and substantially different physics which makes simulating them using most FEL codes impossible or prohibitively computationally expensive
- The off-axis ICL is a multiscale problem in both the transverse (beam size vs oscillation amplitude) and longitudinal (beam length vs radiation wavelength) dimensions
- In general the ICL must be simulated using boosted frame PIC simulations, although we are working on running time independent simulations in Puffin





X. Davoine *et al.* (2018)



ICL Theory I: Slowly Varying Quantities

Equations of Motion

$$\mathcal{H} = \sum_{j=1}^{N_e} \frac{1 + (p_{c,x,j} + a(\boldsymbol{x}_{\perp,j}, \zeta_j, z))^2}{2\gamma_j} + \frac{1}{4}x_j^2$$

Turn off Fields and Solve

Single Particle Motion

$$x_j(z) = r_{m,j}\cos(k_{\beta,j}z + \varphi_j)$$
$$p_{x,j}(z) = -K_j \sin(k_{\beta,j}z + \varphi_j)$$
$$\zeta_j(z) = \zeta_{0,j} - \frac{1 + \frac{K_j^2}{2}}{2\gamma_j^2}z - \frac{K_j^2}{8\gamma_j^2 k_{\beta,j}}\sin(2\varphi_j) + \frac{K_j^2}{8\gamma_j^2 k_{\beta,j}}\sin(2(k_{\beta,j}z + \varphi_j))$$
$$\gamma_j(z) = \gamma_j$$

Construct Slowly Varying Quantities

Energy Detuning

$$\eta_j \equiv \frac{\gamma_j - \gamma_r}{\gamma_r}$$

Pondermotive Phase

$$\theta_j \equiv k_{\beta,r}z + k_{1,r}\left(\zeta_{0,j} - \frac{1 + \frac{K_j^2}{2}}{2\gamma_j^2}z - \frac{K_j^2}{8\gamma_j^2 k_{\beta,j}}\sin(2\varphi_j)\right)$$

Undulator Parameter Detuning

$$\delta_j \equiv \frac{K_j - K_r}{K_r}$$

Betatron Phase Detuning

$$\vartheta_j \equiv \varphi_j + k_{\beta,j}z - k_{\beta,r}z$$

Turn fields back on, get equations for slowly varying quantities

$$\eta_j' = \frac{K_r}{\gamma_r^2}\sin(k_{\beta,r}z + \vartheta_j)\left.\frac{\partial a}{\partial \zeta}\right|_j$$
$$\delta_j' = \frac{1 + K_r^2}{2K_r^2}\eta_j'$$
$$\theta_j' = 2k_{\beta,r}\left(\eta_j - \frac{K_r^2}{2 + K_r^2}\delta_j\right)$$
$$\vartheta_j' = -\frac{1}{2}k_{\beta,r}\eta_j$$

Where

a : normalized magnetic vector potential

$\zeta = z - t(z)$

$K_j = \gamma_j k_{\beta,j} r_{m,j}$

$k_{\beta,j} = 1/\sqrt{2\gamma}$

j subscript : jth particle in beam

r subscript : "resonant" / "reference particle" value

ICL Theory II: Period Averaging

Period Averaging of the Energy Detuning Equation

$$\eta_j' = \sum_{h\in\mathbb{N}^+}\int_{\nu\approx h} d\nu \frac{i\nu k_{1,r}K_r}{\gamma_r^2} e^{i\Delta\nu k_{\beta,r}z} e^{i\Delta\nu\theta_j}\overline{\mathcal{A}_h(\boldsymbol{x}_{\perp,j}(z),\nu,z)\sin(k_{\beta,r}z+\vartheta_j)e^{i\nu k_{1,r}\zeta_j(z)}} + \text{c.c.}$$

Period Averaging of the Field Equation

$$\left[\frac{\partial}{\partial z} + i\Delta\nu k_{\beta,r} - \frac{i}{2\nu k_{1,r}}\nabla_\perp^2\right]\mathcal{A}_h(\boldsymbol{x}_\perp,\nu,z) = \frac{e^{-i\Delta\nu k_{\beta,r}z}}{i\nu I_A}\sum_{j=1}^{N_e}\overline{x_j'(z)\delta^2(\boldsymbol{x}_\perp - \boldsymbol{x}_{\perp,j}(z))e^{-i\nu k_{1,r}\zeta_j}}$$

Where

$\mathcal{A}(\boldsymbol{x}_\perp,\nu,z)$: Fourier transform of slowly varying field envelope
$\nu = k_1/k_{1,r}$: radiation frequency normalized to fundamental
h : integer harmonic number
$\Delta\nu = \nu - h$: frequency detuning
$I_A = 17$ kA : Alfvén Current

Introduce particle distribution function and write Maxwell-Klimontovich Equations → Separate distribution function into background and perturbation → Perform Van Kampen normal mode expansion to obtain dispersion relation

