Sudah

1-qubit gates

Two parts: "1-qubit", "gate"
1 2

1-qubit

1: a single

qubit → $\boxed{\text{qu}}$antum $\boxed{\text{bit}}$

* either 0 or 1
* two values
* binary

* new behavior

QUANTUM

⟹ allows for "Superposition"

the ability to be in multiple states "at once".

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

basis vectors

$|\alpha|^2 + |\beta|^2 = \alpha\alpha^* + \beta\beta^* = 1$

$|\alpha|^2$ = Probability of being in $|0\rangle$

$|\beta|^2$ = probability of being in $|1\rangle$

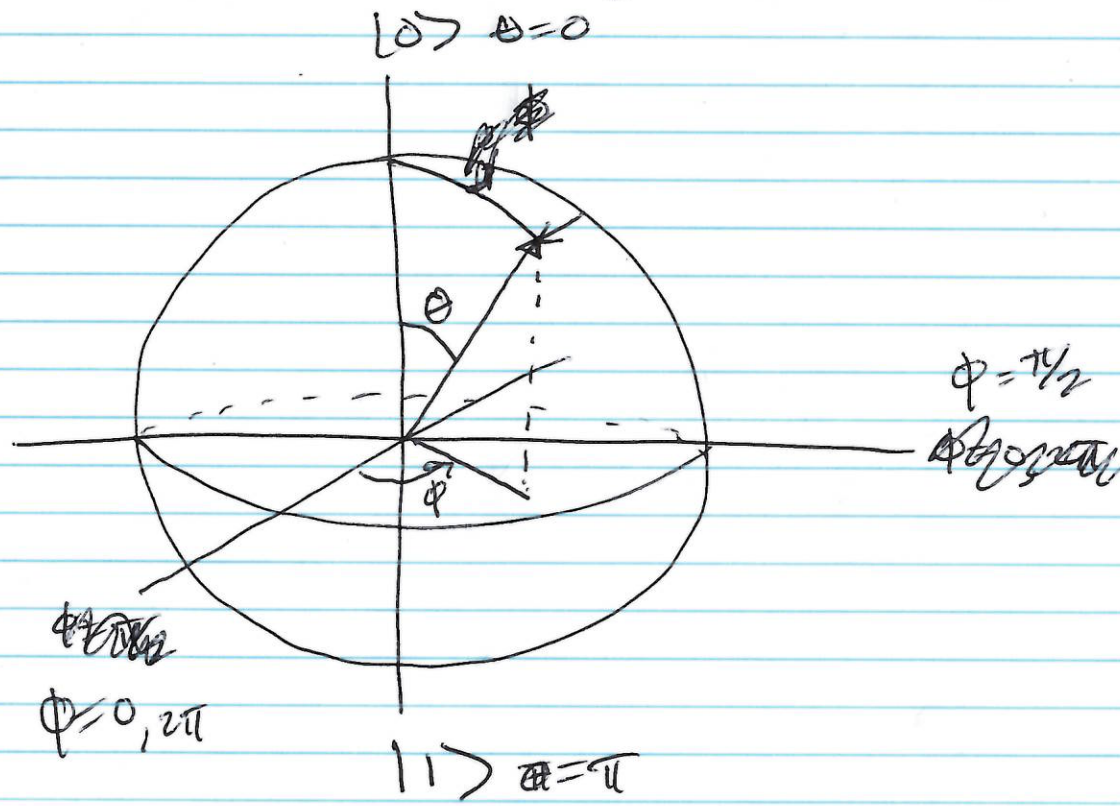
* $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$

- normalize for all θ, ϕ

$\theta \in [0, \pi]$

θ, ϕ are two angles

$\phi \in [0, 2\pi]$



our states: 0 and 1

or

$|0\rangle$ and $|1\rangle$

* these are the kets!

