Invertible and non-invertible generalized anomalies for pedestrians: a view from the torus

Erich Poppitz (Toronto)

based on works with

Andrew Cox and F. David Wandler (Toronto)

Mohamed Anber (Durham)

this talk: 2106.11442, 2305.14425

the big picture:

problem of determining the IR phases of gauge theories is complex

matching of anomalies (’t Hooft) constrains any IR fantasy one might have

old story, eg massless QCD: pions!; preons; Seiberg dualities…

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problem of determining the IR phases of gauge theories is complex

matching of anomalies ('t Hooft) constrains any IR fantasy one might have

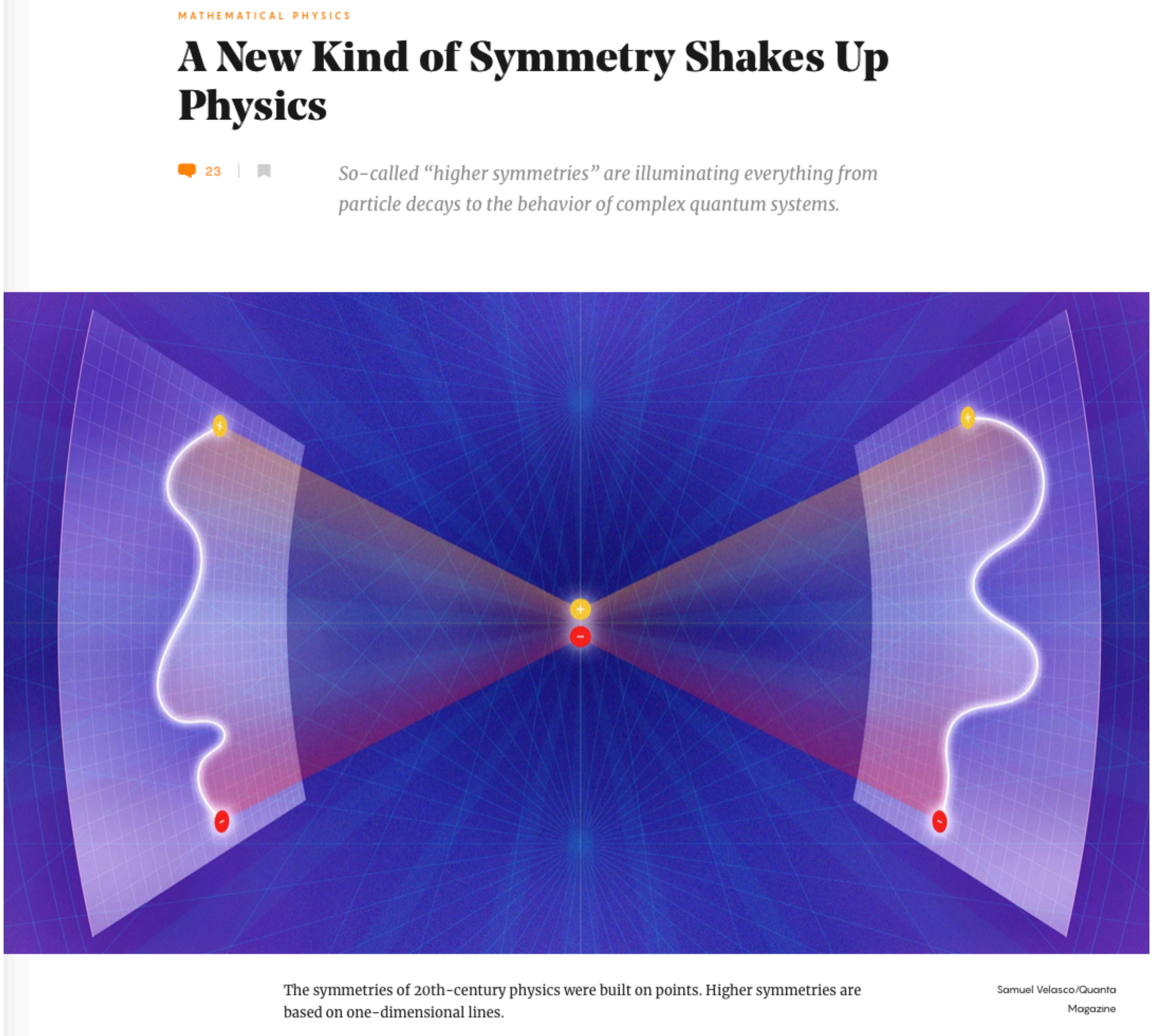
old story, eg massless QCD: pions!; preons; Seiberg dualities…

the new stuff:

there are new 't Hooft anomalies,
thus new constraints on IR behavior,
that were missed in the 1980s,
involving higher form symmetries

Gaiotto, Kapustin, Komargodski,Seiberg: 2014-… [GKKS+]

the hype:



MATHEMATICAL PHYSICS

A New Kind of Symmetry Shakes Up Physics

23

So-called “higher symmetries” are illuminating everything from particle decays to the behavior of complex quantum systems.

The symmetries of 20th-century physics were built on points. Higher symmetries are based on one-dimensional lines.

Samuel Velasco/Quanta Magazine

“Quanta”
April 18, 2023

any hype aside, this is exciting from a general QFT point of view as it gives a new nonperturbative tool to study gauge theories

this talk:

1. **HOW MIXED ANOMALIES BETWEEN CHIRAL (invertible or not) AND CENTER SYMMETRY ("1-form") ARISE IN HILBERT SPACE OF GAUGE THEORY ON TORUS**

2. **WHAT THEY IMPLY**

the main points to make

1. generalized anomalies between discrete symmetries (parity or chiral symmetry) and center symmetry (“1-form”) can be understood using canonical quantization on $\mathbb{T}^3$ with appropriate background fields (= twisted b.c. or “’t Hooft fluxes”)

2. quite unusually, they imply exact degeneracies in the Hilbert space at any finite size $\mathbb{T}^3$, thus also in the infinite volume limit!

