Tianlai Cylinder Array Multi-Frequency Angular Power Spectrum (MAPS) Estimation and Analysis

2023-10-24

Shifan Zuo

Multi-Frequency Angular Power Spectrum (MAPS)

$$\langle a_{lm}(\nu)\, a^{*}_{l'm'}(\nu') \rangle = C_l(\nu, \nu')\, \delta_{ll'}\, \delta_{mm'},$$

The MAPS $C_l(\nu, \nu')$ completely quantifies the second order statistics of the sky signal under the assumption that the signal is statistically homogeneous and isotropic on the sky.

This however does not assume that the signal is ergodic or statistically homogeneous along the frequency axis.

The power spectrum $P(k)$ has some intrinsic drawbacks as a summary statistic for the 21-cm signal. The strongly non-Gaussian character of the 21-cm signal has for example prompted the exploration of the bispectrum as an additional summary statistic. Another complication for the power spectrum is caused by the so-called light-cone (LC) effect, i.e. the evolution of the signal along the frequency direction, which breaks the statistical homogeneity (or ergodicity) along the line-of-sight (LoS) direction.

m-mode MAPS Estimation Method

We start from the m-mode measurement equation for a transit interferometer array,

$$\boldsymbol{v}^m = \boldsymbol{B}^m \boldsymbol{a}^m + \boldsymbol{n}^m, \tag{1}$$

where the superscript m indicates that the equation is satisfied for a specific m. Write in another way, this is

$$v_\alpha^m(\nu) = \sum_l B_{\alpha l}^m(\nu) a_l^m(\nu) + n_\alpha^m(\nu), \tag{2}$$

where we have used a single index α to indicate a baseline (i, j).

We calculate the correlations of the measured m-modes,

$$\begin{aligned}
\langle v_\alpha^m(\nu) v_{\alpha'}^{m'*}(\nu')\rangle &= \sum_{ll'} B_{\alpha l}^m(\nu) \langle a_l^m(\nu) a_{l'}^{m'*}(\nu')\rangle B_{l'\alpha'}^{m'*}(\nu') + \langle n_\alpha^m(\nu) n_{\alpha'}^{m'*}(\nu')\rangle \\
&= \sum_{ll'} B_{\alpha l}^m(\nu) B_{l'\alpha'}^{m'*}(\nu') C_l(\nu, \nu') \delta_{ll'} \delta_{mm'} + \langle n_\alpha^m(\nu) n_{\alpha'}^{m'*}(\nu')\rangle \\
&= \sum_{l} B_{\alpha l}^m(\nu) B_{l\alpha'}^{m*}(\nu') C_l(\nu, \nu') \delta_{mm'} + \langle n_\alpha^m(\nu) n_{\alpha'}^{m'*}(\nu')\rangle,
\end{aligned} \tag{3}$$

from which we see we need to only consider $m = m'$, that is

$$\langle v_\alpha^m(\nu) v_{\alpha'}^{m*}(\nu')\rangle = \sum_l B_{\alpha l}^m(\nu) B_{l\alpha'}^{m*}(\nu') C_l(\nu, \nu') + \langle n_\alpha^m(\nu) n_{\alpha'}^{m*}(\nu')\rangle. \tag{4}$$

For notational simplicity, we rewrite Eq. 4 as

$$v_{(\alpha\alpha')}^m(\nu, \nu') = \sum_l B_{(\alpha\alpha')l}^m(\nu, \nu') C_l(\nu, \nu') + v_{(\alpha\alpha')}^m(\nu, \nu'), \tag{5}$$

where $v_{(\alpha\alpha')}^m(\nu, \nu') = \langle v_\alpha^m(\nu) v_{\alpha'}^{m*}(\nu')\rangle$, $B_{(\alpha\alpha')l}^m(\nu, \nu') = B_{\alpha l}^m(\nu) B_{l\alpha'}^{m*}(\nu')$, $n_{(\alpha\alpha')}^m(\nu, \nu') = \langle n_\alpha^m(\nu) n_{\alpha'}^{m*}(\nu')\rangle$ and we have merged $(\alpha\alpha')$ as a single index. We see different ms actually give the same solution of $C_l(\nu, \nu')$, so it's beneficial to use all available ms to better constraint $C_l(\nu, \nu')$, that is

$$v_{(m\alpha\alpha')}(\nu, \nu') = \sum_l B_{(m\alpha\alpha')l}(\nu, \nu') C_l(\nu, \nu') + v_{(m\alpha\alpha')}(\nu, \nu'), \tag{6}$$

here we have merged $(m\alpha\alpha')$ as a single index. This gives a linear mapping between $v_{(m\alpha\alpha')}(\nu, \nu')$ and the MAPS $C_l(\nu, \nu')$.

