Fermionic anomalies and topological phases on the lattice:
The Ginsparg-Wilson relation and its generalizations

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The Standard Model of Particle physics is an enormously successful theory.

But it has a rather embarrassing problem.

The problem is that the Standard Model is a *chiral guage theory*.

And we do not know how to do computer simulations of a chiral gauge theory.

- This is not a “technical” problem about algorithms or hardware.
- It means that we really do not how to *define* the Standard Model in a nonperturbative way.

There is no known nonperturbative lattice construction of a nonabelian chiral gauge theory

- Recall that the massless Dirac fermion action in $d = 4$,

$$S = \int d^4x \, \bar{\psi} \not{D} \psi \tag{1}$$

has both a vector and chiral $U(1)$ symmetry

$$\psi \xrightarrow{U(1)_V} e^{i\theta}\psi \tag{2}$$

$$\psi \xrightarrow{U(1)_\chi} e^{i\theta\gamma_5}\psi \tag{3}$$

- The chiral matrix γ^5 lets us define left and right handed Weyl fermions

$$\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi. \tag{4}$$

Fermion doubling problem

What goes wrong when you try to put a Weyl fermion on the lattice?

- Let's look at 2D Weyl fermions
- Choose the basis $\gamma^1, \gamma^2 = \sigma^1, \sigma^2, \sigma^3$ and $\gamma^5 = -i\gamma^1\gamma^2 = \sigma^3$
- Dirac operator $\not{D} = \gamma^\mu \partial_\mu = \sigma^1 \partial_1 + \sigma^2 \partial_2$
- Action for a Dirac fermion

$$S = \int d^2x \, \bar{\psi} \, \not{D} \psi \tag{5}$$

$$= \int d^2p \, \begin{pmatrix} \bar{\psi}_L & \bar{\psi}_R \end{pmatrix}_{-p} \begin{pmatrix} 0 & p_1 - ip_2 \\ p_1 + ip_2 & 0 \end{pmatrix} \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}_p \tag{6}$$

$$= \int d^2p \, \underbrace{\bar{\psi}_R (p_1 + ip_2) \psi_R}_{\text{keep this}} + \underbrace{\bar{\psi}_L (p_1 - ip_2) \psi_L}_{\text{throw this away}} \tag{7}$$

Fermion doubling problem

- We are left with

$$S = \int d^2p \, \bar{\psi}_R (p_1 + ip_2) \psi_R \tag{8}$$

- Now, discretize this by replacing the derivatives with finite differences

$$\text{position space:} \qquad \partial_1 \psi(x) \to \frac{1}{2a} \left(\psi_{i+1} - \psi_{i-1} \right) \tag{9}$$

$$\text{momentum space:} \qquad ip_1 \psi(p) \to \frac{i}{a} \sin(ap_1) \psi(p) \tag{10}$$

- This gives the lattice action

$$S = \frac{1}{a} \sum_p \bar{\chi}(-p) \left[\sin(ap_1) + i \sin(ap_2) \right] \chi(p) \tag{11}$$

$$= \frac{1}{a} \sum_p \bar{\chi}(-p) D(p) \chi(p) \tag{12}$$

- $D(p)$ has a zero at $p = (0, 0)$, which is good.
- But $D(p)$ also has zeros at $p = (\pi, 0)$, $p = (0, \pi)$ and $p = (\pi, \pi)$, which is not so good!

Fermion doubling problem

So we got 4 Weyl fermions instead of 1!

OK, but if they are all right-handed, we still have a chiral theory, right?

Fermion doubling problem

- Actually, it's worse.

$$D(\mathbf{p}) \sim \sin(p_1) + i\sin(p_2) \tag{13}$$

- Near the corners of the Brillouin zone, $p_\mu = q_\mu + n_\mu \pi/a$ ($n = 0, 1$)

$$\mathbf{p} \sim (0,0) \qquad D(q) \approx q_1 + iq_2 \qquad \text{RH} \tag{14}$$

$$\mathbf{p} \sim (0,\pi) \qquad D(q) \approx q_1 - iq_2 \qquad \text{LH} \tag{15}$$

$$\mathbf{p} \sim (\pi,0) \qquad D(q) \approx -q_1 + iq_2 \qquad \text{RH} \tag{16}$$

$$\mathbf{p} \sim (\pi,\pi) \qquad D(q) \approx -q_1 - iq_2 \qquad \text{LH} \tag{17}$$

- So instead of 4 RH Weyl fermions, we have 2 Dirac fermions!
- This is vector-like theory, not a chiral one.
- For $d = 2k$ dimensions, you find 2^{d-1} Dirac fermions when you discretize a Weyl fermion
- Also, note that this doubling occurs for Dirac fermions as well ($1 \to 2^d$)

Fermion doubling problem

But there are infinitely many ways to discretize the free fermion action.