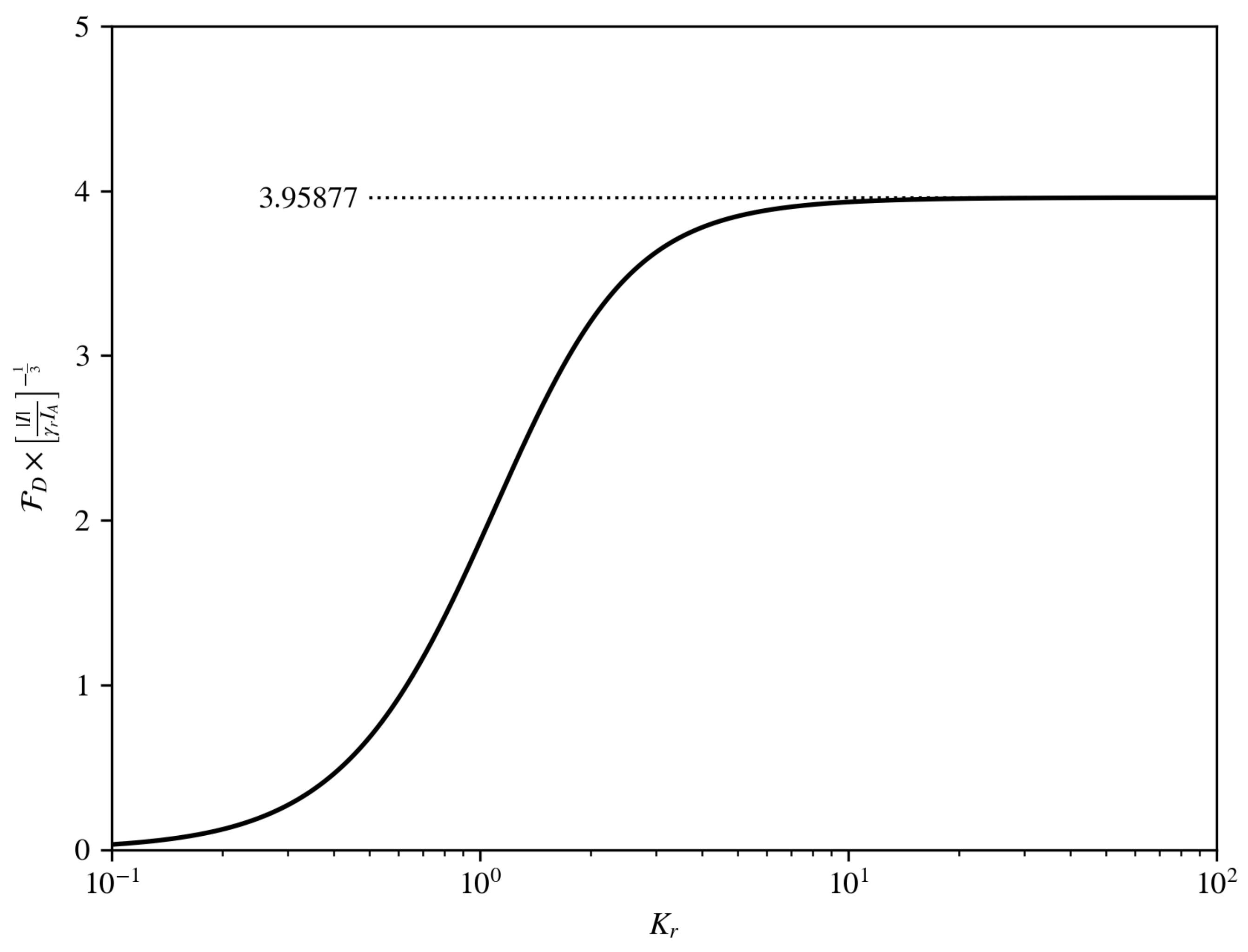
	Oscillation	Particle Motion	Period Average
FEL	$r_m \ll \sigma_{\text{radiation mode}}$	Oscillates locally, explores radiation mode over time	$\mathbf{E}(\mathbf{x}_{\perp,j}(z),z)\times\overline{\cos(k_u z)e^{ihk_1\zeta_j(z)}}$
ICL	$r_m \sim \sigma_{\text{radiation mode}}$	Oscillates across radiation mode every oscillation	$\overline{\mathbf{E}(\mathbf{x}_{\perp,j}(z),z)\cos(k_{\beta,j}z)e^{ihk_1\zeta_j(z)}}$

ICL Theory III: Dispersion Relation

Dispersion Relation

$$\left[\mu_\ell - \hat{\Delta\nu} + \mathcal{F}_D^{-1}\hat{\boldsymbol{\nabla}}_\perp^2\right]\hat{\mathcal{A}}_\ell(\hat{\boldsymbol{x}}_\perp) - \pi\mathcal{V}(\mu_\ell)\hat{\mathcal{W}}(\hat{\boldsymbol{x}}_\perp)\int d^2\hat{\boldsymbol{x}}'_\perp\hat{\mathcal{W}}(\hat{\boldsymbol{x}}'_\perp)\hat{\mathcal{A}}_\ell(\hat{\boldsymbol{x}}'_\perp) = 0$$

