Entanglement

Introduction:

Quantum entanglement = correlation between multiple quantum objects (here usually qubits)

→ hidden quantum information with no classical equivalent

Example: • consider two classical coins:
possible outcomes: $\underbrace{HH,\ HT,\ TH,\ TT}_{\text{probability of each} = 25\%}$

• analogue with two quantum coins:

$$|\psi\rangle = \frac{1}{2}|HH\rangle + \frac{1}{2}|HT\rangle + \frac{1}{2}|TH\rangle + \frac{1}{2}|TT\rangle$$

• but can also prepare quantum coins in state like

$$|\psi\rangle = \frac{1}{\sqrt{2}}|HH\rangle + \frac{1}{\sqrt{2}}|TT\rangle$$

⇒ possible outcomes only HH or TT

Imagine observing first coin first:

If H ⇒ second coin must later be measured as H
If T ⇒ second coin <u>must</u> later be measured as T

But the outcome of the first coin was random.

→ HOW does the second coin know what to do???

Do the coins communicate??

NO! Still works if distance too large for light-speed signal between measurements

Are the outcomes secretly predetermined in a correlated way??

NO! → more about this later ("hidden variable theories")

Instead: coins in this "entangled" state share non-classical information

Multi-qubit states:

- General one-qubit state: $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ with $\alpha_0, \alpha_1 \in \mathbb{C}$ and $|\alpha_0|^2 + |\alpha_1|^2 = 1$

- Generalization to two qubits:

$$|\psi\rangle = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle$$

orthonormal basis states of two-qubit system

with $\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11} \in \mathbb{C}$ and $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$

(straightforward generalization to more than two qubits)

Probability of measuring one of these four outcomes
= corresponding $|\alpha|^2$

Example: $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

$$\text{Prob}(\text{both}=0) = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

$$\text{Prob}(\text{both}=1) = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\text{Prob}(\text{first}=1, \text{second}=0) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$\text{Prob}(\text{first}=0, \text{second}=1) = 0$$

Now measure <u>only</u> the first qubit:

$$\text{Prob}(\text{first}=1) = \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{1}{2}$$

State of two-qubit system after measuring the first qubit as 1:

→ pick contributions with 1 in first place:

$$\rightarrow |\psi'\rangle \propto \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

"proportional to"

After adjusting overall normalization:

$$|\psi'\rangle = \frac{1}{\sqrt{2}}|10\rangle - \frac{1}{\sqrt{2}}|11\rangle$$

(change of state can be expressed via projection operator

$$|\psi'\rangle = \frac{\langle 1_A|}{\sqrt{\langle\psi|1_A\rangle\langle 1_A|\psi\rangle}}|\psi\rangle \;)$$

Exercise:

Start from $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$:

- Probability of measuring first qubit as 0?
- State after measurement?

Start over from $|\psi\rangle$:

- Probability of measuring second qubit as 0?
- State after measurement?

Start over from $|\psi\rangle$:

- Probability of measuring second qubit as 1?
- State after measurement?

Answers:

$\text{Prob (first} = 0) = \frac{1}{2}$, $|\psi'\rangle = |00\rangle$

$\text{Prob (second} = 0) = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4}$

$|\psi'\rangle = \frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{2}|10\rangle\right) = \sqrt{\frac{2}{3}}|00\rangle + \frac{1}{\sqrt{3}}|10\rangle$

$\text{Prob (second} = 1) = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}$

$|\psi'\rangle = |11\rangle$

What makes a two-qubit state entangled?

Some two-qubit states can be factored into a term describing the first qubit times a term describing the second qubit.

Example: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle = |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$

tensor product

Such a state is called separable.
If a state is <u>not</u> separable, it is <u>entangled</u>.

I.e. a two-qubit state $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$ is entangled if it <u>cannot</u> be written as

$$\underbrace{(\alpha_0|0\rangle + \alpha_1|1\rangle)}_{\text{first qubit}} \otimes \underbrace{(\beta_0|0\rangle + \beta_1|1\rangle)}_{\text{second qubit}}$$

with some $\alpha_0, \alpha_1, \beta_0, \beta_1$.

Is our example from earlier entangled?

Let's check: Ansatz for product state:

$$(\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle)$$

$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

$$\stackrel{!}{=} \frac{1}{\sqrt{2}}|00\rangle + 0|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

$\Rightarrow \alpha_0\beta_1 = 0 \Rightarrow \alpha_0 = 0$ or $\beta_1 = 0$

If $\alpha_0 = 0 \Rightarrow \alpha_0\beta_0 = 0$ ↯

If $\beta_1 = 0 \Rightarrow \alpha_1\beta_1 = 0$ ↯

$\Rightarrow |\psi\rangle$ cannot be written as a product state.
$\Rightarrow |\psi\rangle$ is entangled.