Can you be clever with the discretization and avoid this problem?

A no-go theorem

- Consider the free Dirac fermion action on the lattice

$$S = \int_{-\pi/a}^{\pi/a} \frac{d^{2k}p}{(2\pi)^{2k}} \bar{\Psi}_{-p} D(p) \Psi_p \tag{18}$$

- Nielsen and Ninomiya (1981) showed the following 4 conditions cannot all hold simultaneously for a Dirac fermion

  1. $D(\mathbf{p})$ is a periodic, analytic function of $\mathbf{p}$
  2. $D(\mathbf{p}) \propto \gamma^\mu p_\mu$ for $a|p| \ll 1$
  3. $D(\mathbf{p})$ is invertible everywhere except $\mathbf{p} = 0$
  4. $\{D(\mathbf{p}), \gamma^5\} = 0$

- So you cannot be clever, at least if you want all the above to be true.
- Intuition: if all the above were true, then you would get a Dirac fermion without a $U(1)$ chiral anomaly

Fermion doubling problem

To get a lattice theory free of doublers, we need to violate at least one of the assumptions of Nielsen-Ninomiya.

But which one?

Wilson's idea

Here's an example of how to get rid of the doublers.

- Add a momentum-dependent mass term so that the doublers become heavy and decouple.

$$S = \int \bar{\psi} \left[\sum_{\mu} \gamma^{\mu} \sin(p_{\mu}) + mF(p) \right] \psi \tag{19}$$

- Choose $F(p)$ such that $F(0) = 0$ but $F(p) \sim 1$ at the corners of BZ.

$$F(p) = \sum_{\mu} 1 - \cos(p_{\mu}) \tag{20}$$

- Doublers are gone, but so is the exact chiral symmetry
- The $U(1)$ chiral symmetry of the action is recovered only in the continuum limit

In lattice QCD simulations, not having chiral symmetry means massless quarks need fine tuning.

The continuum theory has an exact chiral symmetry of the action, but the lattice theory does not.

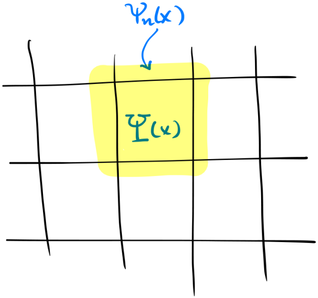
Ginsparg and Wilson (1982) asked:
If you obtain lattice theory is obtained by “blocking” a continuum theory, what happens to the chiral symmetry?

Ginsparg-Wilson equation

Start with a continuum theory and construct a lattice theory by RG blocking:

$$e^{-\bar{\chi}\mathcal{D}\chi} = \int d\Psi\, d\bar{\Psi} e^{-S(\bar{\Psi},\Psi)}\, e^{-(\bar{\psi}-\bar{\chi})m(\psi-\chi)} \tag{21}$$

where $$\psi_{\mathbf{n}} = \int d^d\mathbf{x}\ \Psi(\mathbf{x}) f(\mathbf{x} - \mathbf{n}a) \tag{22}$$



GW found that the lattice Dirac operator $\mathcal{D}$ satisfies

$$\{D, \bar{\gamma}\} = 0 \to \boxed{\{\mathcal{D}, \bar{\gamma}\} = a\mathcal{D}\bar{\gamma}\mathcal{D}}$$

This is the Ginsparg-Wilson relation.

Ginsparg-Wilson equation

$$\boxed{\{\mathcal{D}, \bar{\gamma}\} = a\mathcal{D}\bar{\gamma}\mathcal{D}} \tag{23}$$

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A remnant of chiral symmetry on the lattice

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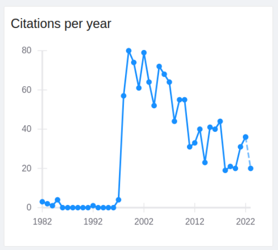


What happened in 1997?