Fresnel Parameter: strength of diffraction



$$\mathcal{F}_D = 16\nu\frac{K^2}{2+K^2}\rho_0 = \frac{\sqrt{3}}{4}\frac{z_r}{L_{G,0}}$$

V function: finite energy and undulator parameter spread

$$\mathcal{V}(\mu) = \frac{A}{\mu^2}I\left(\frac{\Sigma}{\mu}\right)$$

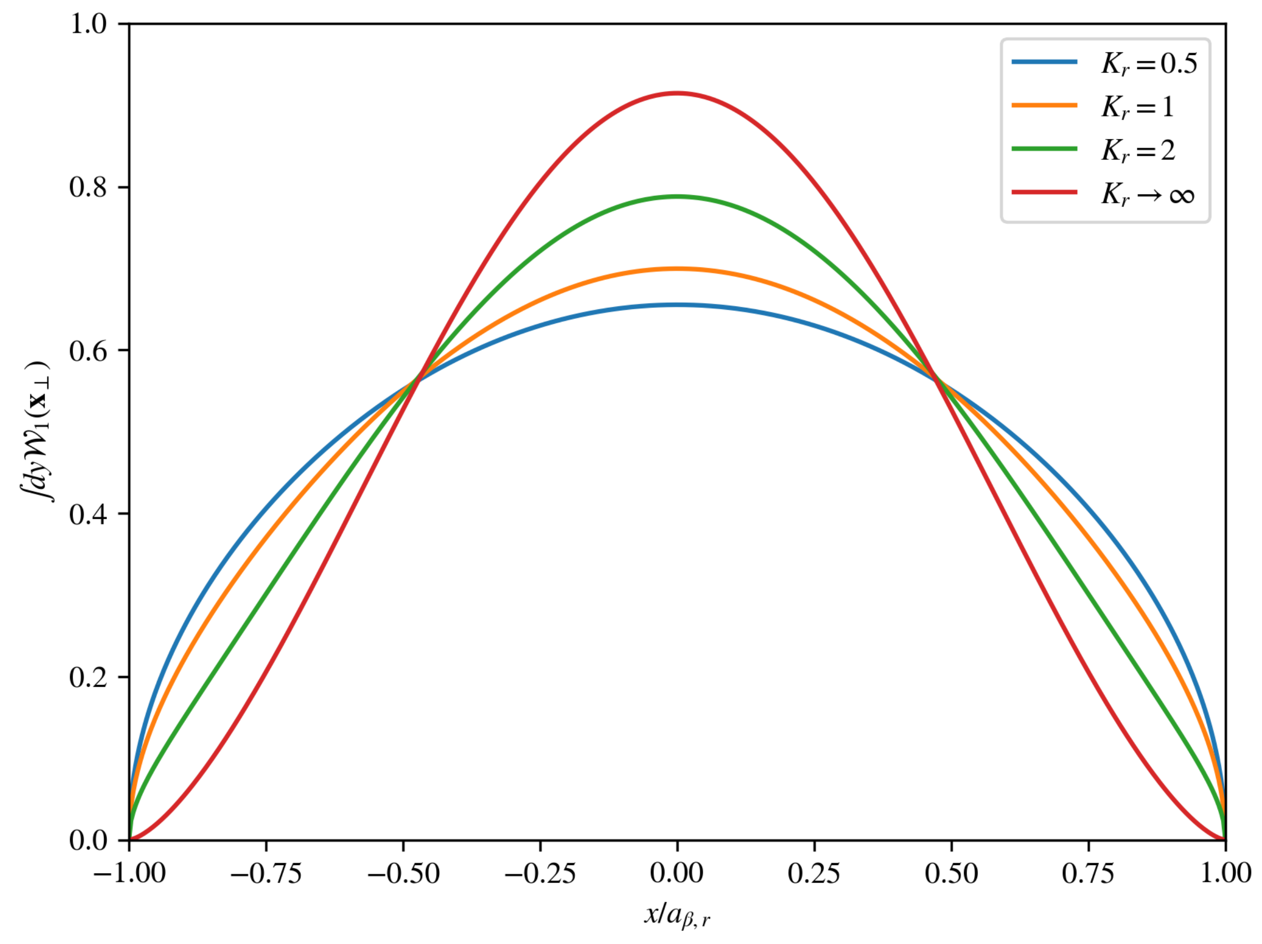
$$A = \left(1 + \frac{2(3+K_r^2)}{4+K_r^2}\Delta\nu\right)$$