3. such degeneracies occur for anomalies involving both invertible and “noninvertible” symmetries

comments/disclaimers/excuses:

will use “old-fashioned” language, one that was around by 1980

not focused on applications, but on gaining simple understanding

study examples; no pretense of generality, no theorems

there are many details that I can’t explain in an hour

plan

1. reminder of old-fashioned "poor man's" language (~1980)
   - 1.1 center symmetry in gauge theories on $\mathbb{T}^3$ ("1-form symmetry")
   - 1.2 electric flux sectors in Hilbert space on $\mathbb{T}^3$
   - 1.3 "magnetic" fluxes on $\mathbb{T}^3$

     ("'t Hooft fluxes" or "'t Hooft twisted b.c."
     "2-form background field for 1-form symmetry")

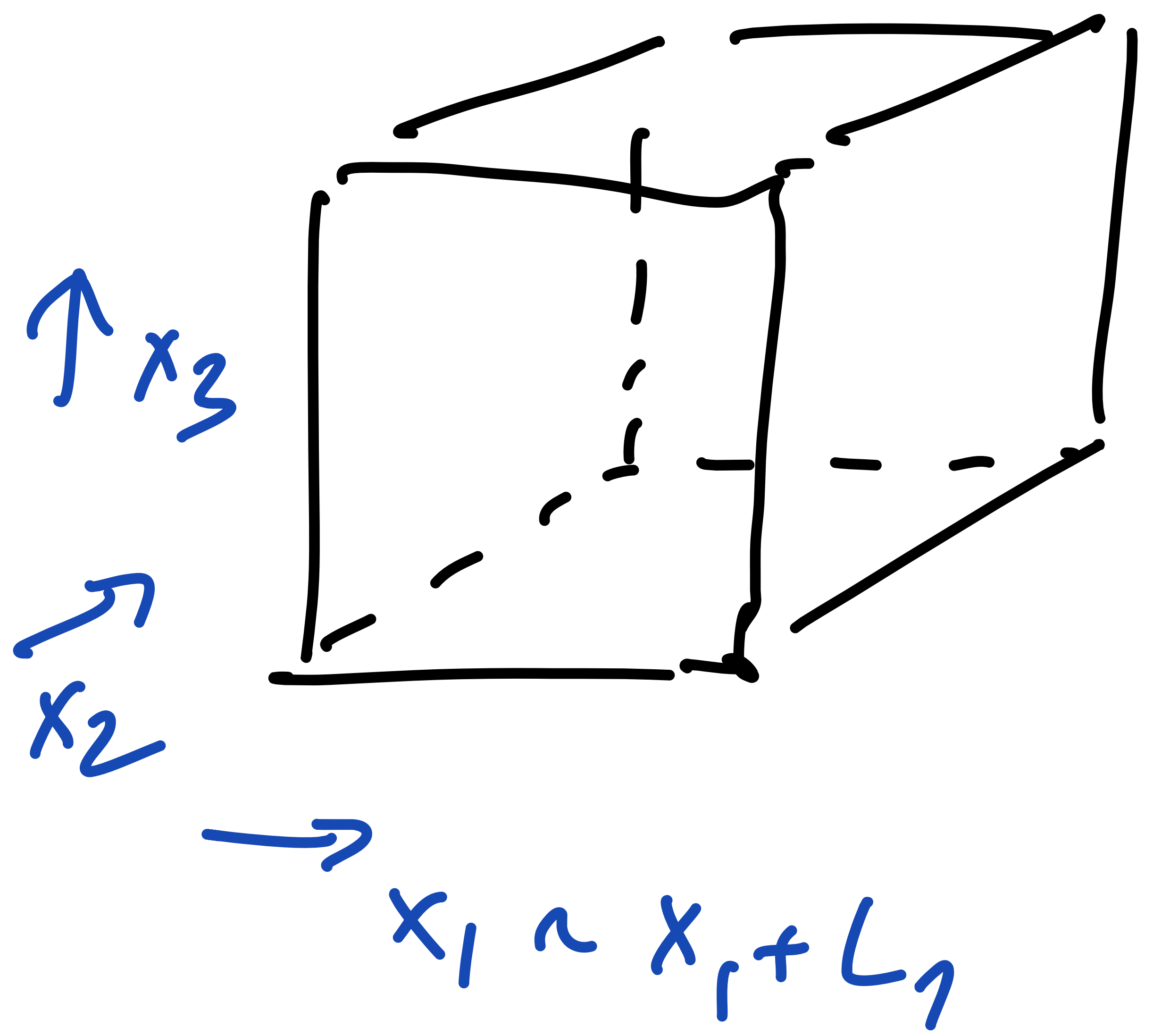
2. the basic anomaly: $\theta-$periodicity anomaly [à la 't Hooft/van Baal -> GKKS+]

3. examples of mixed anomalies & implications
   - 3.1 invertible symmetry in SU(N) with adjoint quarks (like super-YM)
   - 3.2 noninvertible symmetry in $SU(N) \times U(1)$ with two-index S/AS Dirac quarks

1. reminder of old-fashioned language (~1980)

use Hamiltonian quantization on $\mathbb{T}^3$:

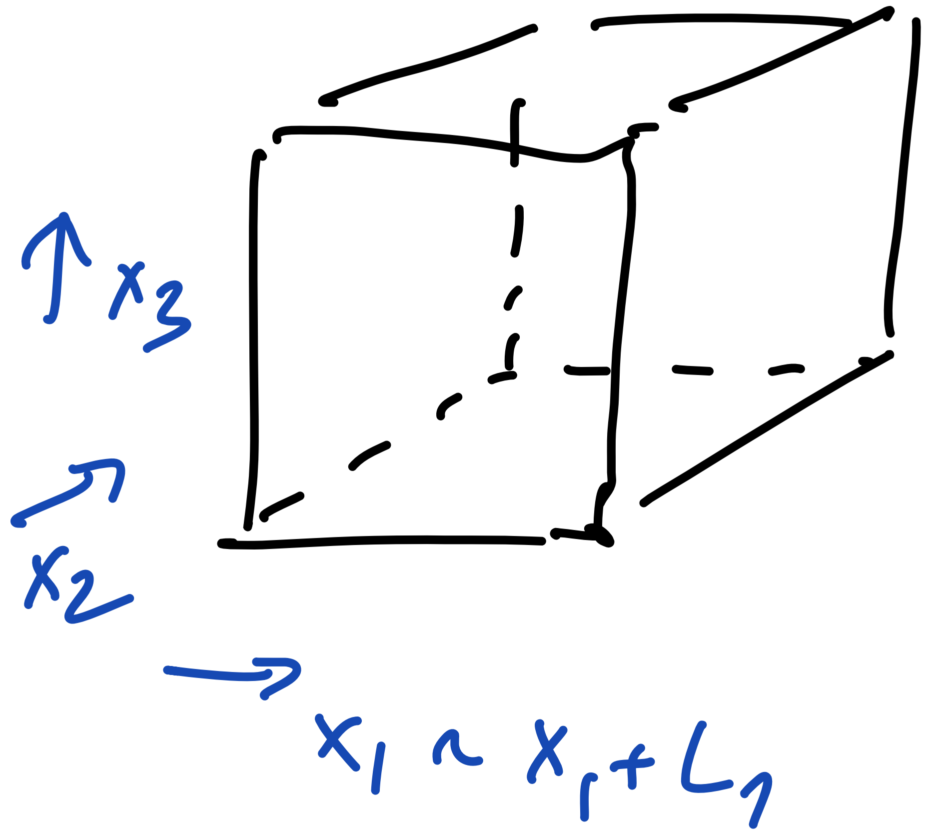
$A_0 = 0$ gauge, states $\Psi[A]$ invariant under time-independent gauge transforms (Gauss' law)



1. reminder of old-fashioned language (~1980)

1.1 center symmetry: $\hat{T}_i, i = 1{,}2{,}3$: “gauge” transforms periodic in x_i up to a center element

$$\hat{T}_i(\vec{x} + \vec{e}_j L_j) = \hat{T}_i(\vec{x})\; e^{i\frac{2\pi}{N}\delta_{ij}}$$



only acts on winding Wilson loops in fundamental

$$\hat{W}_i = \mathrm{tr}_F \mathscr{P}\; e^{i\int_0^{L_i} \hat{A}_i dx^i} \longrightarrow \hat{T}_i \hat{W}_j \hat{T}_i^{-1} = e^{i\frac{2\pi}{N}\delta_{ij}}\; \hat{W}_j$$

- time-direction version familiar from deconfinement transition in pure YM

- modern language: $\mathbb{Z}_N^{(1)}$ 1-form symmetry, only acts on line operators, not on local gauge invariants like $\mathrm{tr}\, F_{\mu\nu}F_{\lambda\sigma}\ldots$

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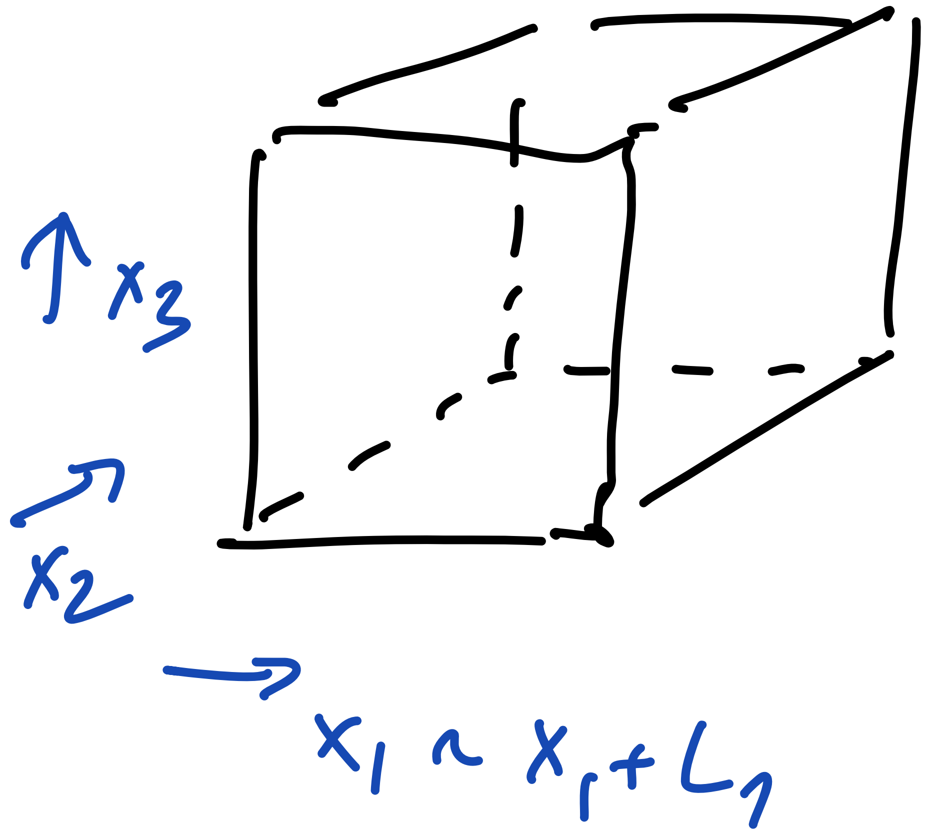