Writting in matrix-vector form, Eq. 6 is

$$\boldsymbol{v} = \boldsymbol{B}\boldsymbol{c} + \boldsymbol{n}. \tag{7}$$

m-mode MAPS Estimation

Further calculation gives

$$\hat{C}_l(\nu,\nu') = \left[\sum_m \Big(\boldsymbol{B}_m^\dagger(\nu)\boldsymbol{B}_m(\nu)\Big) \odot \Big(\boldsymbol{B}_m^\dagger(\nu')\boldsymbol{B}_m(\nu')\Big)^*\right]^{-1} \left[\sum_m \Big(\boldsymbol{B}_m^\dagger(\nu)\boldsymbol{v}_m(\nu)\Big) \odot \Big(\boldsymbol{B}_m^\dagger(\nu')\boldsymbol{v}_m(\nu')\Big)^*\right]$$

The Tikhonov-regularized solution is

$$\hat{C}_l(\nu,\nu') = \left[\sum_m \Big(\boldsymbol{B}_m^\dagger(\nu)\boldsymbol{B}_m(\nu)\Big) \odot \Big(\boldsymbol{B}_m^\dagger(\nu')\boldsymbol{B}_m(\nu')\Big)^* + \varepsilon\boldsymbol{I}\right]^{-1} \left[\sum_m \Big(\boldsymbol{B}_m^\dagger(\nu)\boldsymbol{v}_m(\nu)\Big) \odot \Big(\boldsymbol{B}_m^\dagger(\nu')\boldsymbol{v}_m(\nu')\Big)^*\right]$$

where the regularization parameter $\varepsilon > 0$.

MAPS Analysis

The MAPS $C_l(\nu, \nu')$ of the sky can be written as the sum of several different components,

$$C_l(\nu, \nu') = C_l^{\rm fg}(\nu, \nu') + C_l^{21}(\nu, \nu') + C_l^{\rm n}(\nu, \nu'). \tag{17}$$

A common paramterization of the MAPS for the foreground is [1]

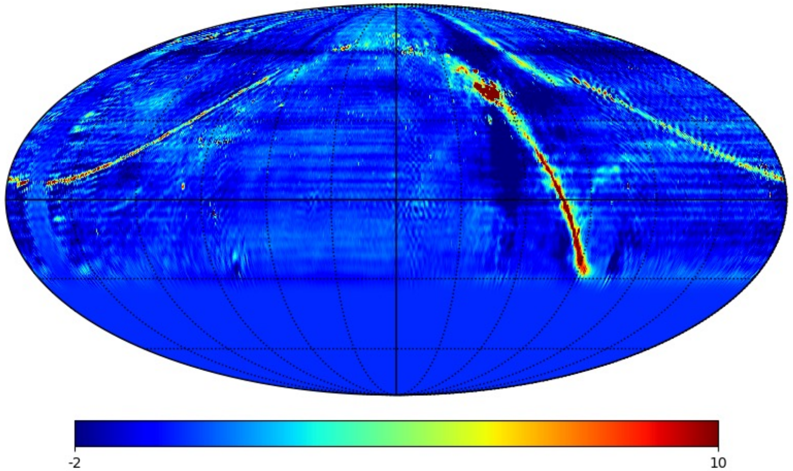
$$C_l^{\rm fg}(\nu, \nu') = \sum_i A_i \left(\frac{l}{l_0}\right)^{-\alpha_i} \left(\frac{\nu\nu'}{\nu_0^2}\right)^{-\beta_i} \exp\left(-\frac{(\log\nu - \log\nu')^2}{2\zeta_i^2}\right), \tag{18}$$