Implications of the GW equation

- Immediately following Hasenfratz's rediscovery of the GW relation, it was realized all the lattice fermions with nice chirality properties (overlap fermions, perfect actions, domain-wall fermions) satisfy this equation
- Lüscher (1998) showed that the GW equation infact implies an exact "chiral" symmetry of the action

An exact chiral symmetry on the lattice

- Lüscher (1998) showed that the GW equation infact implies an exact symmetry of the action
- The exact symmetry is

$$\delta\psi \to \gamma^5(1 - aD/2)\psi \tag{24}$$

$$\delta\bar{\psi} \to \bar{\psi}(1 - aD/2)\gamma^5 \tag{25}$$

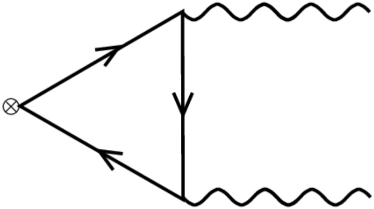
- The variation in the action is

$$\begin{aligned} \delta(\bar{\psi}D\psi) &= (\delta\bar{\psi})D\psi + \bar{\psi}D(\delta\psi) && (26) \\ &= \bar{\psi}(1 - aD/2)\gamma^5 D\psi + \bar{\psi}D\gamma^5(1 - aD/2)\psi && (27) \\ &= \bar{\psi}\left(\{D, \gamma^5\} - aD\gamma^5 D\right)\psi = 0. && (28) \end{aligned}$$

- This means that any Dirac operator satisfying the GW equation automatically will not have additive mass renormalization, and will reproduce the correct anomaly on the lattice!
- Consequently, there are no doublers!
- But wait, isn't the chiral symmetry supposed to be anomalous?

Chiral anomaly

Recall that the free 4D Dirac fermion has chiral symmetry. However, in the presence of $U(1)$ gauge fields, the chiral symmetry is afflicted by a famous anomaly.



- This means that even though the chiral symmetry is a symmetry of the action, it is not a symmetry of the measure

$$\langle \partial_\mu J_5^\mu \rangle \sim \int F \wedge F \tag{29}$$

How does this show up for GW formulation?

Anomaly for GW fermions

The exact symmetry is:

$$\delta\psi \to \gamma^5(1 - aD/2)\psi \tag{30}$$
$$\delta\bar{\psi} \to \bar{\psi}(1 - aD/2)\gamma^5 \tag{31}$$

This implies that the measure transforms with the Jacobian

$$d\psi d\bar{\psi} \to d\psi d\bar{\psi} e^{\varepsilon \mathrm{tr}\,(\gamma^5(1-aD/w)} \tag{32}$$
$$= d\psi d\bar{\psi} \exp(1 - \varepsilon a \,\mathrm{index} D) \tag{33}$$

This is a rather involved calculation in perturbation theory!

A construction of lattice chiral gauge theory

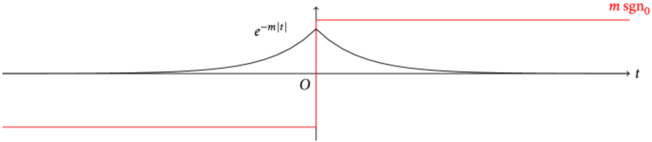
- So, can you gauge this symmetry on the lattice? If so, you would have a chiral gauge theory.
- Lüscher (1999) considered this question
  - A nonperturbative construction can be achieved for abelian chiral gauge theories, precisely when the perturbative anomalies cancel.
- The question for nonabelian chiral gauge theories is open.
- There are other interesting proposals:
  - Single-surface domain-wall (Kaplan, 2023),
  - Symmetric mass generation (Eichten, Preskill; Fidkowski, Kitaev, ...)

Kaplan's idea (1991): Domain-wall fermions

- David Kaplan (1991) suggested to use anomaly-inflow to obtain massless chiral fermions
- Consider a 5D spacetime, with $m(x_5) < 0$ for $x_5 < 0$, and $m(x_5) > 0$ for $x_5 > 0$.
- By solving the Dirac equation

$$\left[\gamma^\mu \partial_\mu + \gamma^5 \partial_5 + m(x_5)\right] \psi(\mathbf{x}, x_5) = 0 \tag{34}$$

- We find that a *massless* 4D Weyl fermion is stuck to the 4D slice $x_5 = 0$

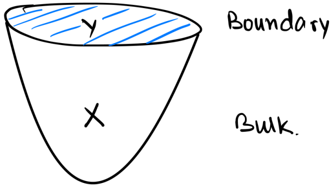


Kaplan's idea (1991): Domain-wall fermions

- This looks like another solution to the doubling problem
- However, it was shown (Neuberger, Narayanan) that in the limit of infinite wall separation $L_5 \to \infty$, the effective domain wall theory is the same as the "overlap solution", which was also a solution to the GW equation