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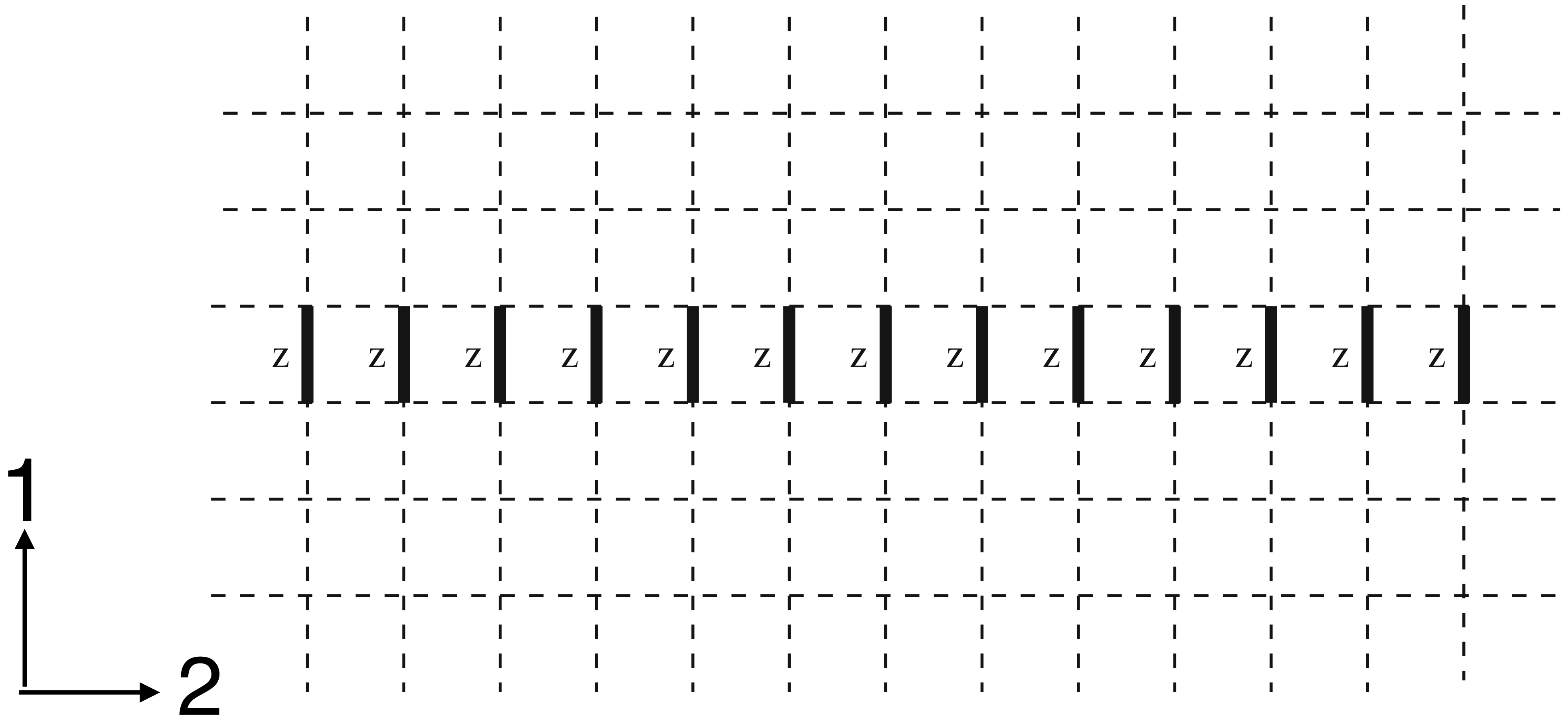
on lattice, $\hat{T}_1$ multiplies by $z = e^{i\frac{2\pi}{N}}$ shown link fields in direction 1 (for all x_3, x_4)

- all nonwinding closed loops invariant
- winding loops transform by z



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if the SU(N) theory <u>has adjoint fields</u> only, $\mathbb{Z}_N^{(1)}$ remains a symmetry, since

$$\hat{\Psi}_{adj} \to \hat{T}_i \hat{\Psi}_{adj} \hat{T}_i^{-1}$$ so transformed field has same b.c. ($\hat{T}$ and $\hat{T}^{-1}$ phases cancel)

if matter representation has nontrivial N-ality (transforms under center), the story changes; need to mention two cases for my examples:

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in SU(N=2k) with two index (S/AS) quarks only $\mathbb{Z}_2^{(1)} \subset \mathbb{Z}_N^{(1)}$ is a symmetry

$\hat{\Psi}_{S/AS} \to \hat{T}_i \hat{\Psi}_{S/AS}\ \hat{T}_i^t$, so transformed field obeys different b.c., only when

$\hat{\Psi}_{S/AS} \to \hat{T}_i^{\frac{N}{2}}\ \hat{\Psi}_{S/AS}\ (\hat{T}_i^{\frac{N}{2}})^t$ transformed under $\mathbb{Z}_2^{(1)}$ is consistent

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in $SU(N) \times U(1)$ with <u>quarks in (N,1)</u> [or (S/AS,1)…] $\mathbb{Z}_N^{(1)}$ remains a symmetry,
with $\hat{T}_i(x)$ $\mathbb{Z}_N$ phase compensated by opposite $\mathbb{Z}_N$ phase in $U(1)$

