Lecture 1:

Fundamentals of Linear Algebra for Quantum Mechanics

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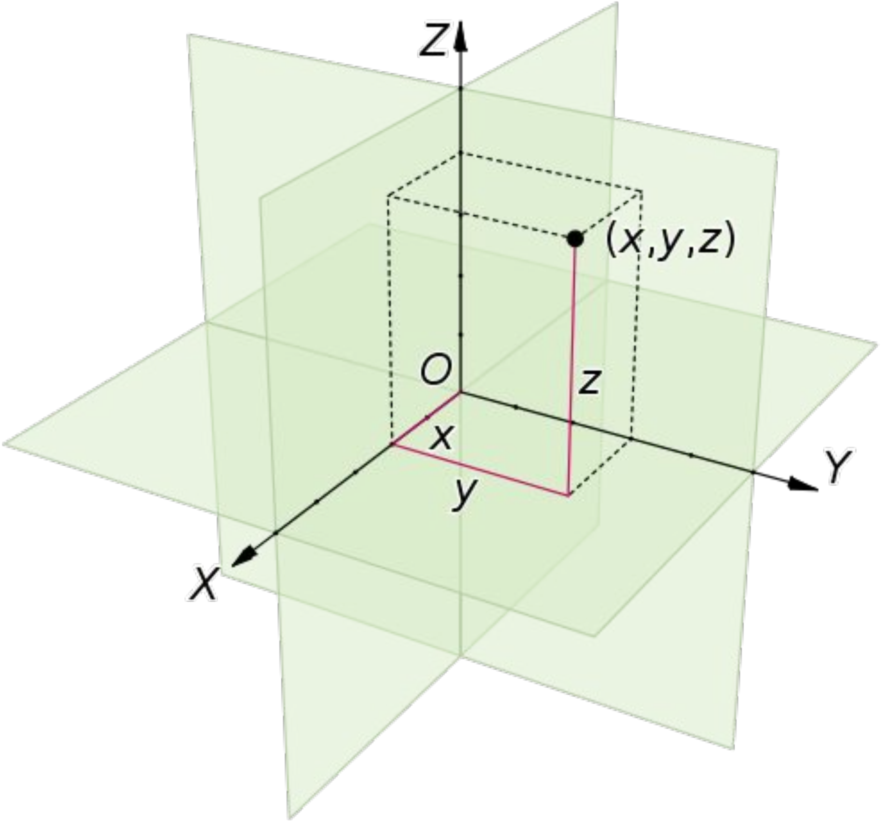
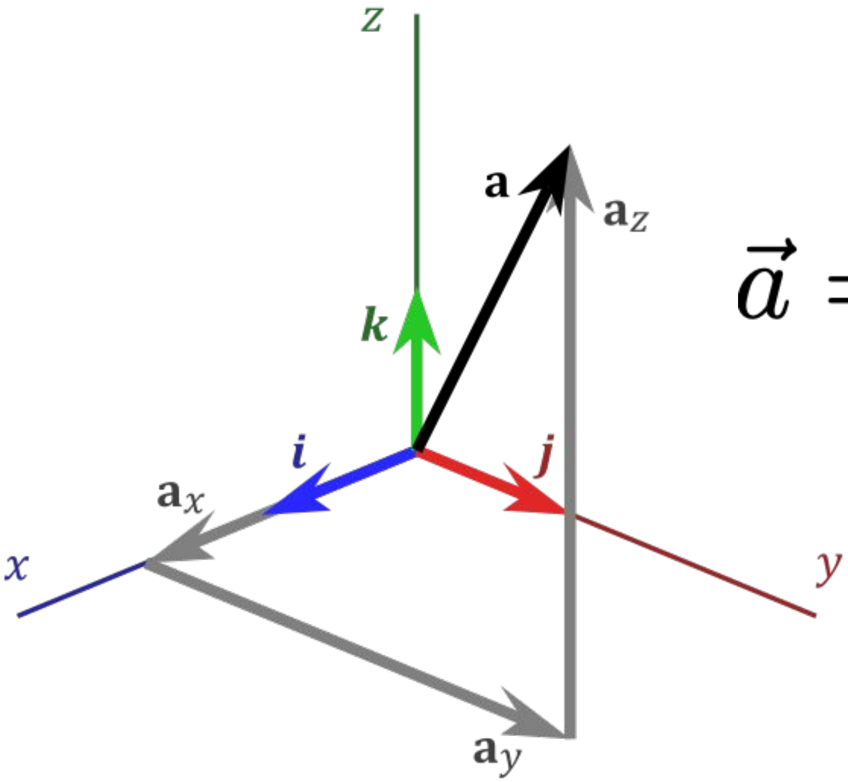
Fermilab
U.S. DEPARTMENT OF ENERGY | Office of Science

1. Vector Spaces



What are vector spaces?

