Creating Superpositions

Part 2/2: Quantum mechanical spin and the Stern-Gerlach experiment

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Overview

What is the **spin** of a quantum particle?

Through spin we will learn about **basis** choices and measurements

Understand the concept of **commutability** and how it relates to simultaneity of measurement

Define **quantum numbers** by a **complete set of commuting observables** (CSCO) which parametrise a state

Brief recap

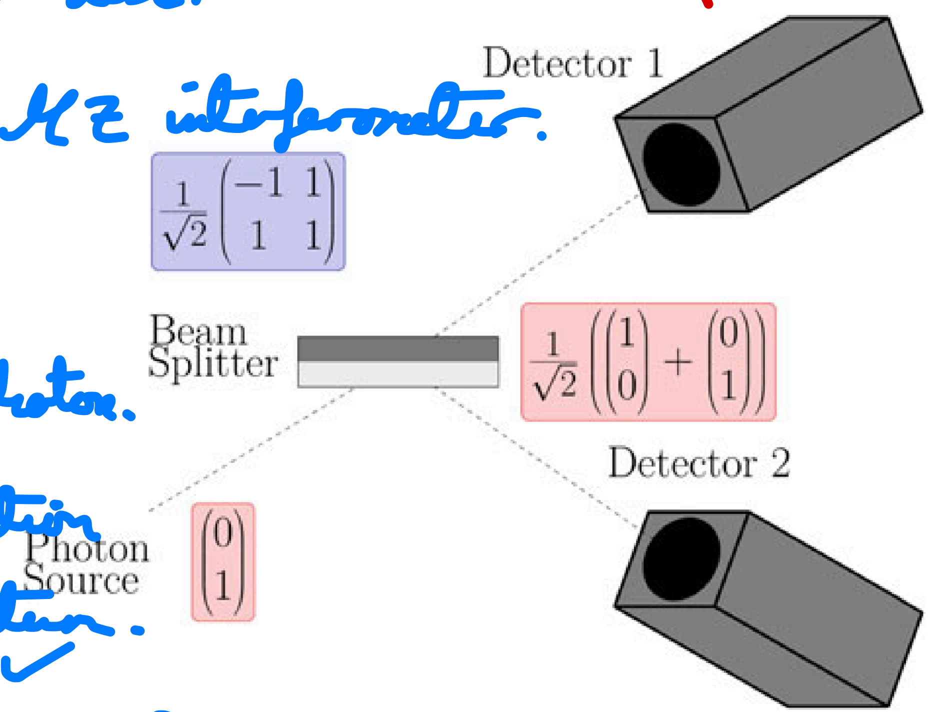
→ Considered B-S in classical & QM desc.

→ More splitters/phase-shifters/mirrors & MZ interferometer.

→ Compared interpretations:
Classical EM wave/particle vs. quantum photon.