$$\Sigma \equiv \sqrt{\left(\frac{3}{4}+\Delta\nu\right)^2\frac{\sigma_\eta^2}{\rho_0^2} + \frac{(1+\Delta\nu)^2 K_r^4}{(2+K_r^2)^2}\frac{\sigma_\delta^2}{\rho_0^2}}$$

$$I(y) = \frac{1}{\sqrt{2\pi}}\int dx\frac{e^{-\frac{1}{2}x^2}}{(1-xy)^2}$$

$$I(0) = 1$$

***W* function:** spacial emission/ interaction strength profile



$$\hat{\mathcal{W}}_h(\hat{\boldsymbol{x}}_\perp) = \sum_{\substack{n\in\mathbb{Z}\\ h-n \text{ odd}}}\frac{[\mathrm{JJ}]_{h-n}}{[\mathrm{JJ}]_h}\mathcal{C}_n(\hat{x})\,\delta(\hat{y})$$

$$\mathcal{C}_n(x) = \begin{cases}\frac{T_n(x)}{\pi\sqrt{1-x^2}} & |x|<1\\ 0 & |x|\geq 1\end{cases}$$

$$[\mathrm{JJ}]_n = J_{\frac{n-1}{2}}(h\xi_r) - J_{\frac{n+1}{2}}(h\xi_r)$$

$$\xi = \frac{K^2}{2(2+K^2)}$$

Solving the Dispersion Relation: Variational Principle

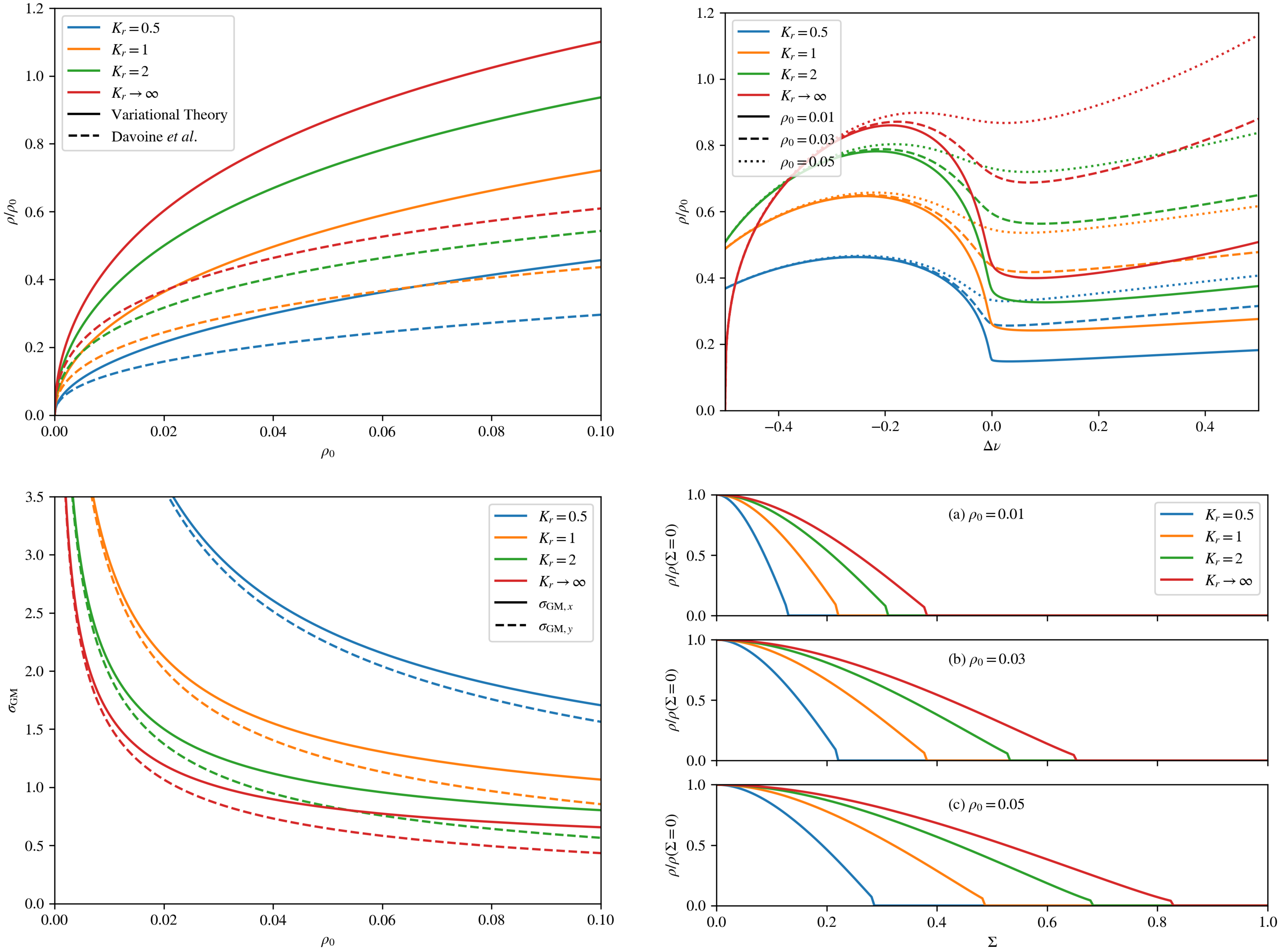
$$\mathcal{J}\left[\hat{\mathcal{A}}_\ell(\hat{\boldsymbol{x}}_\perp)\right] \equiv (\mu_\ell - \hat{\Delta\nu}) \int d^2\hat{\boldsymbol{x}}_\perp \left(\hat{\mathcal{A}}_\ell(\hat{\boldsymbol{x}}_\perp)\right)^2 + \mathcal{F}_D^{-1} \int d^2\hat{\boldsymbol{x}}_\perp \hat{\mathcal{A}}_\ell(\hat{\boldsymbol{x}}_\perp) \hat{\boldsymbol{\nabla}}_\perp^2 \hat{\mathcal{A}}_\ell(\hat{\boldsymbol{x}}_\perp) - \pi \mathcal{V}(\mu_\ell) \left(\int d^2\hat{\boldsymbol{x}}_\perp \hat{\mathcal{W}}(\hat{\boldsymbol{x}}_\perp) \hat{\mathcal{A}}_\ell(\hat{\boldsymbol{x}}_\perp)\right)^2$$

