Theoretical Advances in $g-2$ (lattice)

Shaun Lahert

University of Utah

Beach XV, 6 June 2024

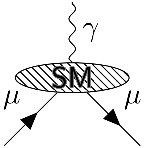
$g-2$ Theory Overview

- Magnetic moment of muon μ:

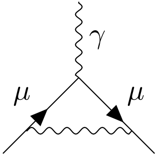
$$\vec{\mu}_\mu = g\frac{e}{2m_\mu}\vec{S}$$

- Anomalous moment from loop effects:

$$a_\mu = (g-2)/2$$



- Computed perturbatively in power series of $\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c} \approx \frac{1}{137}$



$$\Rightarrow \quad a_\mu^{(1)} = \frac{\alpha}{2\pi} \approx 0.00116171491\ldots$$ Schwinger, 1948

- $a_\mu^{(n>1)}$ contributions: mass dependence, QCD, EW, $O\left(10^{n-1}\right)$ diagrams.

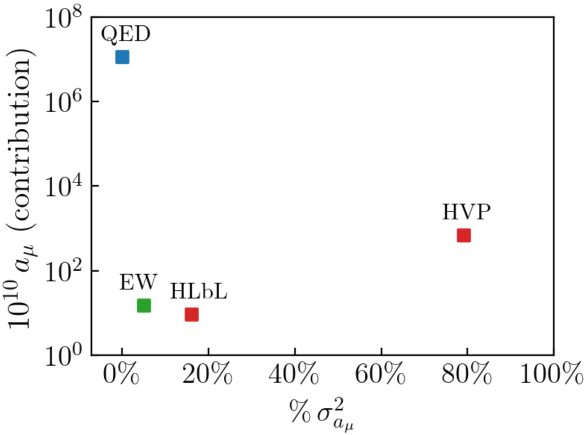
Theoretical result - $a_\mu^{\rm SM}$ (WP 20')[1]

$$10^{10} a_\mu^{\rm SM} = 11659181.0(4.3)$$

$$a_\mu^{\rm SM} = a_\mu^{\rm QED} + a_\mu^{\rm Had.} + a_\mu^{\rm EW}$$

$$a_\mu^{\rm Had.} = a_\mu^{\rm HVP} + a_\mu^{\rm HLbL}$$

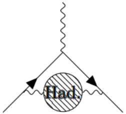
QED computed up to $\mathcal{O}(\alpha^5)$!



[1] T. Aoyama et al., Phys. Rept. 887, 1–166 (2020).

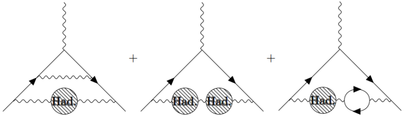
HVP

(LO) :



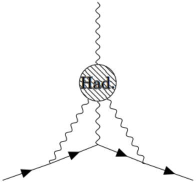
$$\Rightarrow a_\mu^{\mathrm{HVP,LO}} = 4\alpha^2 \int_0^\infty \mathrm{d}Q^2 K_E(Q^2)\hat{\Pi}(Q^2).$$

(NLO):



HLbL

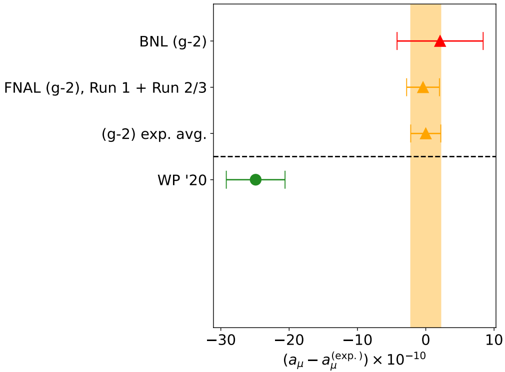
(LO) :



- HVP: $\hat{\Pi}(Q^2)$ convolution with known muon kernels (LO, NLO, NNLO).
- HLbL enters at $\mathcal{O}(\alpha^3)$.

