Two Qubit Gates: CNOT and SWAP

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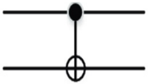
Single Qubit gates review and multi-qubit gates

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \qquad |+\rangle = \frac{1}{2}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{2}(|0\rangle - |1\rangle) \tag{0.1}$$

- X gate – flips $|0\rangle \leftrightarrow |1\rangle$
- H gate – $|0\rangle \mapsto |+\rangle$, and $|1\rangle \mapsto |-\rangle$
- Z gate – $|0\rangle \mapsto |0\rangle$, and $|1\rangle \mapsto -|1\rangle$

Multi-qubits (two qubits)– Most important one **Controlled NOT (CNOT)** gate. Essential in Quantum algorithms and used for entagling qubits. CNOT gate acts on two qubits and results also in two qubits. The first qubit is called the **control** qubit while the second is called the **target** qubit. The gate follows the rule:



1. If the control qubit is $|0\rangle$, then leave the target qubit as the same
2. If the control qubit is $|1\rangle$, then flip the target qubit i.e. on the target, $|0\rangle \leftrightarrow |1\rangle$

The truth table for the CNOT gate is shown:

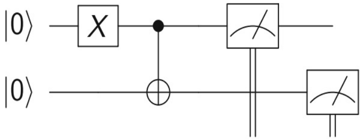
Before		After	
Control bit	Target bit	Control bit	Target bit
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

Hence the matrix form is easily derived as

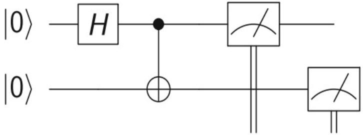
$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Examples

1. The fig shows the quantum circuit sending $|01\rangle$ through CNOT gate, what is the output?



2. In the fig, the control qubit is in a superposition of $|0\rangle$ and $|1\rangle$. What is the effect of the CNOT gate?



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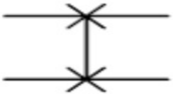
SWAP gate

As the name implies, it swaps the position of the 1st and 2nd qubit. Hence,

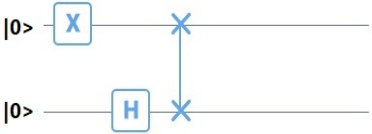
1. $|00\rangle$ and $|11\rangle$ remain unchanged after applting the SWAP gate
2. $|01\rangle$ becomes $|10\rangle$, and $|10\rangle$ becomes $|01\rangle$ after a SWAP gate operation.

The matrix form is given as:

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

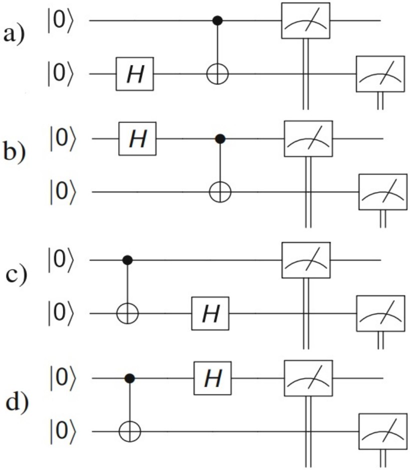


Example: Can you predict whic state is produced by the quantum circuit?



Practice Questions

- Can you predict which states will be produced by these quantum circuits?



Discussion of native 2-qubit gates for different machines. Why are 2-qubit errors different/worse? Solovay-Kitaev Theorem

Native gates: Set of gates physically executable on quantum hardware.

• Not all “textbook" gates are directly implementable on a real device. So such gates are implementated by composing native gates. Some examples...

$$CZ_{q_0,q_1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \text{iSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{0.3}$$

$$CZ\,|00\rangle = |00\rangle\,,\; CZ\,|11\rangle = -\,|11\rangle\,, \qquad iSWAP\,|01\rangle = i\,|10\rangle\,,\; iSWAP\,|10\rangle = i\,|01\rangle \tag{0.4}$$

Controlled-Z (CZ): applies a phase flip to the target qubit if the control qubit is in the state $|1\rangle$.
iSWAP: exchanges the state amplitudes of two qubits, introducing a phase difference.

$$CR(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & 0 & -i\sin\left(\frac{\theta}{2}\right) & 0 \\ 0 & \cos\left(\frac{\theta}{2}\right) & 0 & i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & 0 & \cos\left(\frac{\theta}{2}\right) & 0 \\ 0 & i\sin\left(\frac{\theta}{2}\right) & 0 & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{0.5}$$

Cross-Resonance gate (CR): It is an entangling gate that uses the interaction between qubits to generate entanglement.

$$\mathrm{FSim}(\theta, \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -i\sin(\theta) & 0 \\ 0 & -i\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix} \tag{0.6}$$

FSim: The FSim gate is a tunable two-qubit gate that generalizes the CNOT and CZ gates. It can be used to perform entangling operations with arbitrary angles.