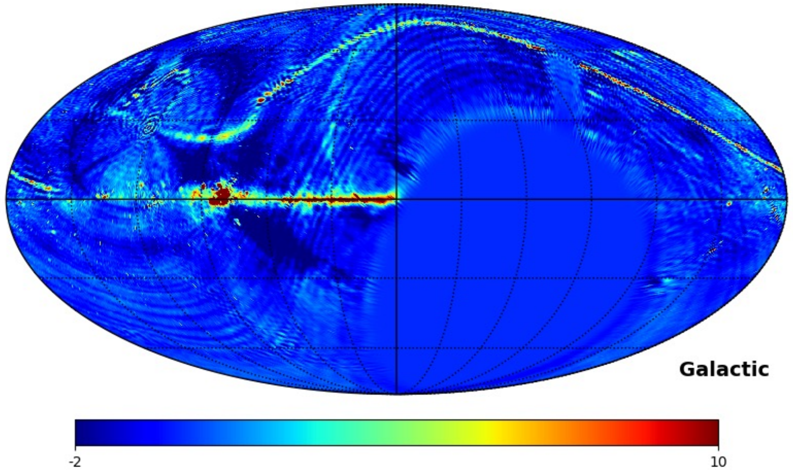
where A_i represents the overall amplitude of a foreground component; α_i determines its angular spectrum, β_i determines its frequency spectrum, and ζ_i controls the degree to which nearby frequency channels are correlated. The statement that foreground emission is spectrally smooth here implies $\zeta_i^2 \gg \log^2(\nu/\nu')$ for each component. This parameterization allows for multiple power-law foreground components and ensures that the covariance matrix is positive definite. If we assume $\zeta_i^2 \gg \log^2(\nu/\nu')$ and there is only a single foreground component, then Eq. 17 can be written as

$$C_l^{\rm fg}(\nu, \nu') = \sqrt{C_l^{\rm fg}(\nu) C_l^{\rm fg}(\nu')}, \tag{19}$$

where $C_l^{\rm fg}(\nu) = C_l^{\rm fg}(\nu, \nu)$ is the single-frequency angular power spectrum.