Contribution	$10^{10} a_\mu$
HVP LO	$693.1(4.0)$
HVP NLO	$-9.83(7)$
HVP NNLO	$1.24(1)$
HLbL	$9.2(1.9)$
Hadronic	$694.6(4.4)$

Muon $g-2$: probe of new physics

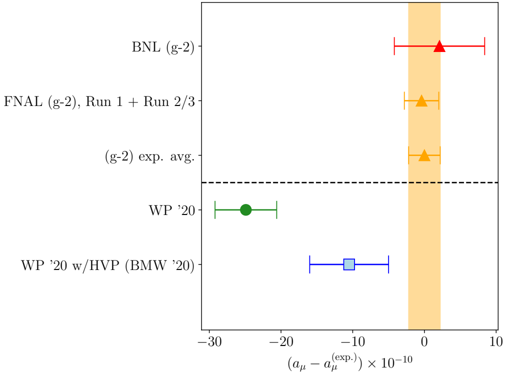


- BNL[2] and FNAL[3] results in excellent agreement.
- WP 20' in $5.1\,\sigma$ tension.
  - Data-driven HVP
  - Lattice + Data-driven HLbL

[2] G. W. Bennett et al., Phys. Rev. D73, 072003 (2006).

[3] B. Abi et al., Phys. Rev. Lett. 126, 141801 (14 2021), D. P. Aguillard et al., Phys. Rev. Lett. 131.16, 161802 (2023).

Muon $g-2$: probe of new physics



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- HVP (0.8%) from BMW[4] (lattice) significantly reduce tensions.

[4] S. Borsanyi et al., Nature, 593.7857, 51–55 (2021).

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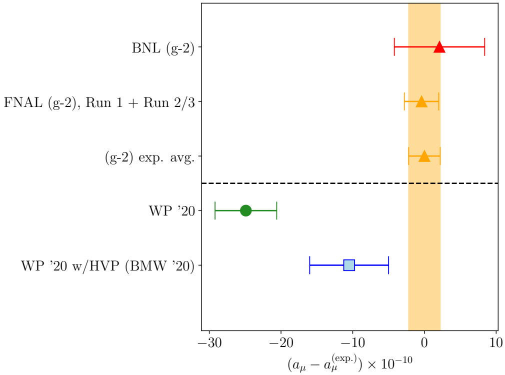


- BNL and FNAL results in excellent agreement.
- WP 20' in $5.1\,\sigma$ tension.
  - Data-driven HVP
  - Lattice $+$ Data-driven HLbL
- HVP (0.8%) from BMW[5] (lattice) significantly reduce tensions.
- New cross-section measurement (data-driven) from CMD3[6] in 3-5σ tension with last 20 years of experiments!

[5] S. Borsanyi et al., Nature, 593.7857, 51–55 (2021).
[6] F. V. Ignatov et al., (Feb. 2023).

Outline

- ~~What is Lattice QCD~~ Talk by Will Jay on Tuesday, backup slides.
- Lattice hadronic vacuum polarization
  - Challenges & progress
  - Euclidean windows
- Lattice hadronic light-by light

Lattice Hadronic Vacuum Polarization

Lattice HVP details

- Lattice a_μ^{HVP} calculations are typically performed in the (Euclidean)time-momentum rep.[7]

$$a_\mu^{\text{HVP,LO}} = 4\alpha^2 \int_0^\infty dt \tilde{K}(t) C(t), \quad C(t) = \frac{1}{3}\sum_i^3 \int d^3x \langle J_i(x) J_i(0) \rangle \ \text{(F.T. of HVP)}$$

$$J_i(x) = \sum_f Q_{q_f} \bar{q}_f(x)\gamma_i q_f(x), \quad Q_u = +\frac{2}{3},\ Q_d = -\frac{1}{3},\ Q_s = -\frac{1}{3}, \ldots$$

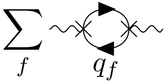
We compute $C(t)$ on the lattice.

[7] D. Bernecker and H. B. Meyer, Eur. Phys. J. A, 47, 148 (2011).