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physically, in each case: center symmetry =>

stability of some flux tubes

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$$\hat{T}_i(\vec{x} + \vec{e}_j L_j) = \hat{T}_i(\vec{x})\, e^{i\frac{2\pi}{N}\delta_{ij}}$$

in each of these cases, the appropriate $\hat{T}_i$ obey

$[\hat{T}_i, \hat{H}] = 0$ so we can label states in $\mathbb{T}^3$ Hilbert space

by "electric flux" quantum numbers $|E, e_1, e_2, e_3\rangle = |E, \vec{e}\rangle$

$\hat{T}_i |E, \vec{e}\rangle = |E, \vec{e}\rangle\, e^{i\frac{2\pi}{N}e_i}$, three (mod N) integers

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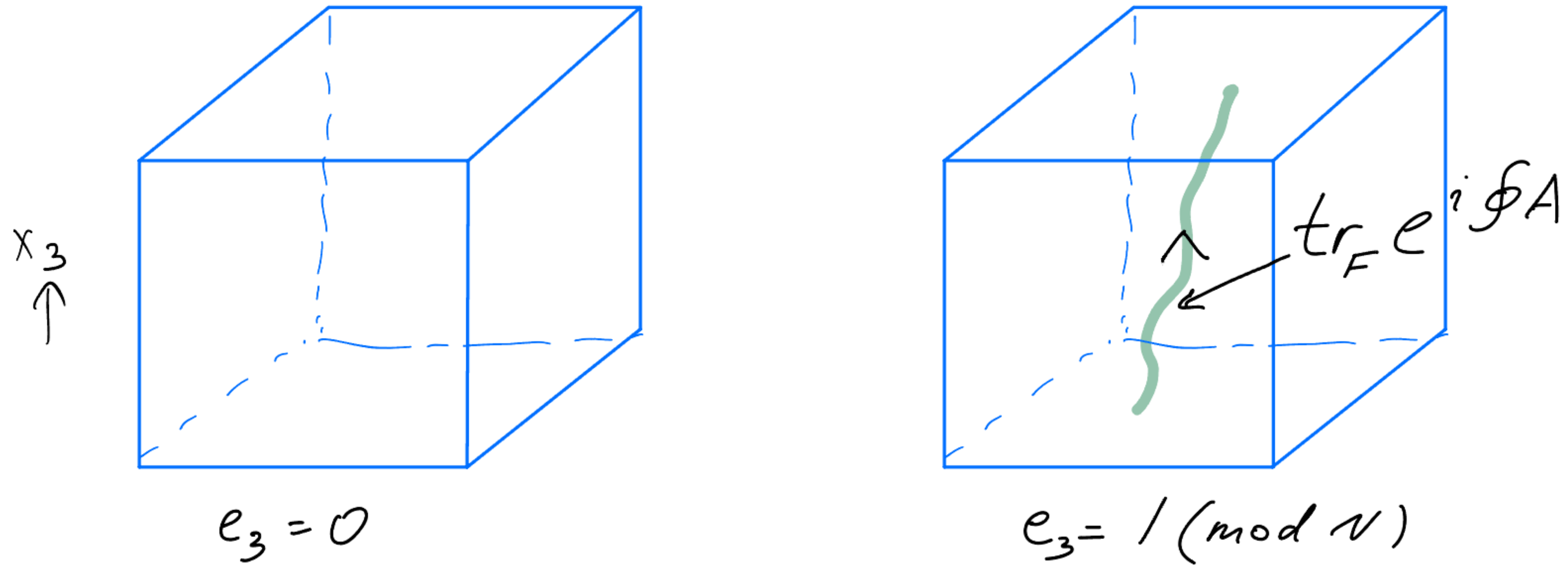
$$[\hat{T}_i, \hat{H}] = 0 \qquad\qquad |E, e_1, e_2, e_3\rangle = |E, \vec{e}\rangle$$

1.2 electric flux sectors in Hilbert space on $\mathbb{T}^3$

value of e_i is changed by one unit by acting with $\hat{W}_i$ on state:

$$\hat{T}_i\,(\hat{W}_i|\vec{e}\rangle) = (\hat{W}_i|\vec{e}_i\rangle)\; e^{i\frac{2\pi}{N}(e_i+1)}$$



in pure YM, at $\theta \neq \pi$, as $L \to \infty$, only one electric flux sector ($\vec{e} = 0$) has finite energy, while all others have energy $\sim L$ with coefficient given by the k-string tension; studied much on and off the lattice:

't Hooft '80, Lüscher '82, van Baal, Witten,…

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("'t Hooft fluxes" or "'t Hooft twisted b.c."
"2-form background field for 1-form symmetry")

"whenever you have global symmetry, it pays to introduce a background gauge field for it"
(Seiberg)

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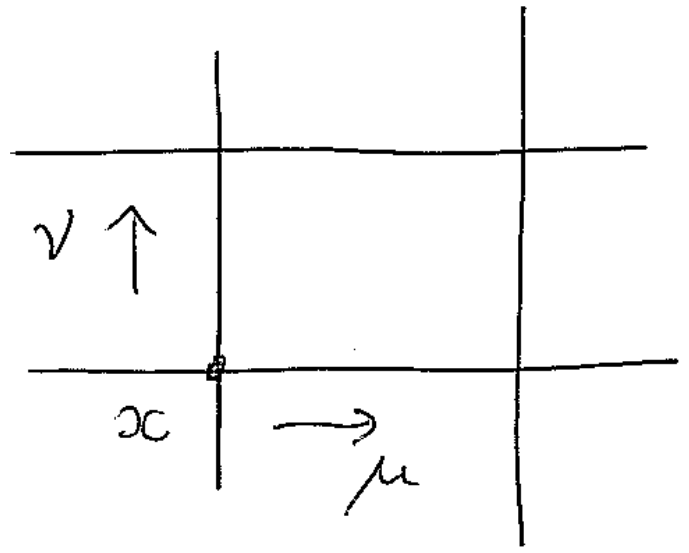
0-form symmetry (usual one, acting on local operators) has 1-form gauge field (link-based)