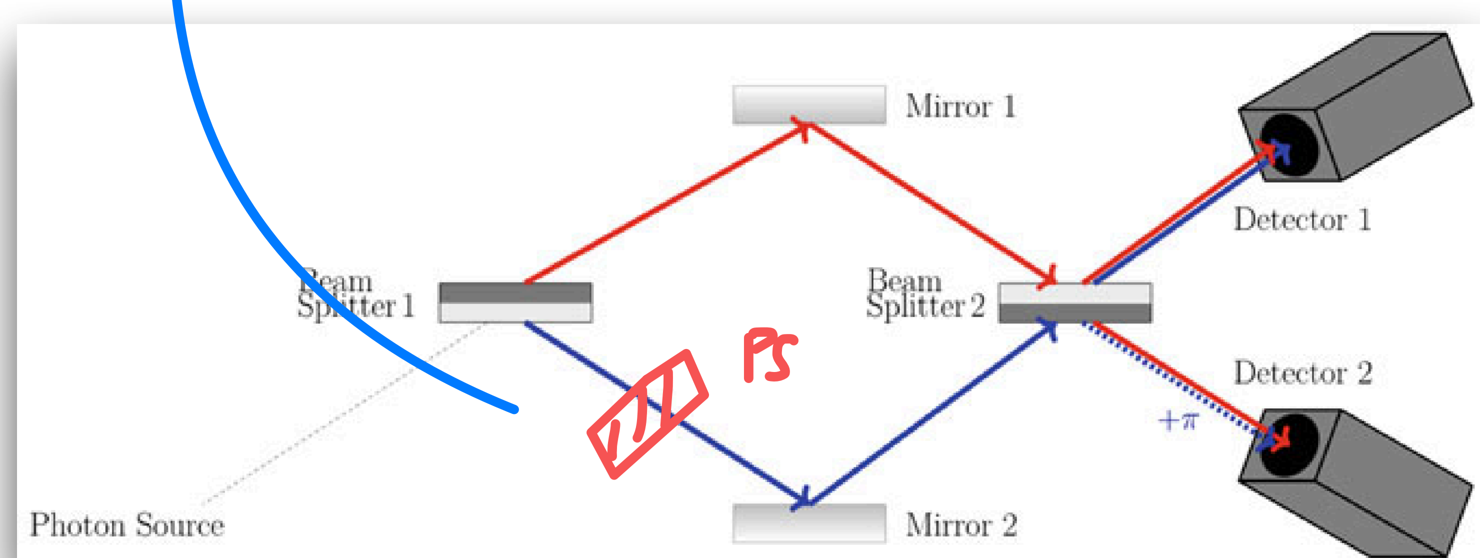
→ Found contradiction in classical explanation
w/ the 2 expt's & no contrad. w/ quantum ✓.

→ MZ ⊕ PS allows one to tune superposⁿ!



Beam Splitter

MZ Interferometer



Quantum spin

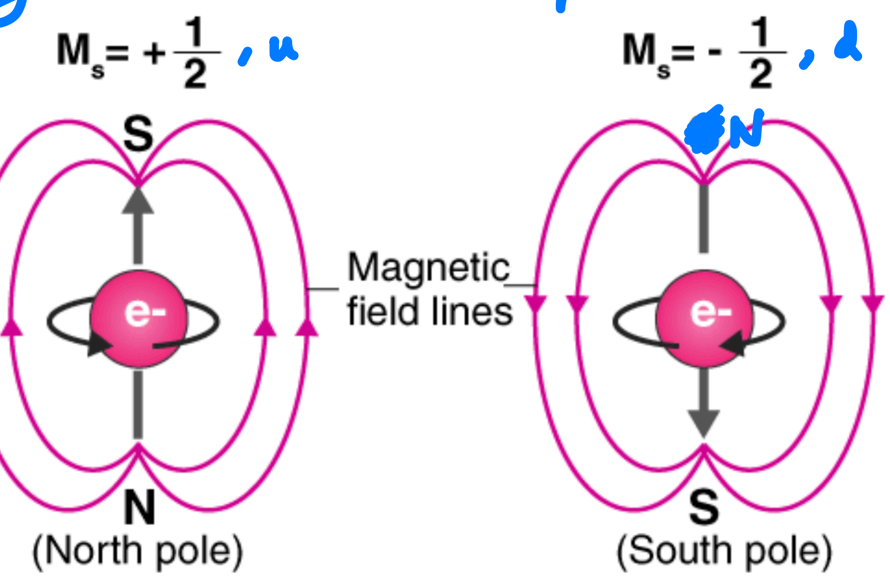
→ Fund. particles (e^-, q, γ, H, ...) have intrinsic property known as spin.

→ Classical spin (E&M):

- Rotating charge ⇒ B-field (RH rule) ⇒ magnet! E.g.? Earth!
- Dir^n of field lines ⟺ spin & RH rule ⇒ spin can point in any dir^n (cont.)
- If e^- is tiny rotating charge ⇒ cont. spin.

→ Quantum spin:

- Spin no longer cont. but discrete.
- E.g. e^- is either spin ↑ or spin ↓?
- How was this observed?