For different quantum (2-qubits) hardware architectures, there are different native gates:

1. **IBM Quantum: Controlled NOT (CNOT) + Controlled-Z (CZ) + iSWAP + Cross-Resonance gate (CR).**
2. **Rigetti Quantum Computing:** The Rigetti quantum computers,such as Aspen-9, provide the following native 2-qubit gates: **CNOT + iSWAP + FSim**.
3. **Honeywell Quantum Solutions:** The Honeywell quantum computers, like the Honeywell H1 system, use the following native 2-qubit gates: **CNOT + iSWAP**.

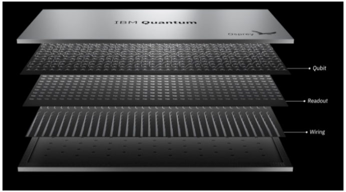


Figure: IBM 433-qubit computer

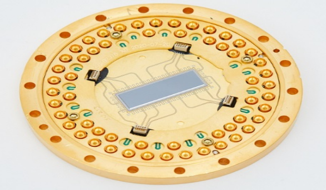


Figure: Rigetti processor

Figure: Honeywell model H1

Why 2-qubit errors are worse/different?

Quantum 2-qubit errors can be worse compared to single-qubit systems errors due to several reasons:

1. **Increased Complexity:** requires precise control over two qubits simultaneously. i.e more complex interactions and synchronization $\Rightarrow$ additional sources of errors.
2. **Error Propagation:** Errors propagate more easily than in single-qubit ones. A single error affects the entanglement between the qubits $\Rightarrow$ error accumulation in larger circuits.
3. **Sensitivity to Noise and Decoherence:** Quantum systems are prone to various sources of noise, such as interactions with the environment and control imperfections. These effects can be more pronounced in 2-qubit systems due to the increased complexity and entanglement involved.
4. **Calibration and Crosstalk:** Achieving precise control over 2-qubit gates requires careful calibration of the control parameters and mitigation of crosstalk (unwanted interactions) between qubits $\Rightarrow$ unintended gate operations or introduce errors.

Research to reduce these errros include error correction codes, error mitigation techniques, improved qubit designs, and enhanced control mechanisms, etc.

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Universality and the Solovey-Kitaev theorem

A universal gate set: set of finite gates to which every unitary operation can be reduced.
Example: 1. $R_x(\theta), R_y(\theta), R_z(\theta)$, Phase gate $P(\phi)$, CNOT
2. CNOT, $H, S = \sqrt{Z}, T = \sqrt{S}$

Technically impossible since number of unitaries is uncountable but the universal set is countably finite. How do we then use the native gates to generalize any unitary operation?
To implement a unitary gate, we would atleast need infinitely many gates in the universal set (and still get some errors though!).

The Solovay-Kitaev theorem – establishes connection between efficient quantum gate synthesis and a universal quantum gate set.

Solovay-Kitaev Theorem

Let $\mathcal{G}$ be a finite set of elements in SU(2) containing its own inverses, such that $\langle \mathcal{G} \rangle$ is dense in SU(2). Let $\varepsilon > 0$ be given. Then there exists a constant c such that for any $U \in SU(2)$, there is a sequence S of gates from $\mathcal{G}$ of length $\mathcal{O}(\log^c(1/\varepsilon))$ such that $||S - U|| \leq \varepsilon$

More formally, given a universal set of gates (a set of gates $\{S\}$ that can generate any unitary gate to arbitrary precision), the Solovay-Kitaev theorem states that for any desired unitary gate U and any error $\varepsilon > 0$, there exists a sequence of gates from the universal set that approximates U up to error ε.

The efficiency is measured in terms of the number of gates required, and the theorem shows that the number of gates needed scales logarithmically with the desired precision.

Example: $R_X(0.8) = HTH$ with error 0.01

Practice

For what values of θ and ϕ is the FSim gate equal to the CZ gate?

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THANK YOU

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