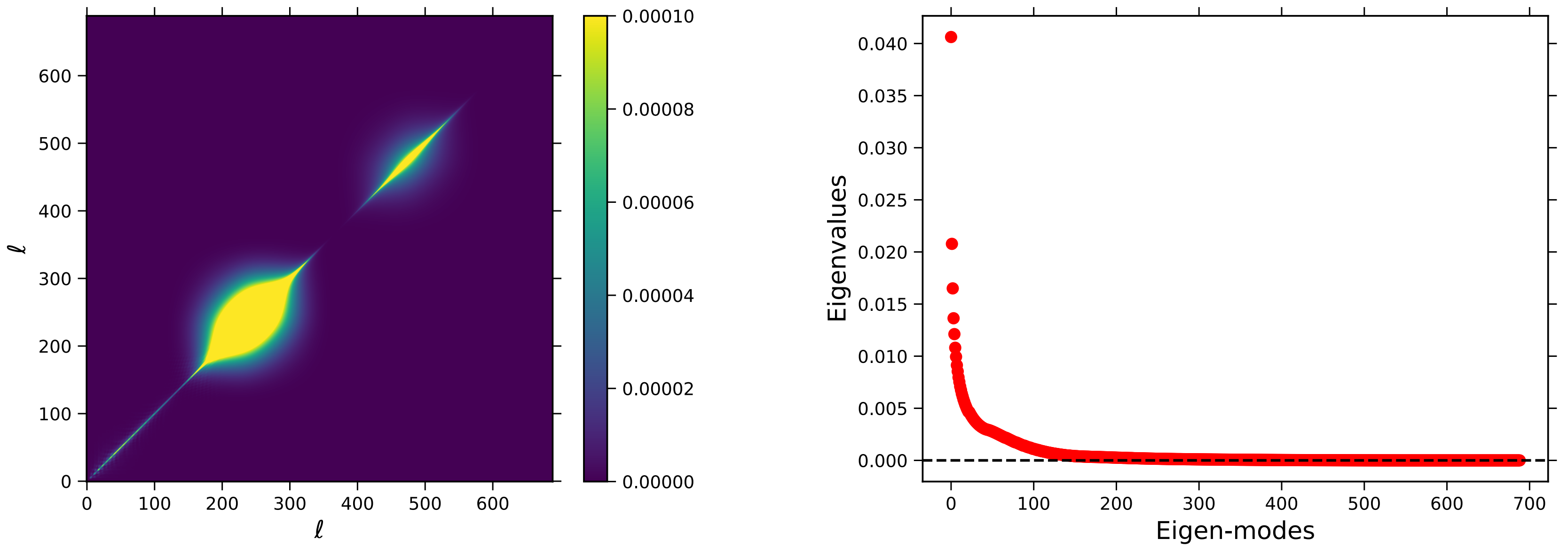
Reconstructed Map





Data obseved in 2023/9/6

Illness of the Matrix



Left: $\sum_m \left(\boldsymbol{B}_m^\dagger(\nu)\boldsymbol{B}_m(\nu)\right) \odot \left(\boldsymbol{B}_m^\dagger(\nu')\boldsymbol{B}_m(\nu')\right)^*$; Right: Its eigenvalues.

C_ℓ and It's fitting

If we assume there is only one component, we have

$$C_\ell(\nu) = A(\frac{\ell}{\ell_0})^{-\alpha}(\frac{\nu}{\nu_0})^{-2\beta}$$

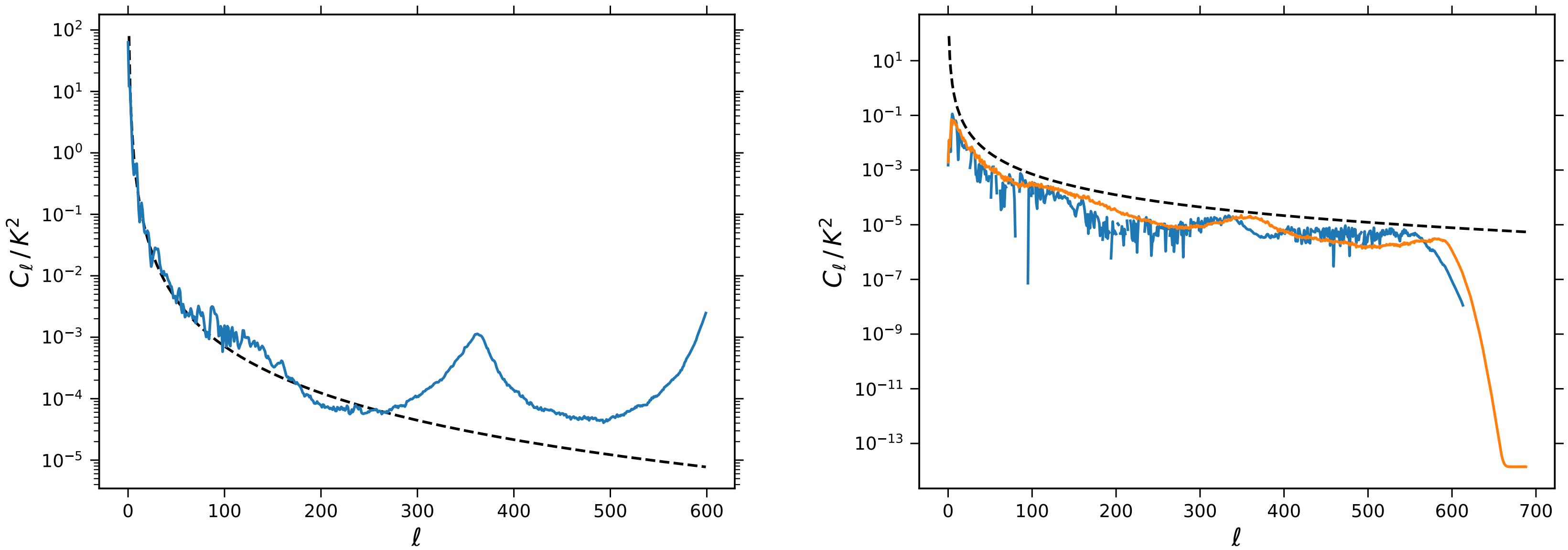


Figure 3: Left: Diagonal approximation solution of C_ℓ; Right: Tikhonov solution of C_ℓ and C_ℓ computed from the reconstructed sky map. The dash line gives a fitting $C_\ell \sim 7.26 \times 10^{-4} \left(\frac{\ell}{100}\right)^{-2.52}$.