2. Bases: The *Standard* Basis and column vectors

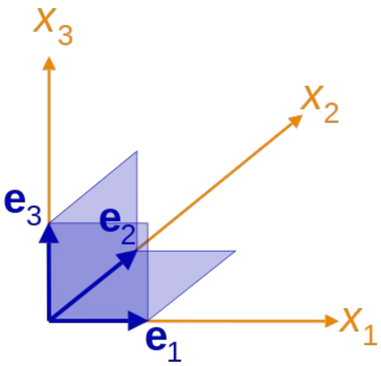


$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

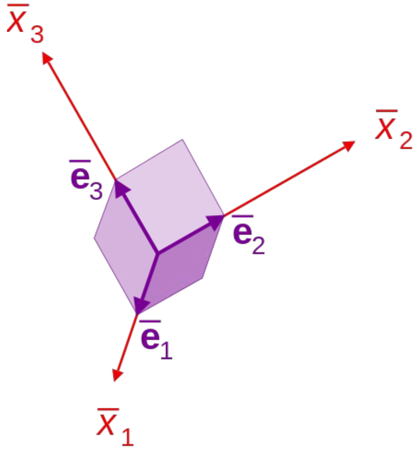
We can represent in 3D column vector form

2. What constitutes a basis?

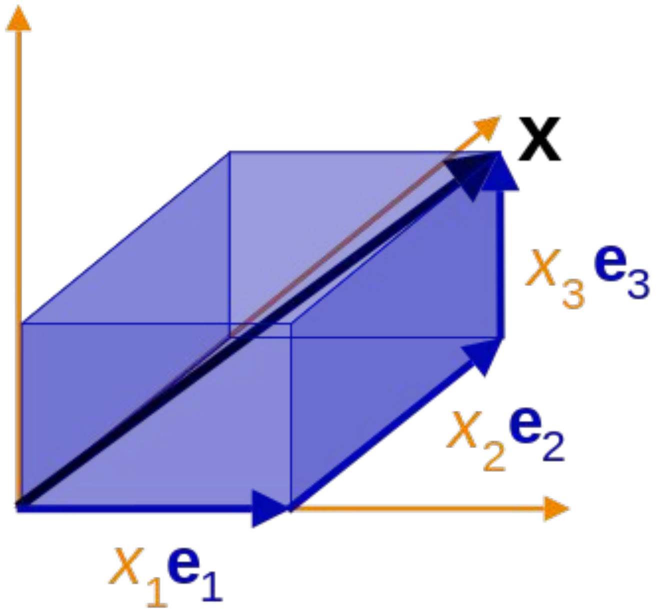
Basis 1:



Basis 2:

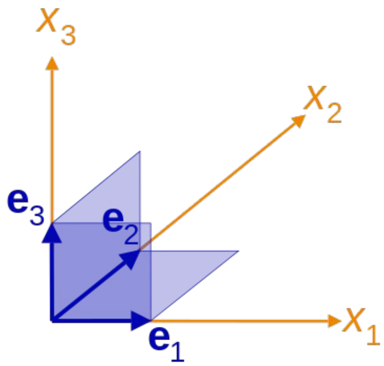


$$\vec{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3}$$

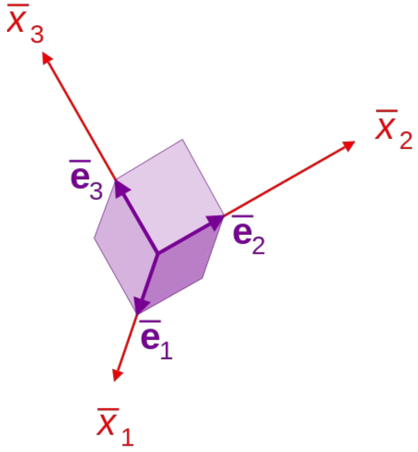


2. What constitutes a basis?

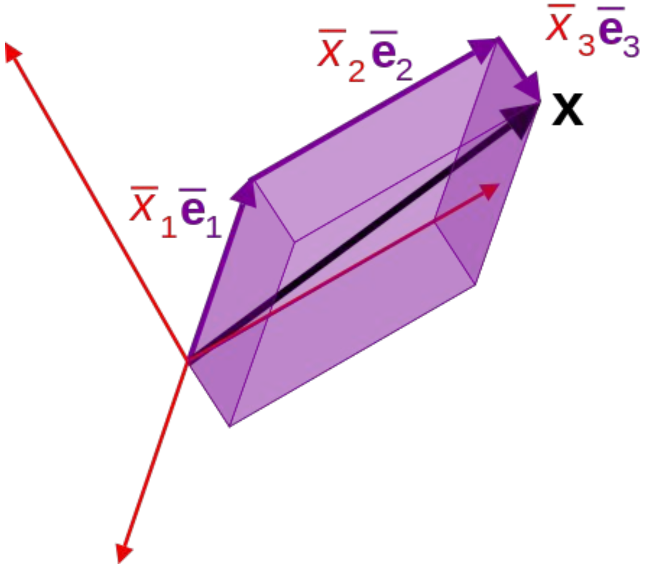
Basis 1:



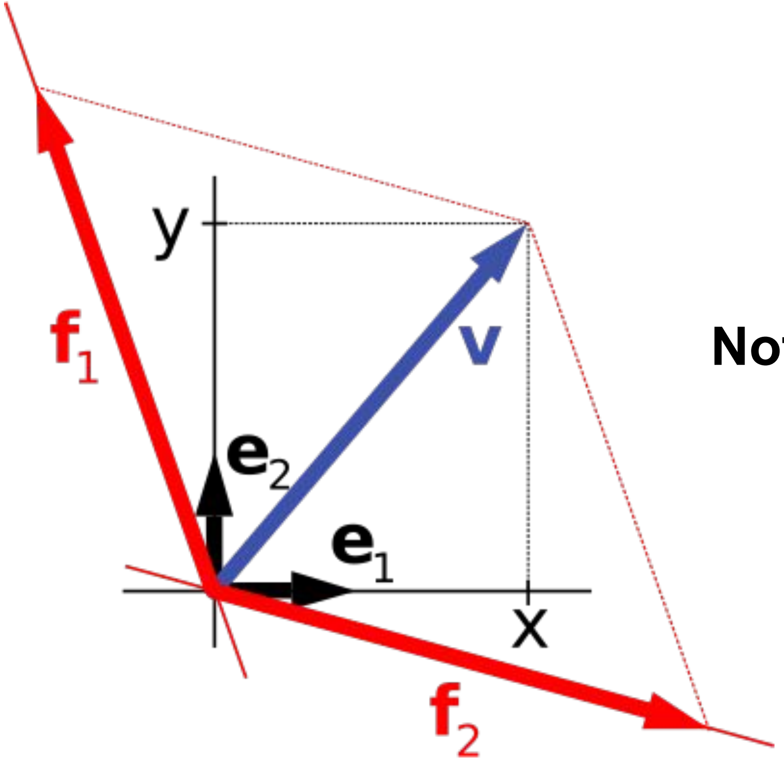
Basis 2:



$$\vec{x} = \bar{x}_1\bar{\mathbf{e}}_1 + \bar{x}_2\bar{\mathbf{e}}_2 + \bar{x}_3\bar{\mathbf{e}}_3$$