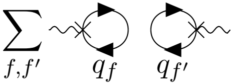
HVP Contributions[8]

- Calculation broken down by flavor, Wick contraction & isospin symmetric/breaking

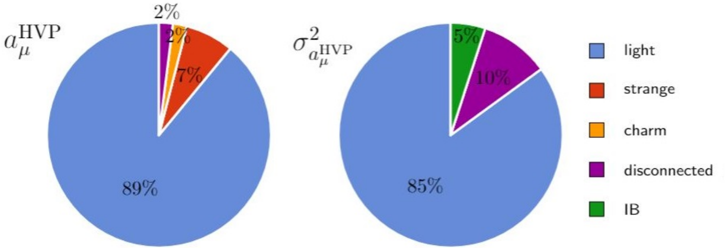
$$a_\mu^{\mathrm{HVP,LO}} = a_\mu^{ll}(\mathrm{conn.}) + a_\mu^{ss}(\mathrm{conn.}) + a_\mu^{cc}(\mathrm{conn.}) + a_\mu^{bb}(\mathrm{conn.}) + \ldots$$



$$+ \; a_\mu(\mathrm{disc.})$$



$$+ \; \Delta a_\mu^{ud}(\mathrm{SIB}) + \Delta a_\mu(\mathrm{QED})$$



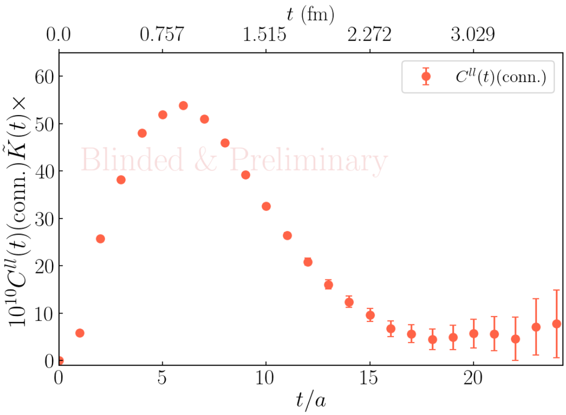
[8] S. Borsanyi et al., Nature, 593.7857, 51–55 (2021).

HVP challenges

- **Light-quark-connected**:
  - Goal: $< 1\%$ determination (BMW 0.8%).
  - Statistics: S2N issue in $C(t)$.
  - Finite volume: Typically 3%.
  - Uncertainty in lattice spacing a: $\frac{\Delta a_\mu}{a_\mu} \approx 1.8 \frac{\Delta a}{a}$
- **Disconnected**: $< 20\%$ determination, LQ physics in tail, expensive.
- **Isospin Breaking**: $\sim 1\%$ effect, expensive.

All uncertainties are systematically improvable with lattice QCD calculations!

The noisy tail (light-quark)



- Noise $\propto e^{-M_\pi t}$, signal $\propto e^{-E_{\pi\pi}(\vec{p}\neq 0)t}$

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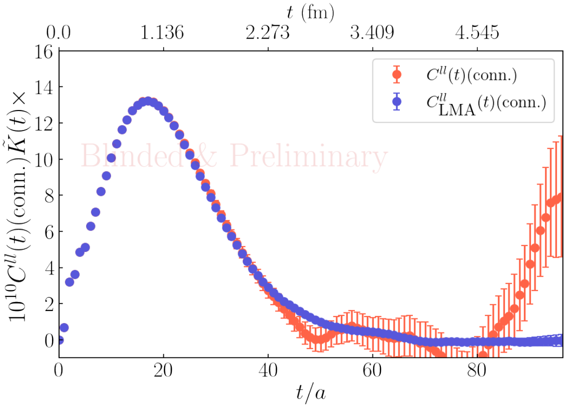

- Noise $\propto e^{-M_\pi t}$, signal $\propto e^{-E_{\pi\pi}(\vec{p}\neq 0)t}$
- Analysis: bounding or fit.
- Computational:
  - Truncated-solver method(TSM)[9]
  - Exact low-lying eigenvectors of Dirac operator (LMA)[10,11]

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[9] J. Gasser, A. Rusetsky, and I. Scimemi, Eur. Phys. J. C, 32.1, 97–114 (Dec. 2003).
[10] L. Giusti et al., JHEP, 04, 013 (2004).
[11] T. A. DeGrand and S. Schaefer, Nucl. Phys. B Proc. Suppl. 140, 296–298 (2005).