* written this way $|\ \rangle$ just tells us whatever is inside is the "state" of the system.

e.g. $|red\rangle$, $|up\rangle$, $|3\rangle$

$|left\rangle$ $|a+b\rangle$

For a qubit: $|0\rangle$, $|1\rangle$

like Stern-Gerlach $|up\rangle, |dn\rangle$

State of qubit $= |0\rangle + |1\rangle$

$|\psi\rangle = |0\rangle + |1\rangle$

* not done, this is without looking

* if I look at it I can only get $|0\rangle$ & $|1\rangle$ $\Rightarrow$ probabilities

$|\psi\rangle \overset{?}{=} p_0|0\rangle + p_1|1\rangle$ no... :(

* $|\psi\rangle \stackrel{?}{=} \sqrt{p_0}\,|0\rangle + \sqrt{p_1}\,|1\rangle$ no...!!

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

with $aa^* + bb^* = 1$ ...

* so a & b are complex

* a & b ~ $\sqrt{p_0}$ & $\sqrt{p_1}$
since $aa^* = p_0$ & $bb^* = p_1$
but $\sqrt{p_0} \neq a$ ... etc.

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

$$\cancel{\Rightarrow = \cos\theta|0\rangle + e^{i\phi}\sin\theta|1\rangle}$$

gate

* a gate is: something you do,
  an operation, an act
  an interaction,
  etc.. to a system.
* Doing nothing can be a gate too.

* in math:

  if A is a gate, & $|\psi\rangle$ the state of a system

  $$A|\psi\rangle = |\phi\rangle$$

  "acts on" — a new state for the same system

if $|\psi\rangle = a|0\rangle + b|1\rangle$ with $|a|^2 + |b|^2 = 1$

$|\phi\rangle = c|0\rangle + d|1\rangle$ and $|c|^2 + |d|^2 = 1$

* A changes a & b into c & d
  while keeping $|c|^2 + |d|^2 = 1$

* this type of change is probability conserving
  → the sums of the probabilities always equals 1

* this is a <u>Unitary</u> transformation

* we write $|\psi\rangle$ with $|0\rangle$ & $|1\rangle$
  what about A?

$$A = \alpha|0\rangle\langle 0| + \beta|0\rangle\langle 1| + \gamma|1\rangle\langle 0| + \delta|1\rangle\langle 1|$$

* these "backward" $\langle 0|$ & $\langle 1|$ are "looking" for $|0\rangle$ or $|1\rangle$

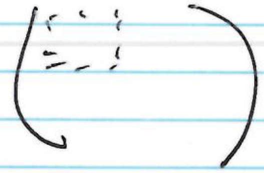
$$A|\psi\rangle = \left(\alpha|0\rangle\langle 0| + \beta|0\rangle\langle 1| + \gamma|1\rangle\langle 0| + \delta|1\rangle\langle 1|\right)\left(a|0\rangle + b|1\rangle\right)$$

$$= a\left(\alpha|0\rangle\cancelto{1}{\langle 0|0\rangle} + \beta|0\rangle\cancelto{0}{\langle 1|0\rangle} + \gamma|1\rangle\cancelto{1}{\langle 0|0\rangle} + \delta|1\rangle\cancelto{0}{\langle 1|0\rangle}\right)$$

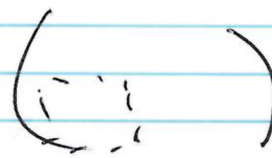
$$+ b\left(\alpha|0\rangle\cancelto{0}{\langle 0|1\rangle} + \beta|0\rangle\cancelto{1}{\langle 1|1\rangle} + \gamma|1\rangle\cancelto{0}{\langle 0|1\rangle} + \delta|1\rangle\cancelto{1}{\langle 1|1\rangle}\right)$$

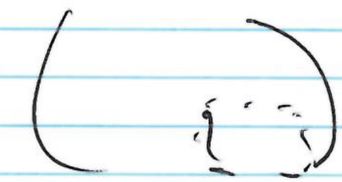
$$= a(\alpha|0\rangle + \gamma|1\rangle) + b(\beta|0\rangle + \delta|1\rangle)$$

5.5

$|0\rangle\langle 0| \longleftrightarrow$ 

$|0\rangle\langle 1| \longleftrightarrow$ 

$|1\rangle\langle 0| \longleftrightarrow$ 

$|1\rangle\langle 1| \longleftrightarrow$ 

$$= \underbrace{(a\alpha + b\beta)}_{c}|0\rangle + \underbrace{(a\gamma + b\delta)}_{d}|1\rangle$$

$$= c|0\rangle + d|1\rangle$$

* how do we know $|c|^2 + |d|^2 = 1$?