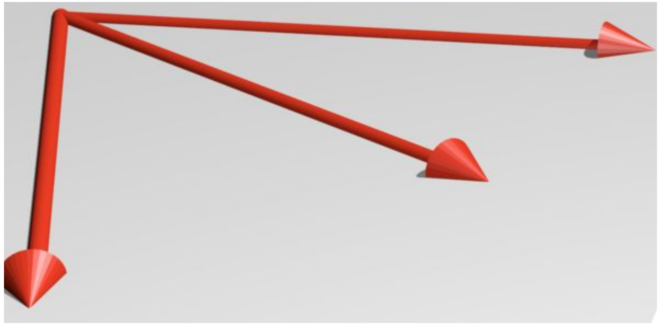


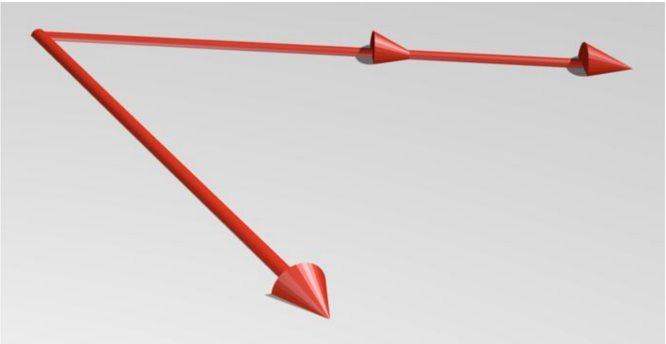
2. Non-orthogonal bases



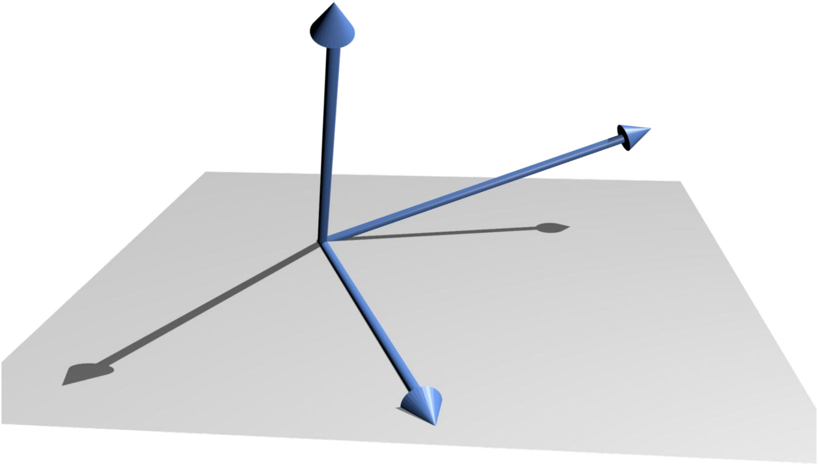
Not all bases are orthogonal

3. Linear Independence





Bases must be independent and span the space!



4. Inner Product: Introducing length

An inner product space induces a norm, that is, a notion of length of a vector.

$$\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}.$$

A norm in a vector space, in turns, induces a notion of distance between two vectors, defined as the length of their difference.

$$dist(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

4. Inner Product: Introducing length

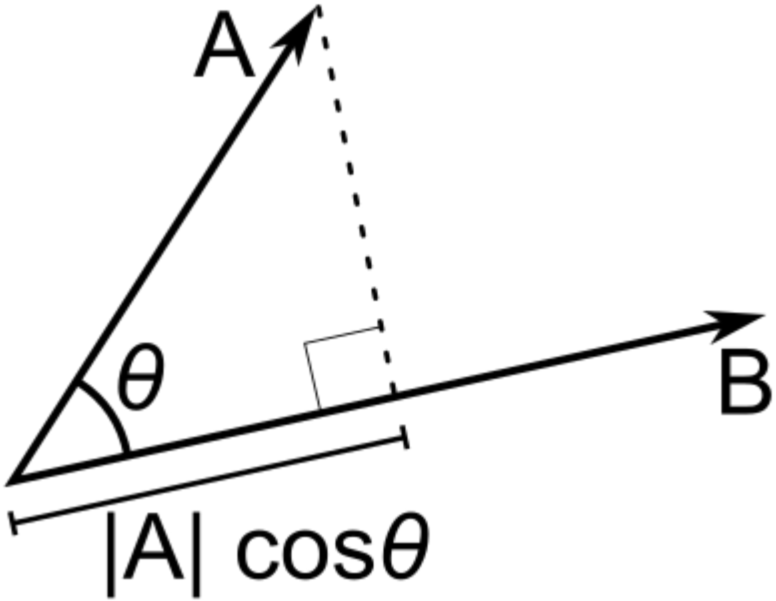
Mathematical definition

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^\mathsf{T} \mathbf{b}$$

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

4. Inner Product: Introducing length

The inner product has a geometrical significance



Group problems (5-10 mins)

Problem 1:

For this problem, use the real vectors

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

Suppose that $\mathbf{v}_4$ is another vector which is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_3$, and satisfying

$$\mathbf{v}_2 \cdot \mathbf{v}_4 = -3.$$

Calculate the following expressions:

(a) $\mathbf{v}_1 \cdot \mathbf{v}_2$.

(b) $\mathbf{v}_3 \cdot \mathbf{v}_4$.

(c) $(2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3) \cdot \mathbf{v}_4$.

(d) $\|\mathbf{v}_1\|, \|\mathbf{v}_2\|, \|\mathbf{v}_3\|$.

(e) What is the distance between $\mathbf{v}_1$ and $\mathbf{v}_2$?