Exercise:
Are the following states entangled?

1) $|\psi_1\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$

2) $|\psi_2\rangle = \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle$

3) $|\psi_3\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

4) $\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$

Answers:

1) $|\psi_1\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |1\rangle \Rightarrow$ not entangled

2) Product state Ansatz leads to $\alpha_0 \beta_0 = 0$
$\Rightarrow \alpha_0 = 0$ or $\beta_0 = 0$
$\Rightarrow$ contradiction with $\alpha_0\beta_1$ and $\alpha_1\beta_1$ both $\neq 0$
$\Rightarrow$ entangled

3) Product state Ansatz leads to $\alpha_0\beta_1 = 0$
$\Rightarrow$ contradiction with $\alpha_0\beta_0 \neq 0$ and $\alpha_1\beta_1 \neq 0$
$\Rightarrow$ entangled

4) $|\psi_4\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$
$\Rightarrow$ not entangled

Separability of state can be difficult to determine in practice.
→ Other entanglement criteria that are easier to compute, but none necessary <u>and</u> sufficient for entanglement

Simple example: probability-based criterion:

If measuring one qubit changes the probability distribution of the other, their state is entangled.

Example: $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

If first = 0, Prob (second = 0) = 100%, Prob (second = 1) = 0%
If first = 1, Prob (second = 0) = 0%, Prob (second = 1) = 100%

Probability-based criterion is <u>sufficient</u>, but not <u>necessary</u> for entanglement

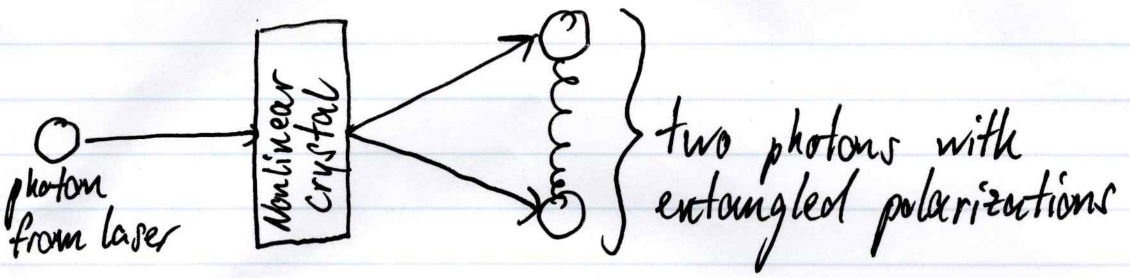
E.g. $|\psi\rangle = \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$
is entangled (i.e. not separable), but measuring one qubit does not change the probability distribution of the other.

Where does entanglement come from?

→ Interactions between quantum systems

Examples:
- electrons in atoms (entangled spins)
- particle decays with conservation laws (spins, momenta etc.), e.g. $h \to \gamma\gamma$

Useful way to produce entangled photon pairs:
spontaneous parametric down-conversion



Quantum mechanics vs hidden variable theories

Outcome of one qubit changing the probability distribution of the second one appears like "spooky action at a distance" when the qubits are spatially separated

Alternative hypothesis: Local "hidden variables" are set when qubits interact before they separate and predetermine outcomes of ~~measurements~~ measurements (Einstein, Podolsky, Rosen, 1935)

i.e. QM is replaced by a "local, realistic theory"

Results do not depend on causally disconnected regions

Physical quantities have reality whether or not they are measured

This alternative hypothesis is testable!

John Stuart Bell, 1964:

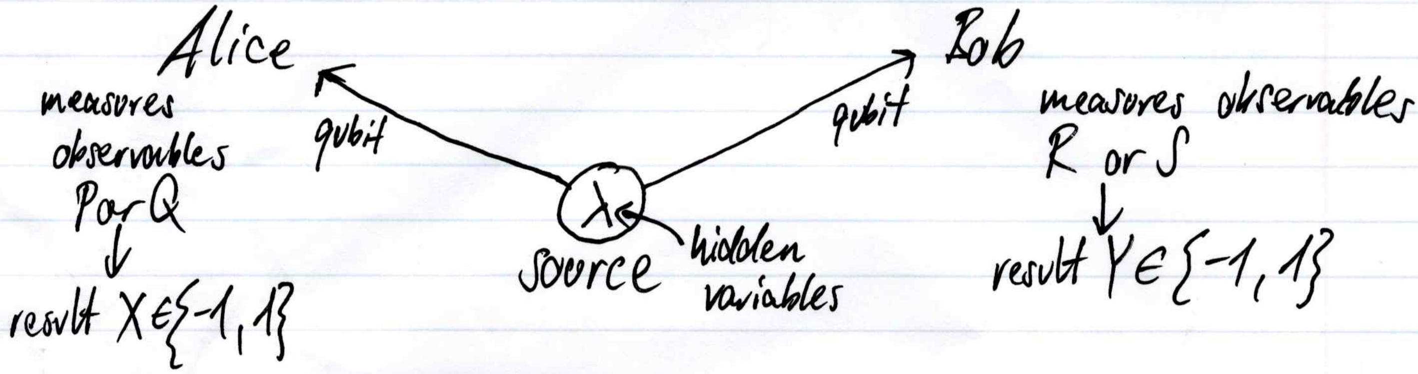
Can derive inequalities for the results of certain measurements that must hold for any theory with local hidden variables but can be violated by quantum mechanics.