Stern-Gerlach Experiment

→ Recall: Total ang. mom. $\underline{J} = \underline{L} + \underline{S}$: $\begin{cases} \underline{L} = \underline{r}\times\underline{p} \leftrightarrow \text{orbital} \\ \underline{S} \leftrightarrow \text{spin} \end{cases}$

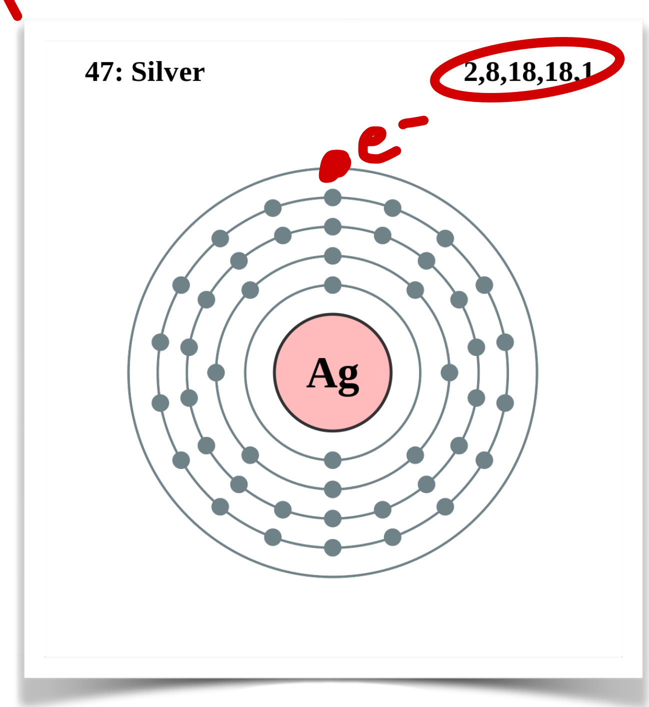
→ Require system equiv. to e^-

⇒ $\underline{L} = 0$ & $\underline{S} = \underline{S}_{e^-}$ ⟷ Silver atoms?

→ Ag^{47} has $\underline{L} = 0$ & all spins average out except for single e^- on outer shell.

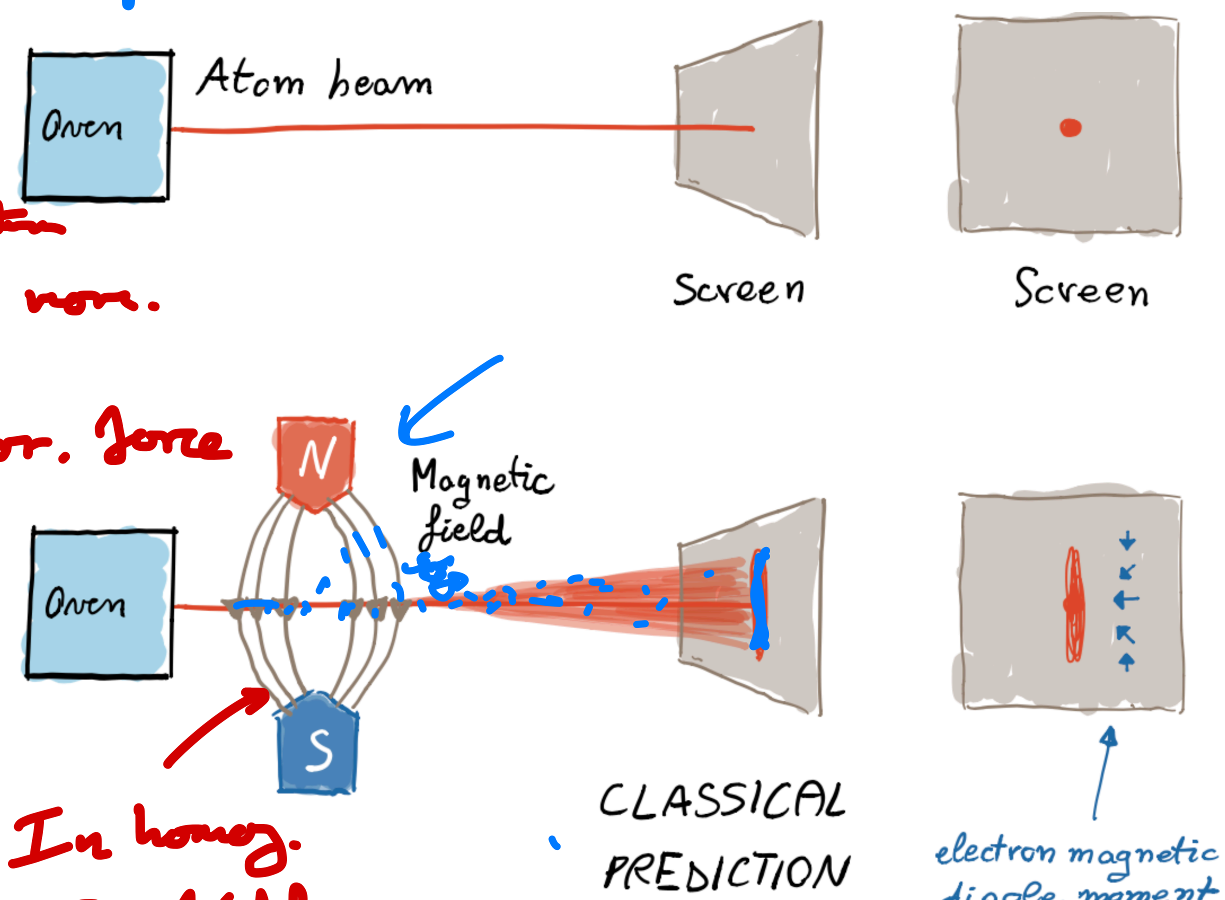
→ Spin ⇒ Spin magnetic moment ↔ $\left[\mu_s = \frac{e}{m_e c}\, \underline{J} \to \underline{S}\right]$ ↔ Displacement by ext. B-field