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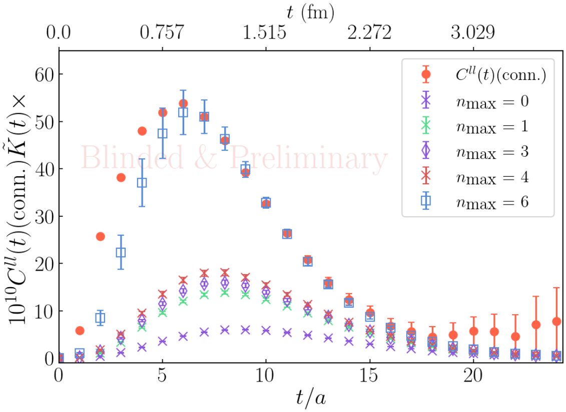

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- Analysis: bounding or fit.
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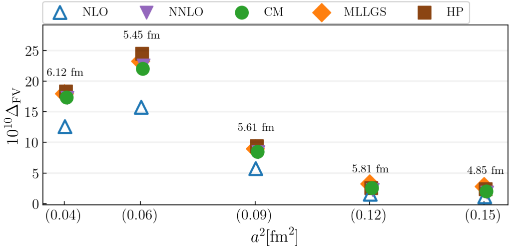
  - Exact low-lying eigenvectors of Dirac operator(LMA/AMA)
- Direct: Two-pion operators & diagonalize correlation matrix[12,13,14]

[12] M. Bruno et al., PoS, LATTICE2019, 239 (2019).
[13] F. Erben et al., Phys. Rev. D, 101, 054504 (5 2020).
[14] S. Lahert et al., PoS, LATTICE2021, 526 (2022).

Finite volume (light-quark)

FV effects: $I = 1$ two-pion states wrapping around the lattice

- Typically: $M_\pi L \approx 4 \rightarrow$ effect $\approx 3\%$.
- EFTs/Models
  - Pions + higher energy effects (ρ) $(\chi$PT(NNLO), MLLGS[15], CM[16], HP[17])
  - Systematic uncertainty: model averaging[18].
- Direct simulation
  - Lattices with $L > 10$ fm $(M_\pi L = 7)$



[15] S. Borsanyi et al., Nature, 593.7857, 51–55 (2021).
[16] B. Chakraborty et al., Phys. Rev. D, 96.3, 034516 (2017).
[17] M. T. Hansen and A. Patella, Phys. Rev. Lett. 123, 172001 (2019).
[18] A. Bazavov et al., (Jan. 2023).

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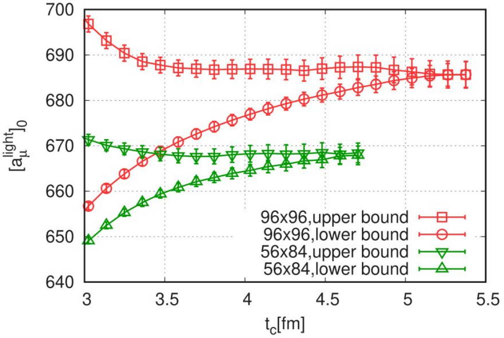
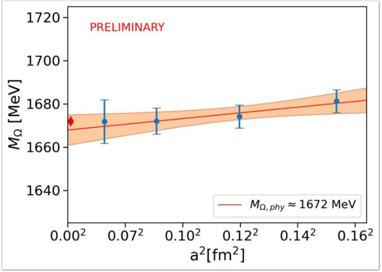
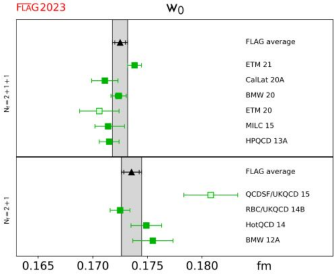


[19] E. Shintani and Y. Kuramashi, Phys. Rev. D, 100.3, 034517 (2019).
[20] S. Borsanyi et al., Nature, 593.7857, 51–55 (2021).

Scale setting

- Uncertainty from a: $\%\,\sigma_{a_\mu^{\mathrm{HVP}}} \approx 1.8\,\%\,\sigma_a$
- Scale from phys. quantities, f_π, $M_\Omega, \ldots$, gradient flow w_0, t_0 (relative)[21]
- Goal for next WP: common SS scheme for all lattice groups.





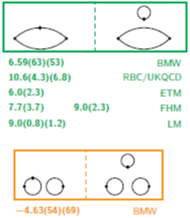
Plots from Alexei Bazavov @ Lattice 2023 & FLAG 2023

[21] S. Borsanyi et al., JHEP, 09.2012, 010 (2012).