- Define functional from the dispersion relation
- Choose a trial family of functions $\hat{\mathcal{A}}_{\text{trial}}(\hat{\boldsymbol{x}}_\perp, \boldsymbol{c})$ parameterized by constants $\mathbf{c}$
- Set $\mathcal{J}[\hat{\mathcal{A}}_{\text{trial}}(\hat{\boldsymbol{x}}_\perp, \boldsymbol{c})] = 0$ and vary $\mathbf{c}$ to maximize gain
- We used a Gaussian trial function which does not account for higher order modes

$$\hat{\mathcal{A}}_{\text{trial}}(\hat{\boldsymbol{x}}_\perp, \boldsymbol{c}) \equiv \frac{e^{-\frac{x^2}{2\sigma_{\text{GM},x}^2}} e^{-\frac{y^2}{2\sigma_{\text{GM},y}^2}}}{2\pi\sigma_{\text{GM},x}\sigma_{\text{GM},y}}$$

Variational Principle Results

- Variational theory shows only a relatively modest decrease in gain due to 3D effects
- Extremely large gain bandwidth
- Uncertainty about the validity of the variational principle



Solving the Dispersion Relation: Matrix Method

- Write the transverse radiation mode as a truncated series of orthonormal basis functions

$$\hat{A}_\ell(\hat{\boldsymbol{x}}_\perp) = \sum_{n=0}^{N-1} (\hat{A}_\ell)_n \phi_n(\hat{\boldsymbol{x}}_\perp)$$

- Convert differential dispersion relation into a matrix equation

$$\int d^2\hat{\boldsymbol{x}}_\perp \phi_m(\hat{\boldsymbol{x}}_\perp)\phi_n(\hat{\boldsymbol{x}}_\perp) = \delta_{mn}$$

- Use methods from linear algebra to solve the generalized eigenvalue problem and obtain growth rates and mode profiles