→ SG Experiment: Classical expect^n?



$5s_1$ → radially symm. ⇔ H atom & no orb. angular mom.

$N_p = N_e$ ⇒ no Lor. force ∴ neutral atom



In homog. B-field

Stern-Gerlach Experiment

→ <u>Quantum interpretation</u>: Spin quantized ⇒ $s_z = +\frac{1}{2}$ <u>or</u> $-\frac{1}{2}$

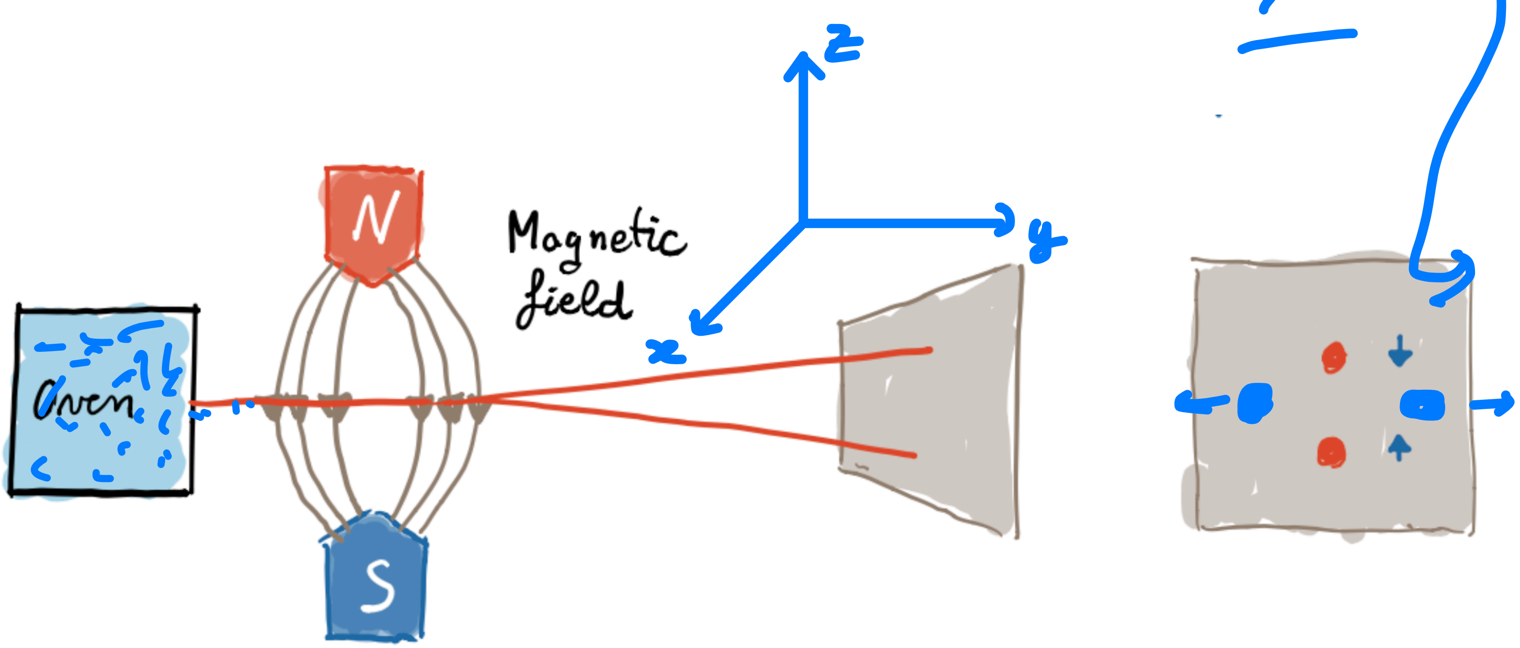
→ Can write <u>state</u> as: $|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$: $|\alpha|^2+|\beta|^2=1$