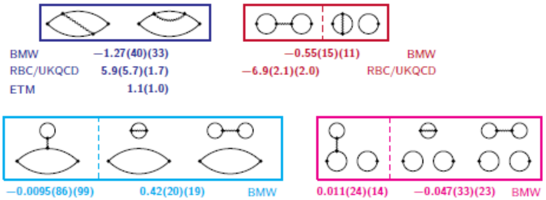
Need to quantify $\Delta a_\mu^{ud}(\mathrm{SIB})$ & $\Delta a_\mu(\mathrm{QED})$ for $< 1\%$ total uncertainty.

Approaches: Perturbation around isosym-QCD, dynamical QCD+QED simulations, IV QED.

SIB $(m_u \neq m_d)$



QED (charged quarks)



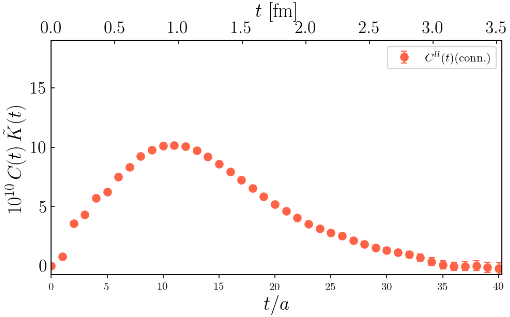
- Five groups within 1σ (conn)

Plots taken from V. Gülpers @ Lattice HVP workshop 2020

- One complete calculation (BMW)
- In progress: Mainz 2206.06582, RBC/UKQCD, Lattice 2022, BMWc, Lattice 2022, FHM 2212.12031, Harris et al., 2301.03995

Lattice HVP windows

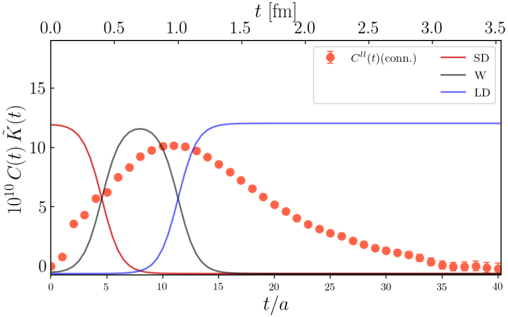
Motivation: Avoid challenging parts of integrand.



[22] T. Blum et al., Phys. Rev. Lett. 121.2, 022003 (2018).

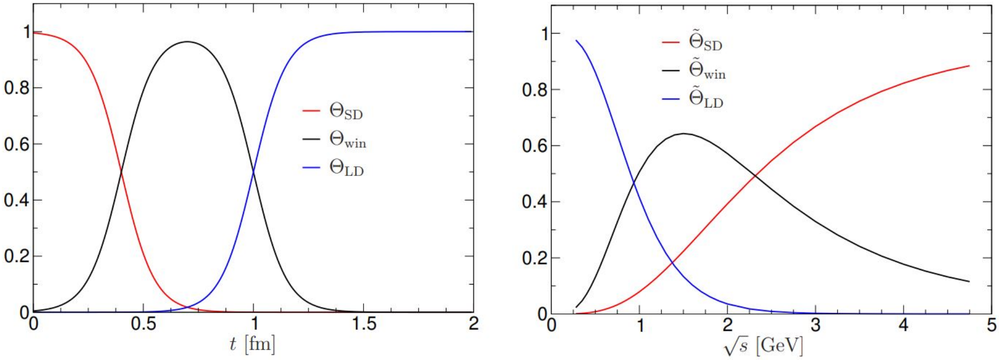
Euclidean windows[23]

Motivation: Avoid challenging parts of integrand.



- 3 complementary windows:
  SD ($\sim 15\%$), W ($\sim 30\%$), LD ($\sim 55\%$).
- W: *Goldilocks* window.
- Lattice community: Focus on W (and SD) while gathering statistics for LD (and full).

[23] T. Blum et al., Phys. Rev. Lett. 121.2, 022003 (2018).