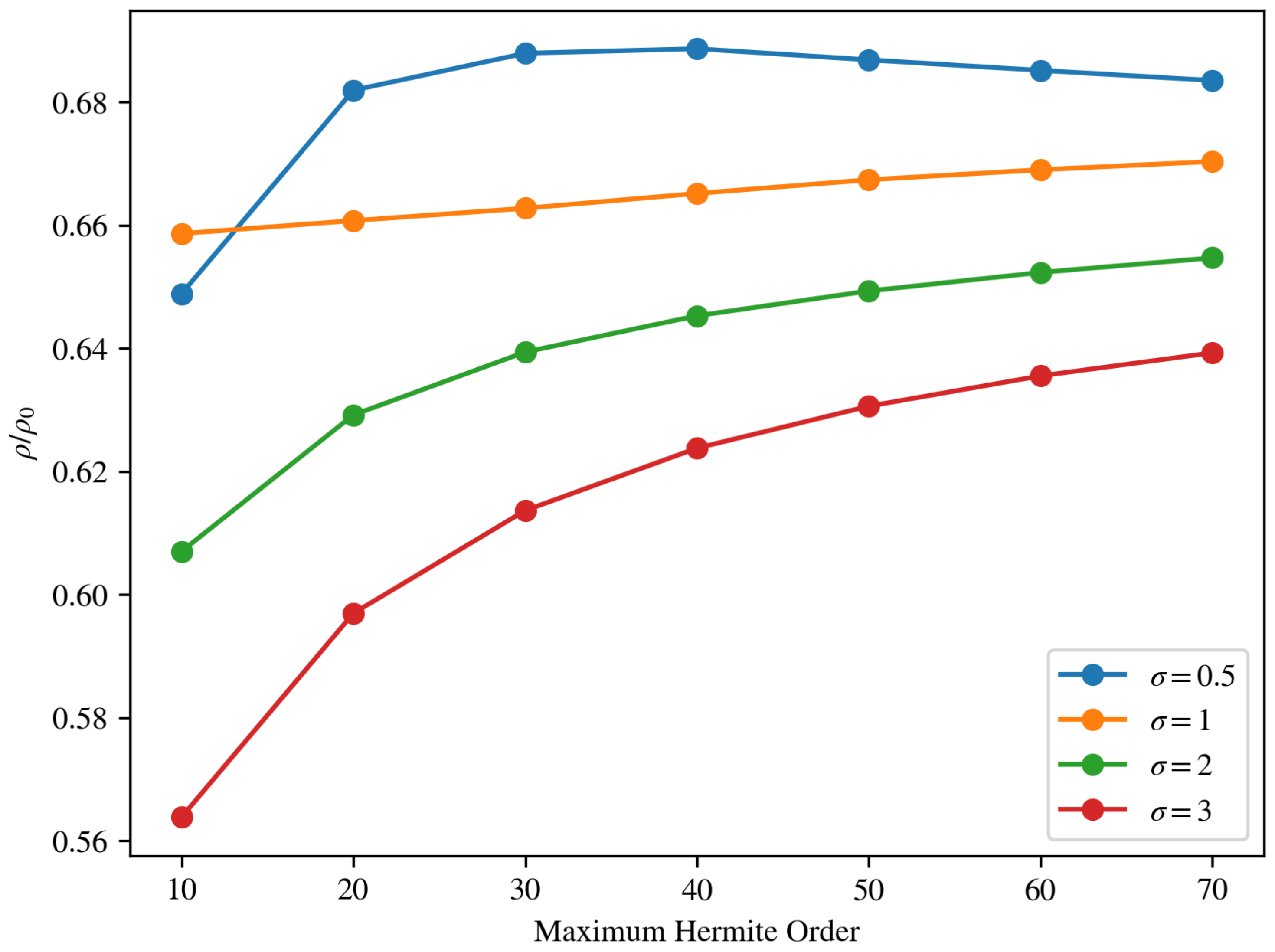
$$\left[\mu_\ell - \hat{\Delta\nu} + \mathcal{F}_D^{-1}\boldsymbol{D}^2 - \pi\mathcal{V}(\mu_\ell)\boldsymbol{\mathcal{W}}\boldsymbol{\mathcal{W}}^\intercal\right]\hat{\boldsymbol{\mathcal{A}}}_\ell = 0$$

$$\phi_n(\hat{\boldsymbol{x}}) = \frac{1}{\sqrt{2^{n_x+n_y} n_x! n_y! \pi\sigma_x\sigma_y}} H_{n_x}\left(\frac{\hat{x}}{\sigma_x}\right) H_{n_y}\left(\frac{\hat{y}}{\sigma_y}\right) e^{-\frac{\hat{x}^2}{2\sigma_x^2}} e^{-\frac{\hat{y}^2}{2\sigma_y^2}}$$