$|\uparrow\rangle \to \begin{pmatrix}1\\0\end{pmatrix}$, $|\downarrow\rangle \to \begin{pmatrix}0\\1\end{pmatrix}$

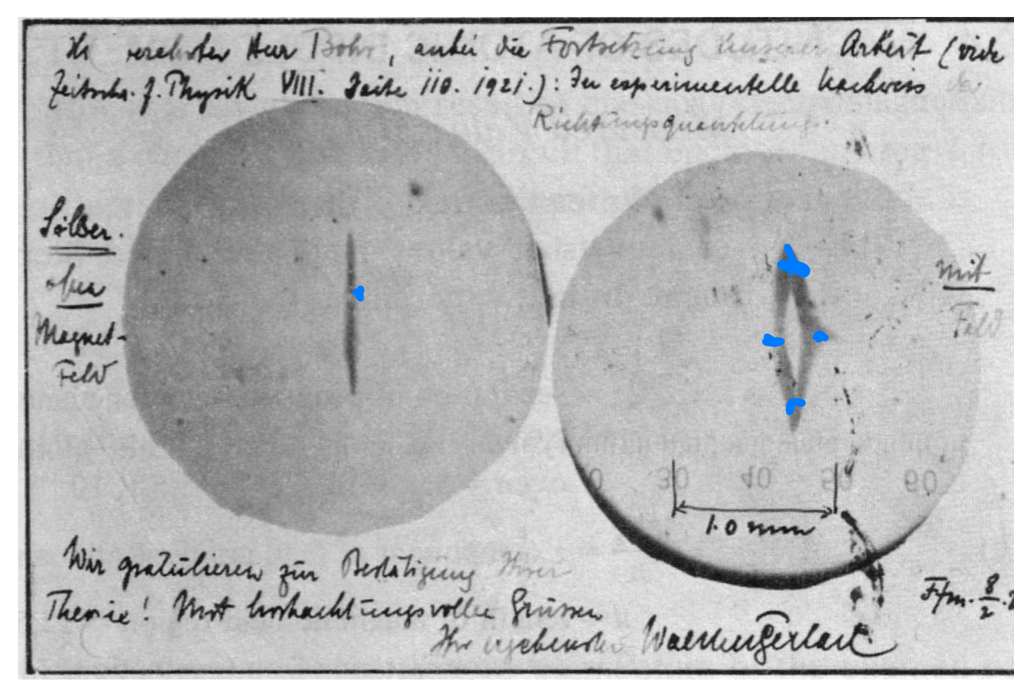
→ <u>Q</u>: What would happen if <u>B</u>-field aligned in x-dirⁿ?