→ $\alpha, \beta, \gamma, \delta$ have to be chosen so that ~~AA^*~~ $AA^\dagger = A^\dagger A = \mathbb{1}$

Matrix & vectors

* the above notation can be simplified into std. matrix / vectors.

* this recreates this

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad |\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

Now

$$A|\psi\rangle = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a + \beta b \\ \gamma a + \delta b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

* lets look at some examples:

$|\psi\rangle = |0\rangle$ probabilities Σ to 1 ✓

$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ ✓

$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{1}{2}|1\rangle$ X

$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$ ✓

$|\psi\rangle = \cos\theta\,|0\rangle + e^{i\phi}\sin\theta\,|1\rangle$ ✓

$\hookrightarrow \cos^2\theta + (e^{i\phi}\sin\theta)\times(e^{-i\phi}\sin\theta)$

$= \cos^2\theta + \sin^2\theta = 1$

for all θ & ϕ. ✓

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad AA^\dagger = A^\dagger A = 1 \checkmark$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

this is $\mathbb{1}$

Pauli matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$ this X gate

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1\cdot 1+1\cdot 1 & 1\cdot 1+1\cdot -1 \\ 1\cdot 1+-1\cdot 1 & 1\cdot 1+-1\cdot -1 \end{pmatrix}$$

H

called Hadamard gate $= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ Z gate

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

Pauli matrix

matrix B

* real crazy, throw up in an exponential

$$A = e^{iB}$$

$a^x \cdot a^y = a^{x+y}$

$$AA^{\dagger} = e^{iB} \cdot e^{-iB^{\dagger}}$$ this is almost

* you can't do this, but if you could . . .

$$e^{i(B-B^{\dagger})}$$ which would be $\mathbb{1}$ if $B = B^{\dagger}$

since $e^{i0} = \mathbb{1}$

"Hermitian"

in fact a unitary matrix can be written as $A = e^{iB}$ <u>if</u> $B = B^{\dagger}$.

then $e^{iB} e^{-iB} = e^{-iB} e^{iB} = e^{0} = \mathbb{1}$ ✓

The action of these gates:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

$$\rightarrow \big(|0\rangle\langle 1| + |1\rangle\langle 0|\big)\big(a|0\rangle + b|1\rangle\big)$$

$$= |0\rangle \overset{0}{\cancel{\langle 1|0\rangle}} a + |0\rangle \overset{1}{\cancel{\langle 1|1\rangle}} b$$

$$+ |1\rangle \overset{1}{\cancel{\langle 0|0\rangle}} a + |1\rangle \overset{0}{\cancel{\langle 0|1\rangle}} b$$

$$= b|0\rangle + a|1\rangle$$

$$X|0\rangle = |1\rangle \quad \& \quad X|1\rangle = |0\rangle$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\big)$$

$$= \frac{1}{\sqrt{2}}\Big(\big[|0\rangle + |1\rangle\big]\langle 0| + \big[|0\rangle - |1\rangle\big]\langle 1|\Big)$$

$$= \overbrace{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)}^{|+\rangle}\langle 0|$$

$$+ \underbrace{\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)}_{|-\rangle}\langle 1|$$

$$= |+\rangle\langle 0| + |-\rangle\langle 1|$$

$$H|\psi\rangle = (|+\rangle\langle 0| + |-\rangle\langle 1|)(a|0\rangle + b|1\rangle)$$

$$= a|+\rangle + b|-\rangle$$

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$Z|\psi\rangle = (|0\rangle\langle 0| - |1\rangle\langle 1|)(a|0\rangle + b|1\rangle)$$