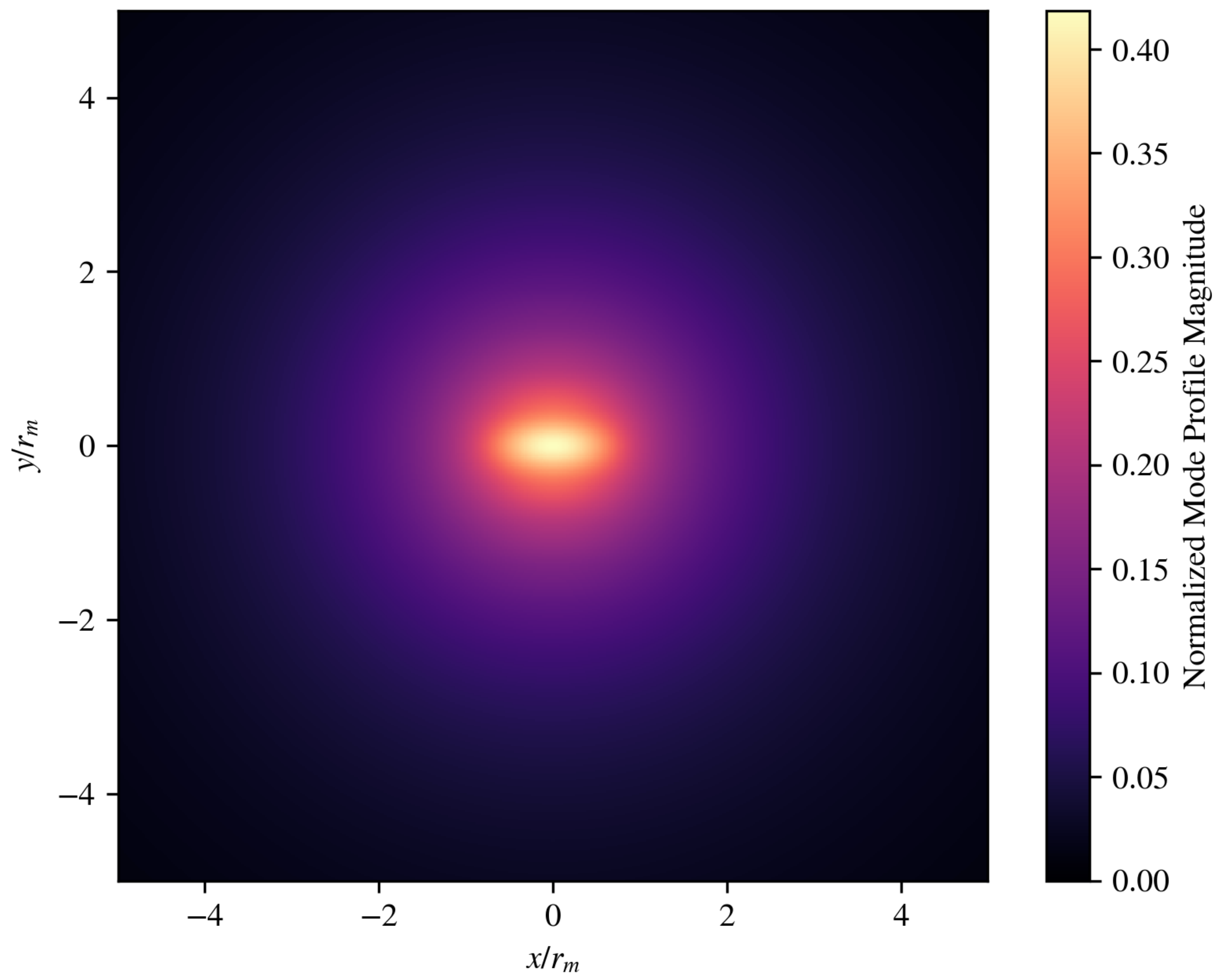
Matrix Method Results

- Yields similar growth rates as the variational method
- Much more reasonable gain bandwidth
- Convergence is very slow and results depend on the σ chosen for the basis functions
- Maximum Hermite Order of 70 requires diagonalizing a 3783x3783 matrix