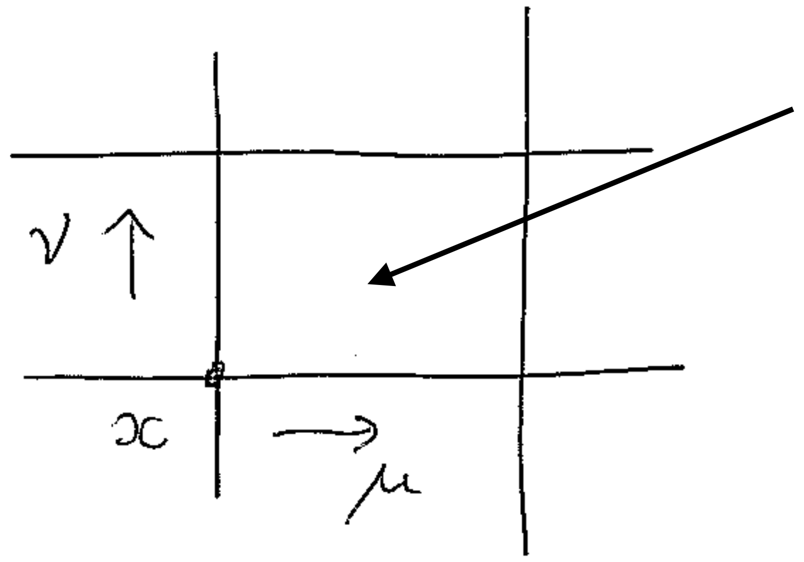
1-form symmetry has 2-form gauge field (plaquette based)



$$U_{x,\mu} \to z_\mu U_{x,\mu}$$

$$z_\mu = e^{i\frac{2\pi}{N} n_\mu}.$$

now, make z_μ x-dependent:



$$e^{iB_{\mu\nu}(x)}$$

plaquette based (2-form) $\mathbb{Z}_N$-valued

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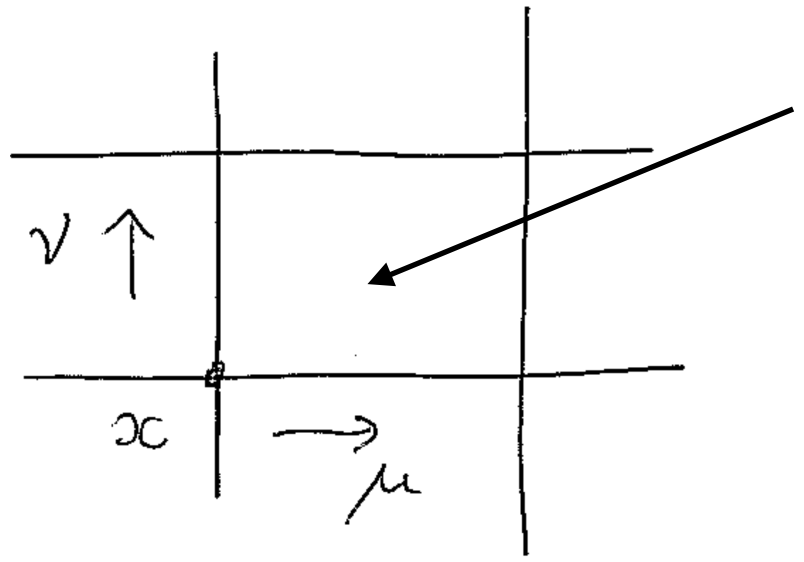
1-form symmetry has 2-form gauge field (plaquette based)

for 1-form gauge field, $\oint A_\mu dx^\mu$ is gauge invariant



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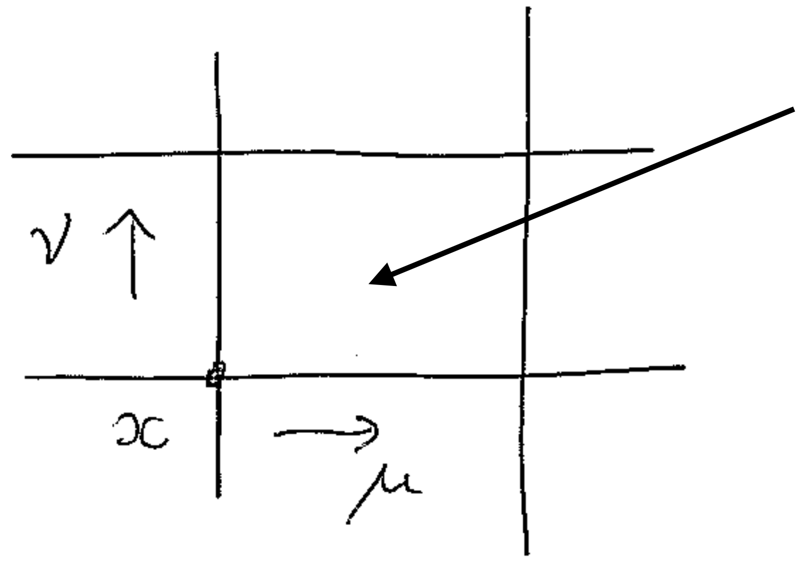
for 2-form abelian/$\mathbb{Z}_N$ gauge field, $\oint B_{\mu\nu} d^2\sigma^{\mu\nu}$

is gauge invariant; on $\mathbb{T}^3$ we can introduce curvature-free background for $\mathbb{Z}_N$ 2-form field