Domain-wall fermions and topological phases

- From developments in condensed-matter physics of topological phases, we now have a more refined understanding of domain wall fermions
- The idea of anomaly-inflow (Callan, Harvey '84) applies generally to SPT phases
- Boundary of SPT (symmetry-protected topological) phase = t'Hooft anomaly

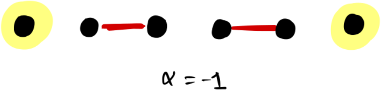


A 2d topological superconductor: the Kitaev chain

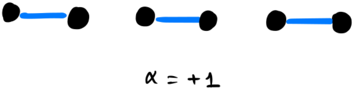
- A 1d chain of N Majorana fermions $\lambda_1, \ldots, \lambda_N$ (Kitaev, 2001):

$$H = \sum_{i \in \text{odd}} (1+\alpha)\, \lambda_i \lambda_{i+1} + \sum_{i \in \text{even}} (1-\alpha)\, \lambda_i \lambda_{i+1} \tag{35}$$

- α decides the strength of bond staggering



nontrivial SPT



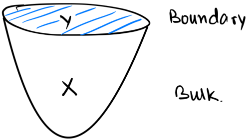
trivial

- 2 isolated Majoranas at the edges = 1 qubit
- Kitaev proposed that this could be used for topological quantum computation

Free fermion topological insulators/superconductors

Symmetry				Spatial Dimension d								
Class	T	C	S	1	2	3	4	5	6	7	8	$\cdots$
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\cdots$
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\cdots$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

Generalizing GW relations?



Anomaly ↕ Topological phase

Ginsparg-Wilson relation ↕ Domain-wall fermion

The traditional domain-wall fermions and GW relation correspond to a 5D topological insulator.

Are there GW relations for *all* topological free-fermion SPT phases?

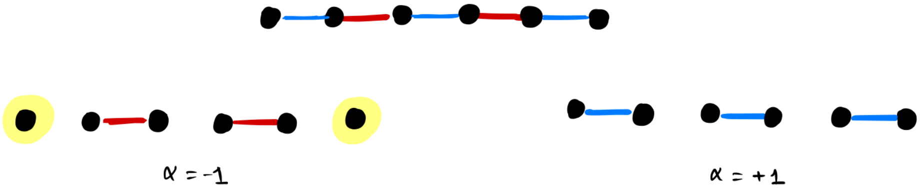
Let's look at the example of Kitaev chain again.

But for simplicity, we will consider the Kitaev chain as itself the boundary of some bulk topological phase.

Field theory for the Kitaev chain

What is the field theory description of the Kitaev chain?

$$H = \sum_{i \in \text{odd}} (1+\alpha)\, \lambda_i \lambda_{i+1} + \sum_{i \in \text{even}} (1-\alpha)\, \lambda_i \lambda_{i+1} \tag{36}$$



Field theory for the Kitaev chain

- A two-component Majorana fermion χ in 2 (Euclidean) spacetime dimensions :

$$S = \int d^2x \ \chi^T \mathcal{T} D \chi + m \chi^T \mathcal{C} \chi \tag{37}$$

- Here, $\mathcal{C}$ and $\mathcal{T}$ are matrices which ensure Lorentz invariance and the antisymmetric nature of $\mathsf{D} = \mathcal{T} D$, and satisfy

$$\bar{\gamma}^T \mathcal{C} \bar{\gamma} = -\mathcal{C}, \quad \bar{\gamma}^T \mathcal{T} \bar{\gamma} = -\mathcal{T} \tag{38}$$

- In the basis, $\gamma_\mu = \{\sigma^1, \sigma^3\}$ with $\bar{\gamma} = \sigma^2$, we have $\mathcal{T} = 1$ and $\mathcal{C} = \sigma^2$.
- This implies a "discrete chiral symmetry" of the massless action, analogous to the $U(1)$ chiral symmetry of the 4d Dirac theory

$$\chi \to \bar{\gamma} \chi \tag{39}$$

Anomaly in a discrete chiral symmetry

- The 4D Dirac fermion has action symmetric under the chiral symmetry, but the measure violates it.
- The discrete chiral symmetry is also anomalously broken by the measure

$$Z_\chi \to (-1)^\nu Z_\chi \tag{40}$$

  where ν is the "mod-2 index" of the Majorana modes of given chirality.
- This anomaly is like Witten's "global anomaly" (Witten 1985, 2016)
  - The change in partition function under a discrete chiral transformation in 2d manifold M can be expressed as the mod-2 index of the dirac operator on $M \times S^1$.

Can such an global anomaly be reproduced on the lattice?