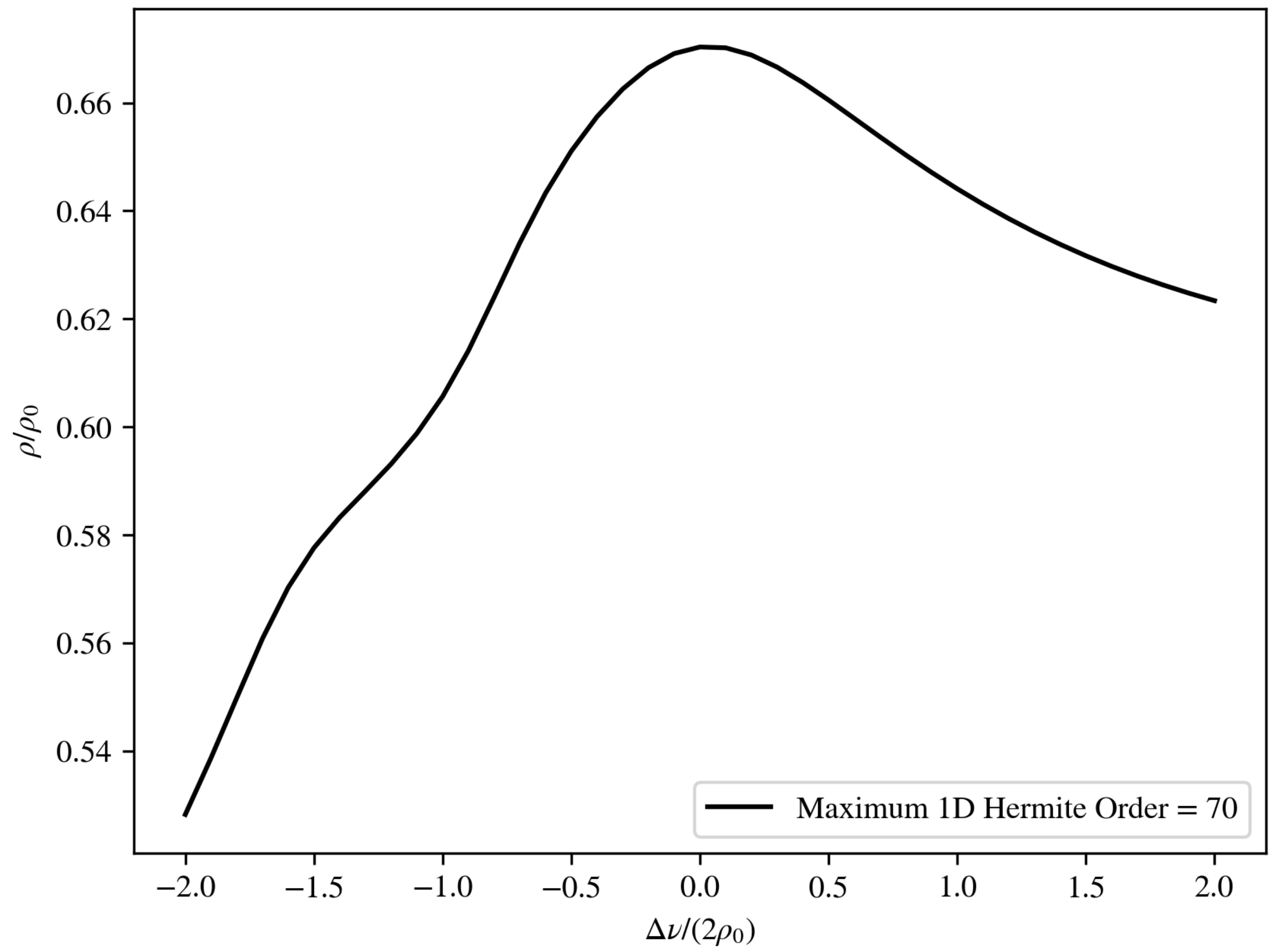
Growth Rate Convergence



Transverse Mode Profile



Detuning Curve



Solving the Dispersion Relation: Baxevanis Method

- A z dependent differo-integral equation for the field evolution can be derived from the Maxwell-Klimontovich equations
- Following P. Baxevanis *et al.* (2013), the field can be expanded in z dependent Gauss-Hermite modes
- Essentially a pseudospectral method for solving the initial value problem
- Work is ongoing!

$$\left[\frac{\partial}{\partial \hat{z}} + i\hat{\Delta}\nu - i\mathcal{F}_D^{-1}\hat{\boldsymbol{\nabla}}_\perp^2\right]\hat{\mathcal{A}}(\hat{\boldsymbol{x}}_\perp,\nu,\hat{z}) = \hat{\mathcal{W}}(\hat{\boldsymbol{x}}_\perp)\int d^2\hat{\boldsymbol{x}}'_\perp \hat{\mathcal{W}}(\hat{\boldsymbol{x}}'_\perp)\int_0^{\hat{z}} d\hat{z}'\Lambda(\hat{z},\hat{z}')\hat{\mathcal{A}}(\hat{\boldsymbol{x}}'_\perp,\nu,\hat{z}')$$

$$\Lambda(\hat{z},\hat{z}') = \frac{\pi}{\mathcal{I}}\iiint d\hat{\eta}d\hat{\delta}d\vartheta e^{\left((\nu-\frac{1}{4})\hat{\eta}-\nu\frac{K_r^2}{2+K_r^2}\hat{\delta}\right)(\hat{z}-\hat{z}')}\left[\frac{\partial}{\partial\hat{\eta}} + \frac{1+K_r^2}{2K_r^2}\frac{\partial_0}{\partial\hat{\delta}}\right]\hat{f}_0(\hat{\eta},\hat{\delta},\vartheta,\hat{z}')$$

$$\mathcal{A}(x_\perp,z) = b\sum_{p,q} C_{p,q}(z)\psi_{p,q}(x_\perp,z)$$

$$\frac{dC_{p',q'}(z)}{dz} + \sum_{p,q} D_{p',q',p,q}(z)C_{p,q}(z) = \sum_{p,q}\int_0^z dz'\Lambda(z,z')X^*_{p',q'}(z)X_{p,q}(z')C_{p,q}(z')$$

$$D_{p',q',p,q}(z) = \int d^2x_\perp \psi_{p',q'}(x_\perp,z)^*\left[\frac{\partial}{\partial z} - \frac{i}{2\kappa}\boldsymbol{\nabla}_\perp^2\right]\psi_{p,q}(x_\perp,z)$$

$$X_{p,q}(z) = \int d^2x_\perp W(x_\perp)\psi_{p,q}(x_\perp,z)$$

Summary & Future Work

- **We developed a 3D theory of the ion channel laser**
  - Accounts for diffraction, transverse guided mode shape, frequency and Betatron phase detuning, and nonzero spread in energy and undulator parameter
  - Obtained a dispersion relation
  - Obtained growth rates and transverse mode profiles using variational and matrix methods with ongoing work adapting the method from Baxevanis *et al.* (2013)
- Ongoing work performing time independent simulations in Puffin and boosted frame PIC simulations in Warp-X
- E306 experiment at FACET-II aims to lase at optical wavelengths in an ICL (See Litos *et al.* (2018))

Questions?

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