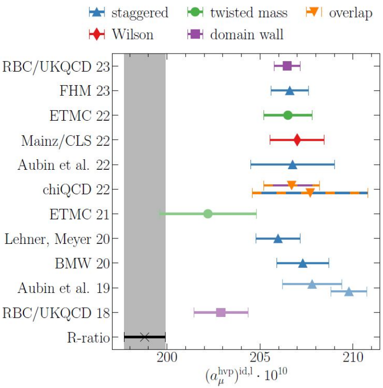


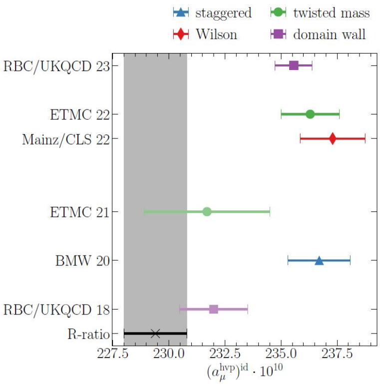
- Long-time $\leftrightarrow$ low-energy.
- SD $\sim$ pQCD.

Plot from Ref.[24]

[24] G. Colangelo et al., Phys. Lett. B, 833, 137313 (2022).

Intermediate window

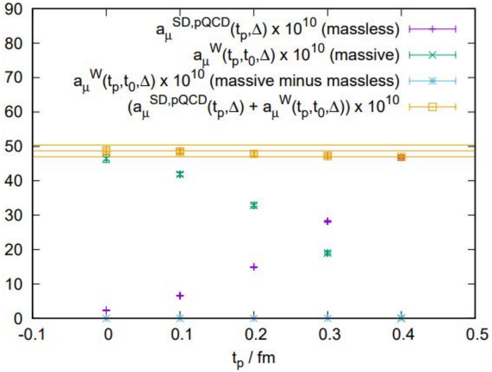




- Left: 7 groups in agreement on light contribution, 2 most recent from blinded analyses.
- Right: 4 groups in agreement.
- 3.8σ tension with R-ratio (WP) 2205.12963, 2311.09523.

Plots: Simon Kuberski @ Lattice 2023

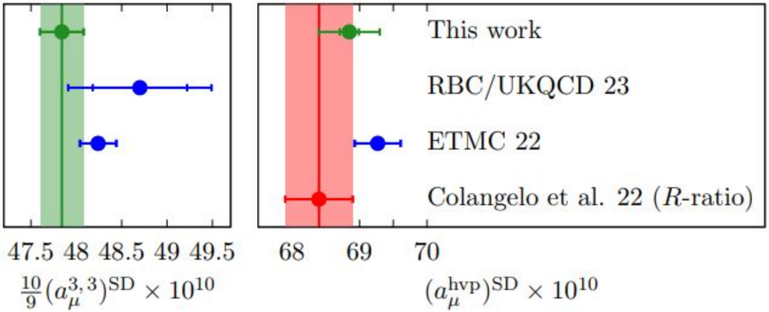
Short-distance window (pQCD)



- a_μ^{HVP} from four-loop pQCD.[25,26]
- Light SD result stable as you replace latt. with massless pQCD.[27]

[25] R. V. Harlander and M. Steinhauser, Comput. Phys. Commun. 153, 244–274 (2003).
[26] K. G. Chetyrkin and A. Maier, Nucl. Phys. B, 844, 266–288 (2011).
[27] T. Blum et al., Phys. Rev. D, 108.5, 054507 (2023).

Short-distance window



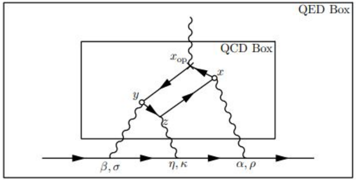
- Plot from Mainz/CLS[28].
- No significant disagreement with R-ratio at SD (high-energy).

[28] S. Kuberski et al., JHEP, 03, 172 (2024).

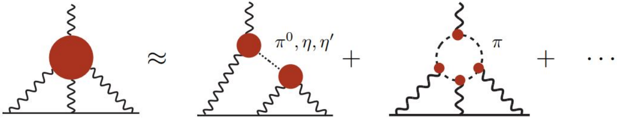
Lattice Hadronic Light-by-Light

Lattice HLbL details

Lattice (direct calculation)



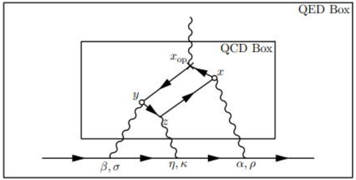
Data-driven dispersive (Lattice form factors)