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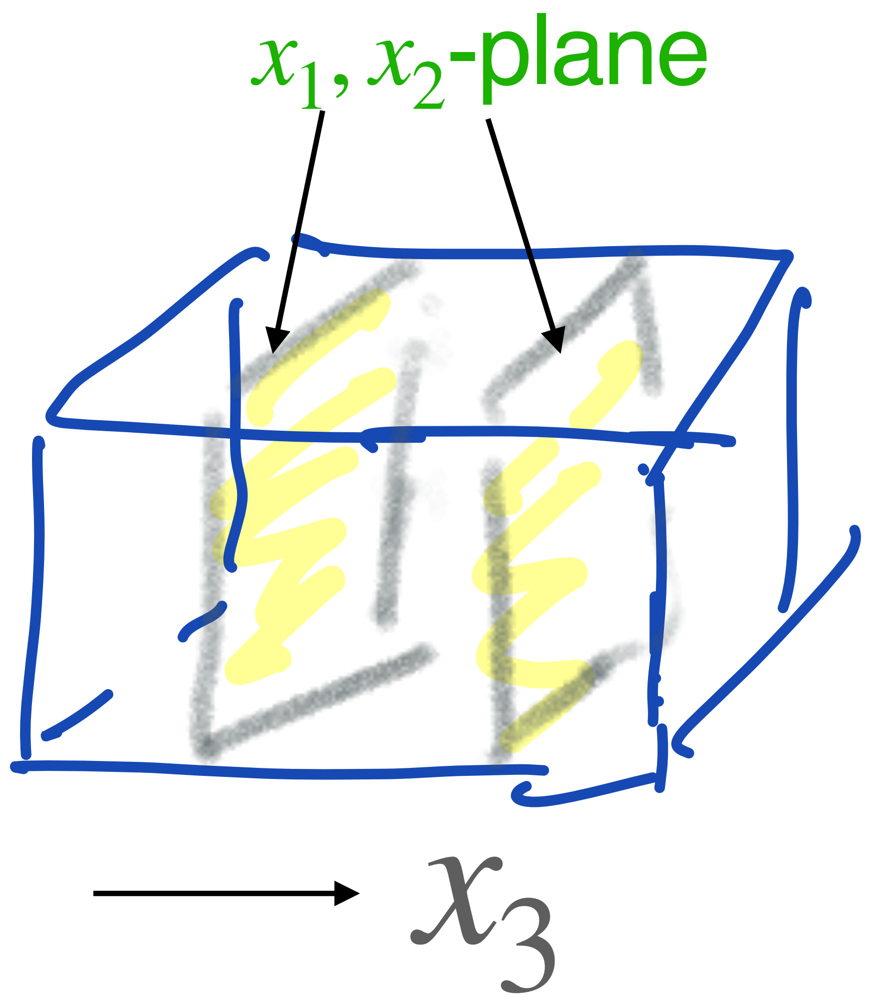
$$\oint dx^1 dx^2 B_{12} = \frac{2\pi m_3}{N} (\mathrm{mod} 2\pi)$$

$$\oint dx^2 dx^3 B_{23} = \frac{2\pi m_1}{N} (\mathrm{mod} 2\pi)$$

$$\oint dx^3 dx^1 B_{31} = \frac{2\pi m_2}{N} (\mathrm{mod} 2\pi)$$



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summary:

in a gauge theory with 1-form symmetry on $\mathbb{T}^3$, introduce a background field, $\vec{m}$, labeling the background (= 't Hooft twist of b.c.; no details…analogy w/ background Wilson loop)

Hilbert space basis is: $|E, \vec{e}\rangle_{\vec{m}}$, with* $\hat{T}_i |E, \vec{e}\rangle_{\vec{m}} = |E, \vec{e}\rangle_{\vec{m}} \; e^{i\frac{2\pi}{N}e_i}$

in thermodynamic limit, usually only $\vec{e} = 0$ have finite energy while dependence on b.c., $\vec{m}$, is expected to be irrelevant, at least for gapped theories

(* at $\theta = 0$)

[check e.g. Teper, Stephenson 1989…]

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2. the basic anomaly: $\theta-$periodicity anomaly [à la 't Hooft/van Baal—> GKKS+]