⇒ <u>Measurement</u>: $s_x = \begin{cases} +\frac{1}{2}, & |\alpha|^2 \\ -\frac{1}{2}, & |\beta|^2 \end{cases}$

<u>A</u>:



OBSERVATION

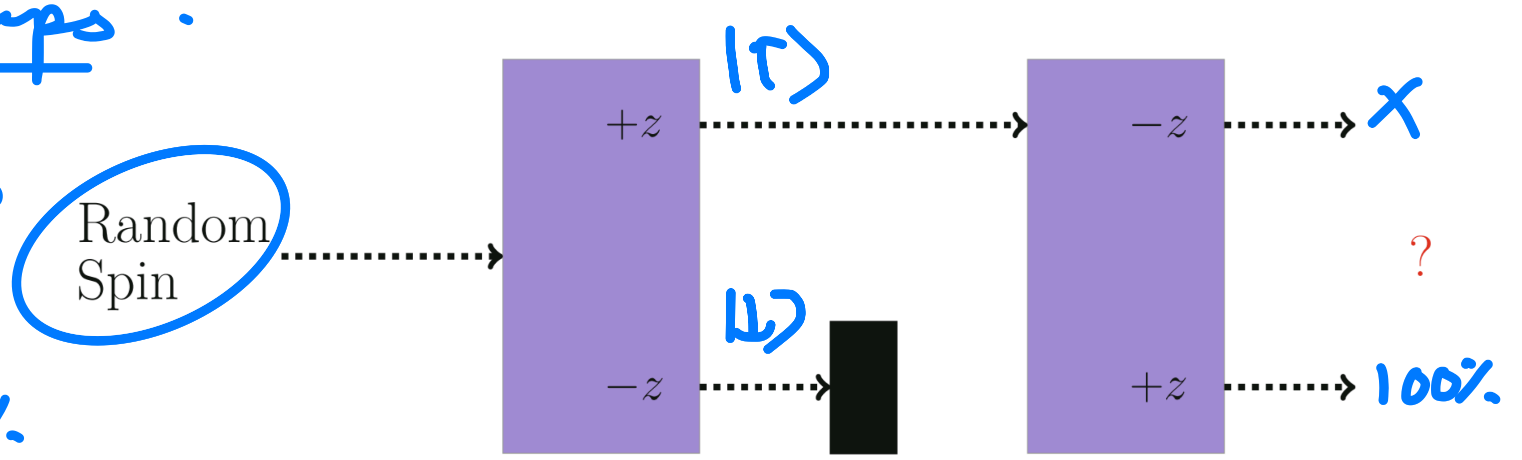


Quantum Explanation

→ From this experiment alone we will discover quantum #'s (e.g. S_z), non-commuting operators & simultaneous measurement.

→ Consider SG-setups:

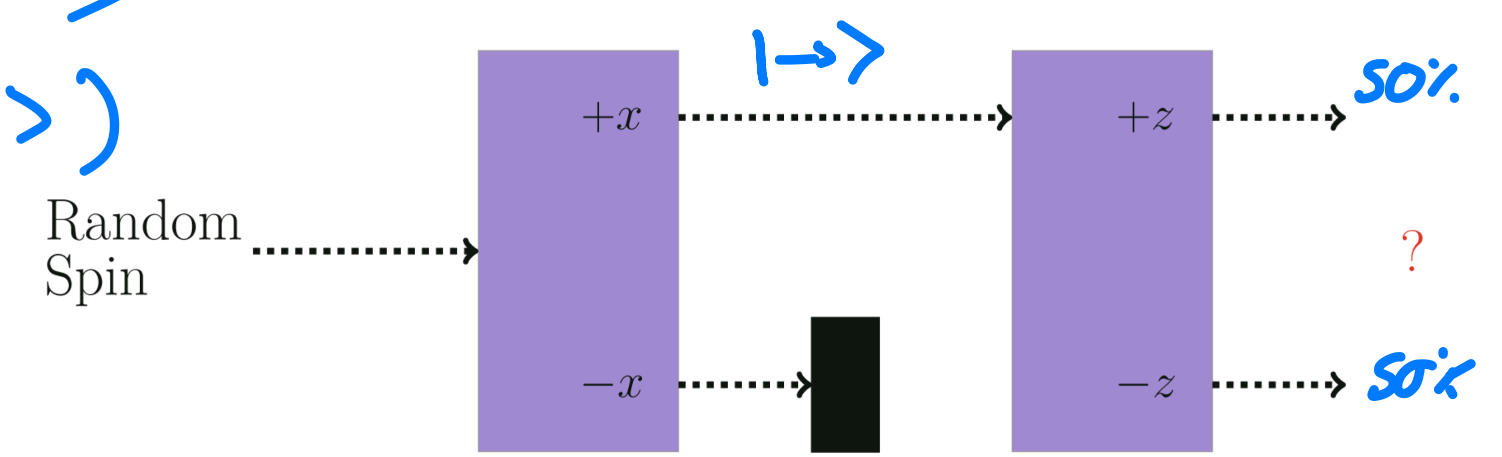
1) $|\psi\rangle = \alpha_z|\uparrow\rangle + \beta_z|\downarrow\rangle$