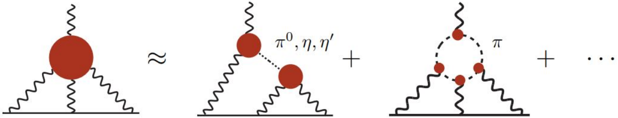


Adapted from talks: Marcus Petschlies, Luchang Jin & Antoine Gérardin @ g-2 TI workshop, Bern 2023

Lattice (direct calculation)

Data-driven dispersive (Lattice form factors)





- RBC-UKQCD 2019: $L_{QED} = L_{QCD} : 4.67 \sim 6.22\text{fm}, m_\pi : 135 \sim 144\text{MeV}$, Domain wall fermion.

  T. Blum et al 2020 (PRL 124, 13, 132002)
- Mainz 2021: $L_{QED} = \infty, m_\pi : 200 \sim 422\text{MeV}$, Wilson fermion.

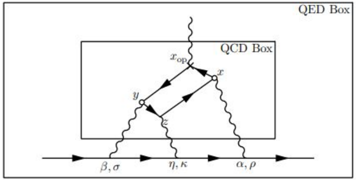
  E.H. Chao et al. 2021 (EPJC 81, 7, 651)
- RBC-UKQCD 2023: $L_{Q\in D} = \infty : 5.5\text{fm}, m_\pi = 139\text{MeV}, a^{-1} = 1.73\text{GeV}$, Domain wall fermion.

  T. Blum et al 2023 (arXiv:2304.04423 [hep-lat])

Adapted from talks: Luchang Jin, Marcus Petschlies & Antoine Gérardin @ g-2 TI workshop, Bern 2023

Lattice HLbL details

Lattice (direct calculation)



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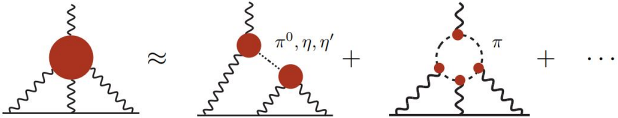


π

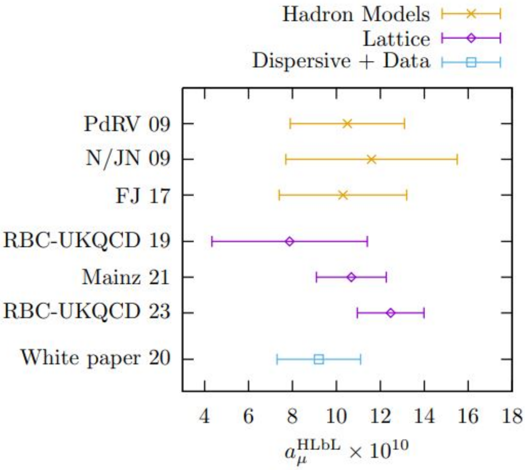
- Mainz - Phys.Rev.D 100 (2019) 3, ETM - Phys.Rev.D 108 (2023) 9

η

- Ongoing work: see talks by Marcus Petschlies & Antoine Gérardin @ g-2 TI workshop, Bern 2023.
- Difficult: $\eta - \eta'$ mixing, noisy quark-disconnected contributions.

Adapted from talks: Luchang Jin, Marcus Petschlies & Antoine Gérardin @ g-2 TI workshop, Bern 2023

Results



- All results in excellent agreement.
- Error: $\approx 15\%$, goal: 10%.

Plot from talk: Luchang Jin @ g-2 TI workshop, Bern 2023

Summary

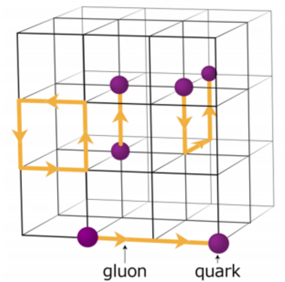
- HVP:
  - Short- and intermediate- windows in good standing w/ more results to come.
  - Tensions with data-driven r-ratio result in W.
  - Work on long-distance and full integrand still underway.
  - Expect multiple $< 1\%$ calculations in the next year.
  - Theory Initiative goal: WP2 with lattice averages.
- HLbL
  - Agreement between dispersive and lattice results.
  - Quickly approaching 10% uncertainty goal.

Thank you

What is Lattice QCD

Lattice perscription: Wick rotate, $t \to it$, and discretize, $a \neq 0$, the path integral.