Since then many variations ("Bell-type inequalities")

Easier to test experimentally than Bell's original inequality:

CHSH (Clauser, Horne, Shimony, Holt) inequality, 1969:

Setup:



Send many rounds of qubit pairs to Alice and Bob.
Consider expectation value of correlation between Alice's and Bob's measurement result:

If hidden variable theory true,

$$E(a, b) = \int_{\Lambda} X(a, \lambda)\, Y(b, \lambda)\, \rho(\lambda)\, d\lambda$$

- E: expectation value
- a: property Alice decides to measure in a particular round ($a \in \{P, Q\}$)
- b: property Bob decides to measure in a particular round ($b \in \{R, S\}$)
- Λ: range of hidden variable λ
- $X(a, \lambda)$: outcome of Alice's measurement
- $Y(b, \lambda)$: outcome of Bob's measurement
- $\rho(\lambda)$: distribution of hidden variable λ

Consider quantity

$$CHSH \equiv E(P,R) + E(Q,R) + E(P,S) - E(Q,S)$$

Can show that $|CHSH| \leq 2$ by using expression for expectation values from above.

(easy to convince yourself of this by ~~[illegible]~~ using that $X, Y \in \{-1, 1\}$ and $\int \rho(\lambda) d\lambda = 1$)

But QM can violate this bound!

For example for $|\Psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

Alice's qubit

Bob's qubit

and $P = \sigma_z^A$, $Q = \frac{1}{\sqrt{2}}\sigma_z^A + \frac{1}{\sqrt{2}}\sigma_x^A$,

$R = \sigma_z^B$, $S = \frac{1}{\sqrt{2}}\sigma_z^B - \frac{1}{\sqrt{2}}\sigma_x^B$

spin operator in z-direction

spin operator in x-direction

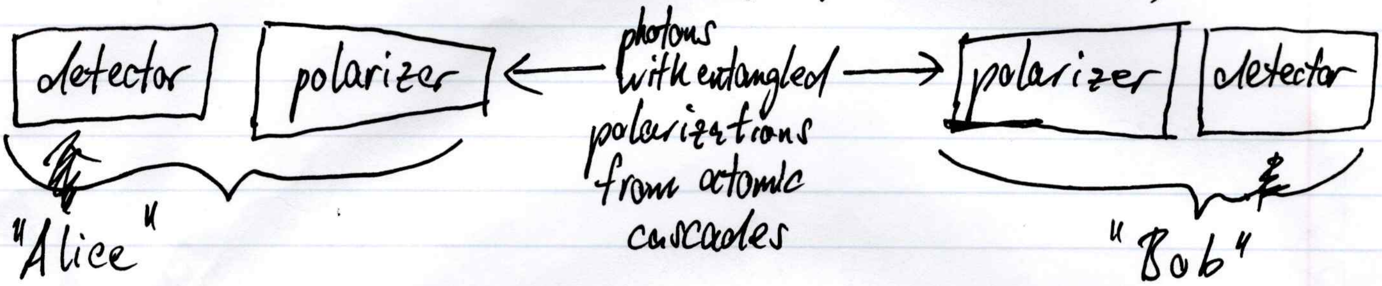
Then in QM: $E(P,R) = \langle\Psi|\sigma_z^A \otimes \sigma_z^B|\Psi\rangle$

analogous for other expectation values above

Find: $|CHSH| = 1 + \sqrt{2} > 2$!!!

Early experimental tests:

- Freedman & Clauser, 1972: (Phys. Rev. Lett. 28)



- Aspect et al., 1981 (Phys. Rev. Lett. 47)

Experiments agree with QM and violate bounds on local hidden variable theories!

Last remaining loopholes closed in 2015 (Hensen et. al, Nature 526)

⇒ Nature is quantum mechanical.
Behavior of entangled objects cannot be explained by hypothetical local hidden variables.

Summary/outlook:

Entanglement: = sharing of non-classical information between multiple qubits
- originates from interactions between qubits
- cannot be explained by local hidden variables
- Technical definition: non-separable state

Entanglement is needed for quantum computers to perform certain computations more efficiently than classical computers.
Possible way to produce entanglement in QC: CNOT gate
→ next lecture