[Clancy, Kaplan, HS, 2023]

Ginsparg-Wilson for chiral symmetry: summary

- Continuum Dirac fermions in $d = 2k$ dimensions.

$$S = \int \bar{\psi}(D + m)\psi \tag{41}$$

- Chiral symmetry for $m = 0$:

$$\psi \to e^{i\varepsilon\gamma^5}\psi, \quad \bar{\psi} \to \bar{\psi}\, e^{i\varepsilon\gamma^5} \tag{42}$$

- GW relation:

$$\{\gamma^5, D\} = aD\gamma^5 D \tag{43}$$

- Lüscher symmetry: $\psi \to \psi + \varepsilon\delta\psi$

$$\delta\psi = \gamma^5(1 - D)\psi \tag{44}$$

$$\delta\bar{\psi} = \bar{\psi}\gamma^5 \tag{45}$$

- Exact anomaly from the noninvariance of the measure

$$d\psi d\bar{\psi} \to d\psi d\bar{\psi}\, e^{-i\varepsilon\, \mathrm{tr}\, \gamma_5 D} = d\psi d\bar{\psi}\, e^{-i\varepsilon(n_+ - n_-)} \tag{46}$$

- GW relation implies
  - ✓ An exact symmetry of the action
  - ✓ Exact anomaly on the lattice (noninvariance of the measure)
  - ✓ No doublers
  - ✓ No additive mass renormlization

Ginsparg-Wilson for 2d Majorana fermions: summary

- Continuum Majorana fermions in $d = 2$ dimensions

$$S = \int d^2x \ \chi^T \mathcal{K} \not{D} \chi + \mu \chi^T \mathcal{M} \chi \tag{47}$$

- Discrete chiral symmetry for $\mu = 0$:

$$\chi \to \bar{\gamma} \chi \tag{48}$$

- GW relation:

$$\{\bar{\gamma}, D\} = aD(\mathcal{M}^{-1}\mathcal{K})\bar{\gamma}D \tag{49}$$

- Lüscher symmetry:

$$\chi \to \bar{\gamma}\sqrt{1 - 2aD\mathcal{M}^{-1}}\,\chi \tag{50}$$

- Exact anomaly from the noninvariance of the measure

$$d\chi \to d\chi \ (-1)^{\nu_-/2} \tag{51}$$

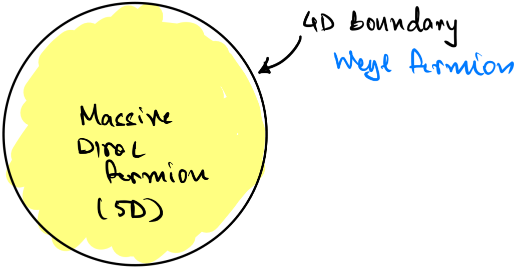
- GW relation implies
  - ✓ An exact symmetry of the action
  - ✓ Exact anomaly on the lattice (noninvariance of the measure)
  - ✓ No doublers
  - ✓ No additive mass renormlization

Generalized GW fermions

- GW relation encodes how the anomalous chiral symmetry manifests on the lattice
- Anomalous theories arise on the boundary of SPT phases
  - This was used by Kaplan (1991) in the construction of domain-wall fermions
- GW relation for (almost) all fermionic anomalies, and therefore a GW relation for each free-fermionic topological phase
  - Discrete symmetries such as time-reversal, reflection, chiral
  - Majorana and Dirac fermions
  - Any dimension

5d GW fermions: an intriguing possible application

- Kaplan's recent proposal (2023) for the construction of a lattice chiral gauge theory is to consider a domain-wall on a single-boundary setup



- Witten-Yonekura (2019): The phase of the partition function for a chiral fermion is

$$Z[A] = |Z| \exp(-i\pi\eta_{5D}[A]) \tag{52}$$

- Another possibility of constructing chiral gauge theories?

Outlook

- A lattice construction of non-abelian chiral gauge theories is currently the biggest open problem towards non-perturbative studies of the standard model.
- How can the study of topological materials shed light on lattice chiral symmetry, and vice versa?
- Can we formulate overlap fermions in Hamiltonian lattice field theory to allow for new approaches to quantum simulation of QCD-like vector gauge theories?
- For certain anomalies (called Dai-Free anomalies), it is still not clear how they appear on the lattice.
- A satisfactory lattice chiral gauge theory construction will also show how it fails when there are anomalies
- Generalizing GW fermions was an important step in clarifying the connections between topological phases and anomalies — an area which continues to have many interesting developments!

Thank you!