$$= a|0\rangle - b|1\rangle$$

$$Z|0\rangle = |0\rangle$$

$$Z|1\rangle = -|1\rangle$$

$$\begin{tikzcd}
{|0\rangle} \arrow[r, leftrightarrow, "X"] \arrow[d, leftrightarrow, "H"'] & {|1\rangle} \arrow[d, leftrightarrow, "H"] \\
{|+\rangle} \arrow[r, leftrightarrow, "Z"] & {|-\rangle}
\end{tikzcd}$$

more advanced gates

* X, Y, and Z are all unitary & hermitian

$$\hookrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

* since they are Hermitian they can be used to make unitaries

Remember ~~B~~ ~~Z~~ $A = e^{iB}$ is unitary
with $B = B^{\dagger}$

So lets consider

$$R_x = e^{iX\theta} \qquad R_y = e^{iY\theta} \qquad R_z = e^{iZ\theta}$$

* these are rotations about the x, y, and z axis

* as matrices

* diagonal

$$R_z = e^{iZ\theta} = e^{i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\theta} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

* Euler identity

$$R_X = e^{iX\theta} = \cos\theta\,\mathbb{1} + iX\sin\theta$$

$$= \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$$

$$R_Y = e^{iY\theta}$$

$$= \cos\theta\,\mathbb{1} + iY\sin\theta$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

* what about a general rotation about some arbitrary axis?

$$R_{\hat{n}} = e^{i\hat{n}\cdot\vec{\sigma}\theta} \qquad \hat{n} = (n_1, n_2, n_3) \qquad \hat{n}\cdot\hat{n} = 1$$

* on an actual machine, we cannot do anything, because the machine has a fixed design.

* usually a few gates available:

Usually e.g.

IBM

X gate, H gate, $\sqrt{X}$ gate, $R_z(\lambda)$

others IonQ

$$GPI = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$$

$$GPI2 = \begin{pmatrix} 1 & -ie^{-i\phi} \\ -ie^{i\phi} & 1 \end{pmatrix} \frac{1}{\sqrt{2}}$$

$$R_z = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$$

* So how can we get any rotation?

Ultimately Solovay-Kitaev, but as a step

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$$

10.5

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = |0\rangle\langle 0| + i\,|1\rangle\langle 1|$$

$$S^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

$$S S^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

$$T = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} = e^{-i\pi/8}|0\rangle\langle 0| + e^{i\pi/8}|1\rangle\langle 1|$$

$$T^2 = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}(1-i) & 0 \\ 0 & \frac{1}{\sqrt{2}}(1+i) \end{pmatrix}$$

$$T T^{\dagger} = \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix}\begin{pmatrix} e^{i\pi/8} & 0 \\ 0 & e^{-i\pi/8} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark$$

Quantum circuits I

- we saw X, Z, H, etc...
- if I want to act on many qubits with 1-qubit gates, ~~the~~ the math gets messy

$\Longrightarrow$ circuit diagram

<u>one qubit</u>

- a line

$|0\rangle$ ——————— $\rightarrow$ a qubit exists & nothing happens $\boxed{\mathbb{1} \text{ gate}}$

zero state

$|0\rangle$ —$\boxed{X}$— $|1\rangle$ $\rightarrow$ 1 state

$\rightarrow$ X gate acts on $|0\rangle$

* notice we act to the left in these diagrams!!!

$|0\rangle$ — $\boxed{H}$ — $|+\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$\rightarrow$ Hadamard gate

$|0\rangle$ — $\boxed{X}$ — $\boxed{H}$ — $|-\rangle$

$\rightarrow$ flip $|0\rangle$ to $|1\rangle$

then Hadamard to $|-\rangle$

$|0\rangle$ — $\boxed{H}$ — $|+\rangle$

$|0\rangle$ — $\boxed{H}$ — $|+\rangle$

$$\rightarrow |0\rangle|0\rangle \xrightarrow{HH} \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right)$$

$$= |+\rangle|+\rangle$$

$|0\rangle$ — $\boxed{H}$ — $\boxed{X}$ — ...

$|0\rangle$ — $\boxed{Z}$ — $\boxed{H}$ — ...

$\vdots \qquad \vdots$

$|0\rangle$ ———— $\boxed{X}$ — ...