Group problems (take home)

Problem 2: **Trickier**

Let $\mathbb{R}^2$ be the vector space of size-2 column vectors. This vector space has an **inner product** defined by $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^{\mathrm{T}}\mathbf{w}$. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is called an **orthogonal transformation** if for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$,

$$\langle T(\mathbf{v}), T(\mathbf{w}) \rangle = \langle \mathbf{v}, \mathbf{w} \rangle.$$

For a fixed angle $\theta \in [0, 2\pi)$, define the matrix

$$[T] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

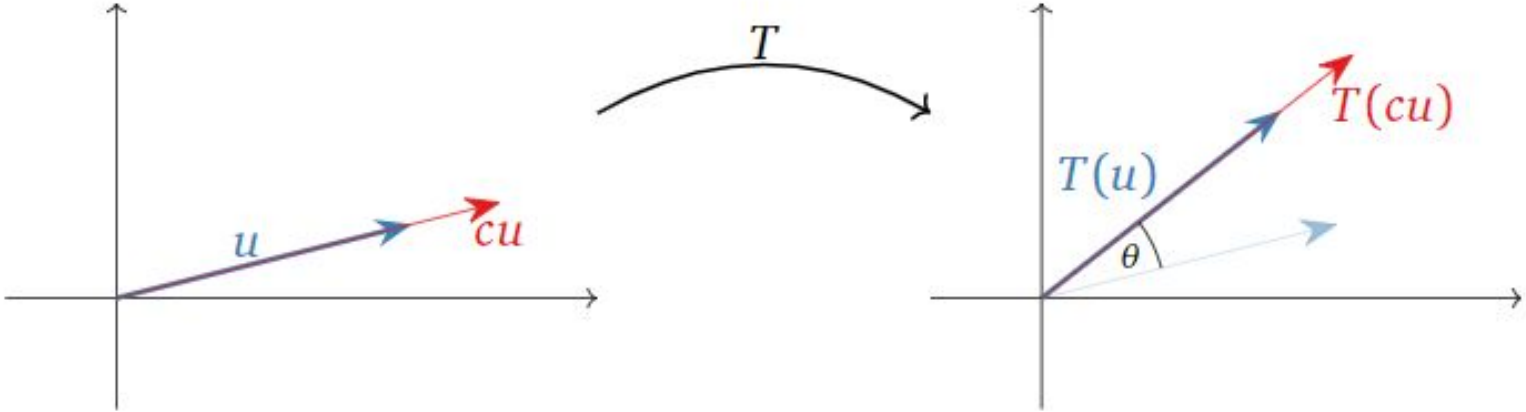
and the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(\mathbf{v}) = [T]\mathbf{v}.$$

Prove that T is an orthogonal transformation.

1. Matrix transformations

$$T\vec{u} = \vec{v}$$



1. Matrix Multiplication

Recall the inner product:

The product of an $m\times n$ matrix and an $n\times p$ matrix is an $m\times p$ matrix

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = a_1x_1 + a_2x_2 + \cdots + a_nx_n.$$

1. Matrix Multiplication

Definition 3.4.2: Entry of the Matrix

Let A be an $m \times n$ matrix. We will generally write a_{ij} for the entry in the ith row and the jth column. It is called the i, j ***entry*** of the matrix.

$$\begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}$$

ith row

jth column

Components are themselves inner products

$$\begin{pmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ — r_m — \end{pmatrix} \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_p \\ | & | & & | \end{pmatrix} = \begin{pmatrix} r_1c_1 & r_1c_2 & \cdots & r_1c_p \\ r_2c_1 & r_2c_2 & \cdots & r_2c_p \\ \vdots & \vdots & & \vdots \\ r_mc_1 & r_mc_2 & \cdots & r_mc_p \end{pmatrix}$$

Practice problems

#1:

Instructions: find AB

$a)$

$$A = \begin{bmatrix} -5 & 5 & -4 \\ -2 & 2 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & -6 \\ 2 & 3 \\ 6 & 1 \end{bmatrix}$$

$b)$

$$A = \begin{bmatrix} 4 & -12 & 1 \\ 1 & 5 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -4 & 1 & 9 \\ -3 & 7 & -6 \end{bmatrix}$$

Do these matrices *commute* i.e. do they satisfy AB = BA?

Group problems (~10 mins)

Problem 1

Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$.

(a) Find all matrices $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $AB = BA$.

(b) Use the results of part (a) to exhibit 2×2 matrices B and C such that $AB = BA$ and $AC \neq CA$.

Problem 2

$$A = \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

(a) $AB^{\mathrm{T}} + \mathbf{v}\mathbf{v}^{\mathrm{T}}$.

(b) $A\mathbf{v} - 2\mathbf{v}$.

(c) $\mathbf{v}^{\mathrm{T}}B$.

(d) $\mathbf{v}^{\mathrm{T}}\mathbf{v} + \mathbf{v}^{\mathrm{T}}BA^{\mathrm{T}}\mathbf{v}$.

NOTE:

Little QM content today :)

Complex phases, Hilbert space, Bra-Ket notation and probability amplitudes will be covered tomorrow!

Thanks!