A : $S_z = +\frac{1}{2}$ 100%

2) $|\psi\rangle = \alpha|\rightarrow\rangle + \beta|\leftarrow\rangle$

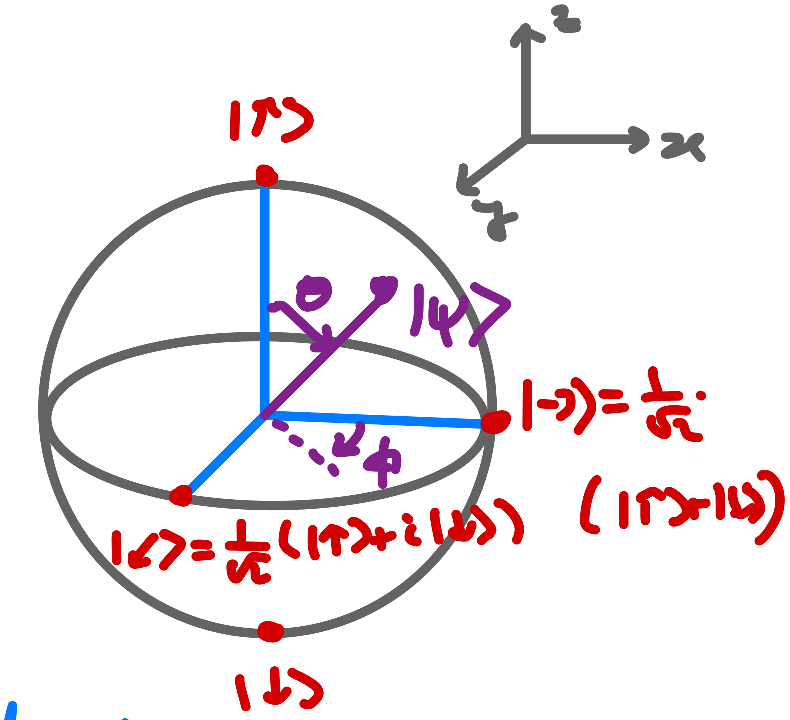
& $|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$



What does this mean?

Is it both S_x & $S_z^{\pm}$?

Quantum Decription



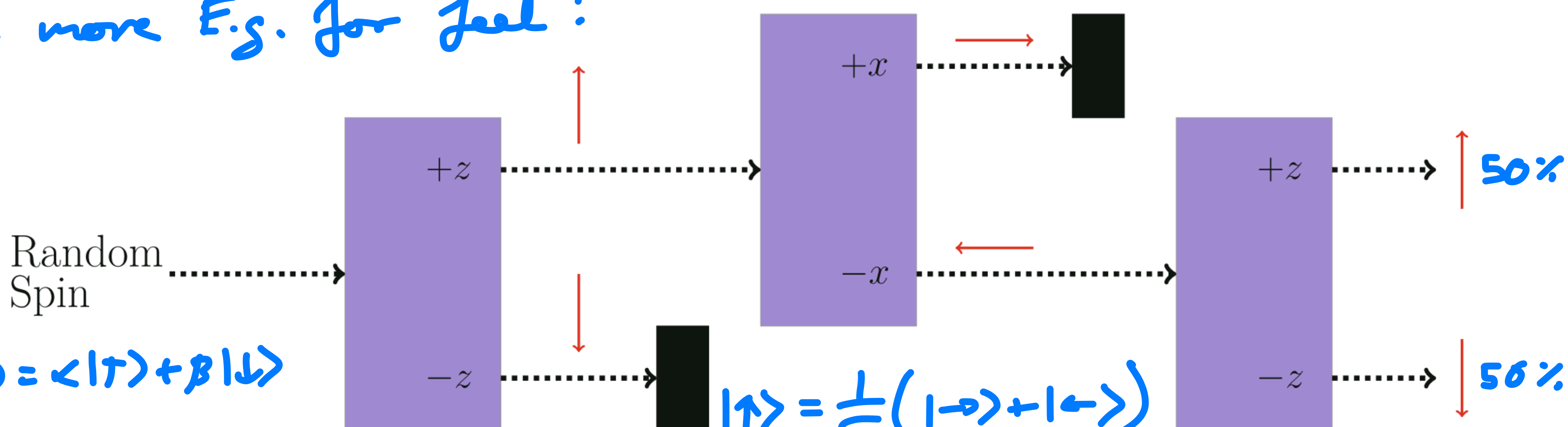
→ Recall: Bloch sphere, can write any qubit (or 2-state) as:

$$|\psi\rangle = \cos\theta/2\,|\uparrow\rangle + e^{i\varphi}\sin\theta/2\,|\downarrow\rangle$$

→ Eigenstates $|\uparrow,\downarrow\rangle$ & $|\rightarrow,\leftarrow\rangle$ lie in orthogonal bases.

$\Leftrightarrow$ Operators $\hat{S}_x$ & $\hat{S}_z$ are non-commuting.

→ One more E.g. for feel:



$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$

$|\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|\rightarrow\rangle + |\leftarrow\rangle\right)$

Check!

$|\leftarrow\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle - |\downarrow\rangle\right)$

Check Bloch sphere!

Basis Transformations

→ What operators represent the S-G 'gates'?

→ Consider z-basis & we can represent:

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \& \quad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

In this basis: $\hat{S}_{x,y,z} = \sigma_{x,y,z}/2$: $\begin{cases} \sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \\ \sigma_y = \begin{pmatrix}0 & i\\-i & 0\end{pmatrix} \\ \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \end{cases}$

(unitary ✓ (check))

→ Operator on a state in a basis must be diagonal ⇒ in z-basis ⇔ $\hat{S}_z$.

→ Measurement of spin: $\begin{cases} \langle\uparrow|\hat{S}_z|\uparrow\rangle = (1\ 0)\frac{1}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = +\frac{1}{2} ✓ \\ \langle\downarrow|\hat{S}_z|\downarrow\rangle = (0\ 1)\frac{1}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2} ✓ \\ \langle\uparrow|\hat{S}_{x,y}|\uparrow\rangle = \langle\downarrow|\hat{S}_{x,y}|\downarrow\rangle = 0 ✓ \text{(check)} \end{cases}$

→ In general a measurement in QM of a property of a quantum state: $\langle\psi|\hat{O}|\psi\rangle$ = quantum #

Basis Transformations

$\rightarrow$ Simultaneity : $\underbrace{[\hat{O}_1, \hat{O}_2]}_{\text{commutator}} |\psi\rangle \equiv (\hat{O}_1\hat{O}_2 - \hat{O}_2\hat{O}_1)|\psi\rangle$

If $[\hat{O}_1, \hat{O}_2] \neq 0 \Rightarrow$ cannot be measured sim'ly.
$\leftrightarrow$ operators orthogonal in any basis.

$\rightarrow$ $\hat{J}_x$ & $\hat{J}_z$ cannot be measured (or known) simultaneously!

$$[\hat{S}_x, \hat{S}_z] = \frac{1}{4}\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \neq 0_{2\times 2} \checkmark$$

check!

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\Rightarrow s_{x,y,z} \leftrightarrow$ unique quantum #'s

$\rightarrow$ Other non-commuting observables?
$\underbrace{x \,\&\, p}$, t & E, etc...

$\Rightarrow$ HUP: $\Delta x \Delta p > \hbar/2$

Summary of lectures

Quantum **superposition** means a state can be linearly composed of two or more **basis states**

Basis states must be chosen with **observable** in mind, always pick appropriate basis to simplify your life! (useful for Q.C.)

Measurements defined by **expectation value** of an operator

Not all observables are measurable **simultaneously**, to check this check that the operators commute when applied to a state

Extras (CSCO)