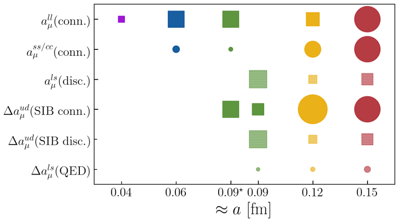
$$\langle O \rangle = \frac{1}{Z_{\text{LOCD}}} \int \mathcal{D}\left[q_f, \bar{q}_f\right] \mathcal{D}[U] \mathrm{e}^{-S_F\left[q_f, \bar{q}_f, U\right] - S_G[U]} O\left[q_f, \bar{q}_f, U\right]$$

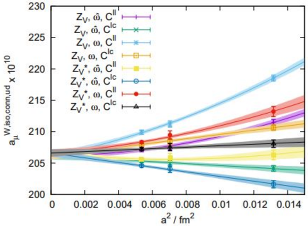


M. Savage,
2015 National Nuclear Physics Summer School

- Quarks on sites, gluons, $U_\mu(n) \equiv \exp\left(\mathrm{i}aG_\mu(n)\right)$, on links.
- Perform path integral:
  - Fermionic: Analytic $\to$ Wick contractions.
  - Bosonic: Markov Chain Monte Carlo (Importance Sampling).
- Observables: $\langle O \rangle(\{a, L, m_q\})$.
- Extrapolate, interpolate, or correct (EFT) to continuum, infinite volume, physical point.

Continuum limit (light-quark)

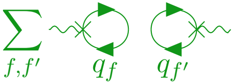




- WP release: physical-mass lattice spacing range $a \approx [0.15, 0.06]$ fm
- Newest lattices w/ $a \approx 0.04$ fm (physical pion)[29], $a \approx 0.03$ fm (unphysical pion).[30]
- Multiple vector current discretizations[31,32,33].

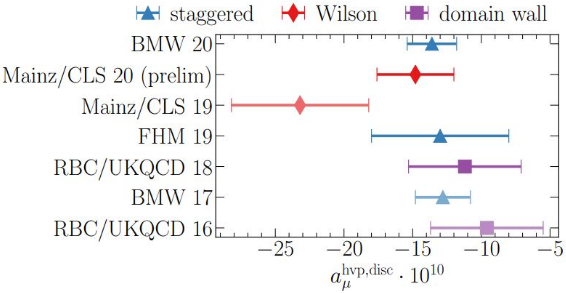
[29] S. Lahert, g-2 TI workshop, Bern 2023, (2023).
[30] S. Kuberski et al., JHEP, 03, 172 (2024).
[31] C. Alexandrou et al., Phys. Rev. D, 107.7, 074506 (2023).
[32] T. Blum et al., Phys. Rev. D, 108.5, 054507 (2023).
[33] S. Kuberski et al., JHEP, 03, 172 (2024).

Disconnected contribution



$$: \quad a_\mu(\text{disc.}) = a_\mu^{I=0} - \frac{1}{9} a_\mu^{I=1}.$$

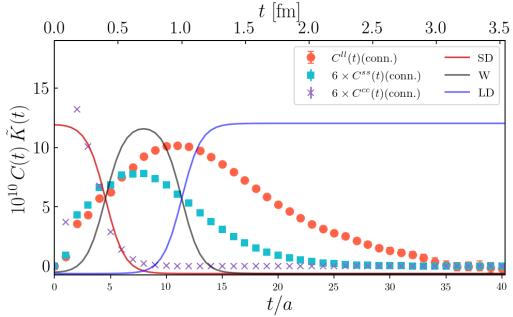
- $I = 1$ channel: Noisy tail and FV effects (like $a_\mu^{ll}(\text{conn.})$).
- Expensive, requires propagators w/ support on all sites.



Plot: Simon Kuberski @ Lattice 2023

Euclidean windows[34]

Motivation: Avoid challenging parts of integrand.



Windows change the relative size of the flavour contributions!

- 3 complementary windows:
  SD ($\sim 15\%$), W ($\sim 30\%$), LD ($\sim 55\%$).
- W: *Goldilocks* window.
- Lattice community: Focus on W (and SD) while gathering statistics for LD (and full).

[34] T. Blum et al., Phys. Rev. Lett. 121.2, 022003 (2018).