3. examples of mixed anomalies & implications

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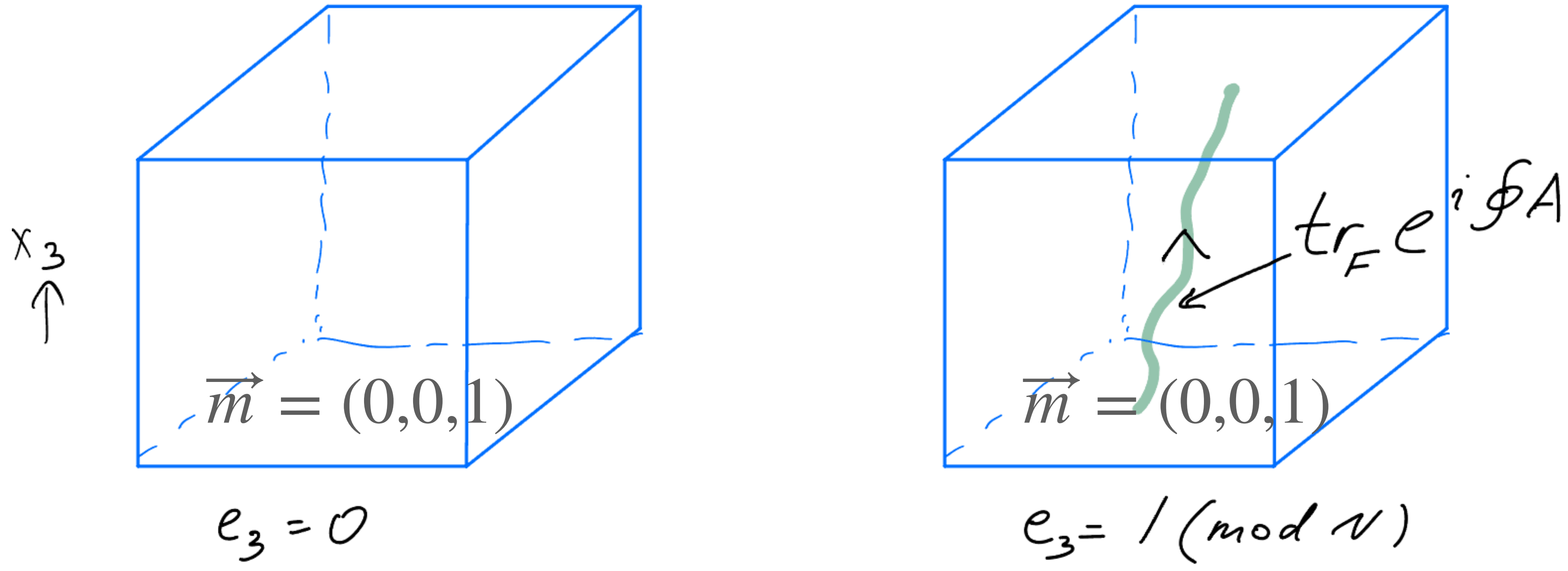

consider a unit “magnetic flux” (twist) in one plane (12, say) only:

$\vec{m} = (0,0,1)$

torus Hilbert space, with or without twists,
splits into N^3 electric flux sectors

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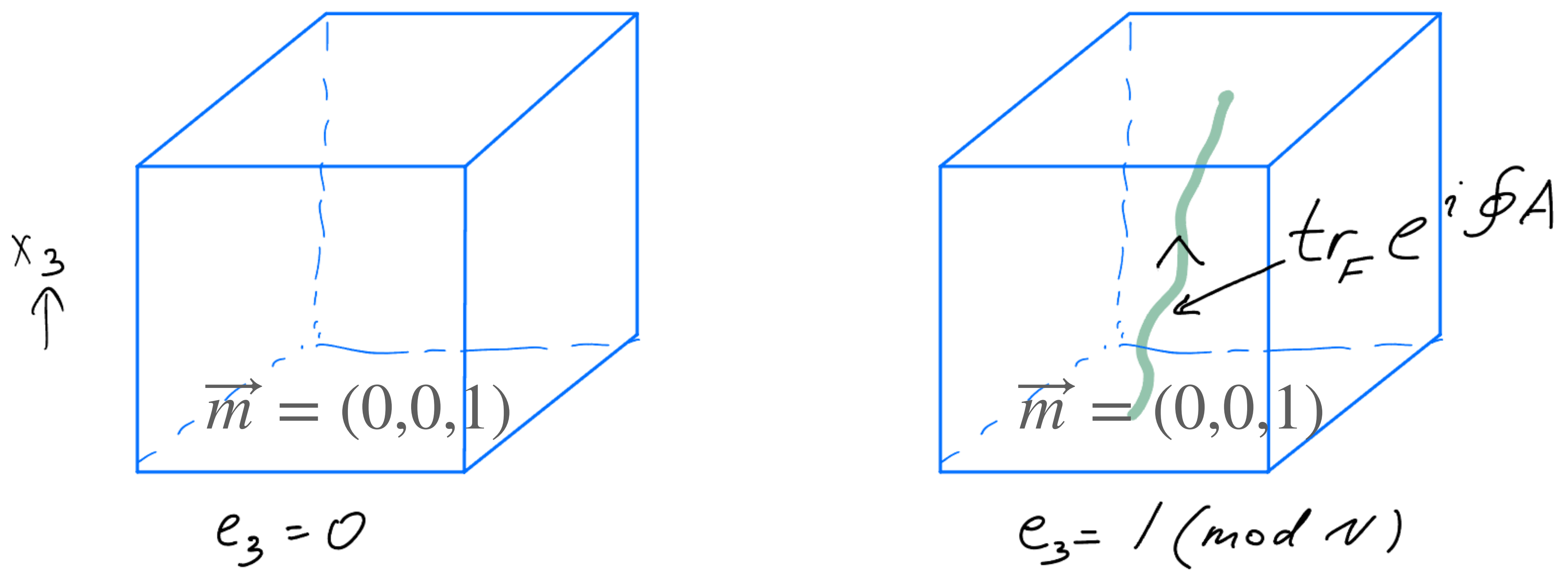

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Crucial observation ('t Hooft ~ 1980)

$\hat{T}_3$, the $Z_N^{(1)}$ generator in the direction orthogonal to the (12) plane of the twist has fractional winding number Q = $\frac{m_3}{N} (\mathrm{mod}\ Z)$

Crucial observation ('t Hooft)

- have to accept (ask later…backup slide)

$\hat{T}_3$, the $Z_N^{(1)}$ generator in the direction orthogonal to the (12) plane of the twist has winding number Q = $\dfrac{m_3}{N}(\mathrm{mod}\ Z)$

[T_3 is a gauge transform, a map from torus to gauge group, so winding makes sense]

3d CS action, $S_{CS} = \int \mathrm{tr}(AdA + \ldots)$, normalized to shift by unity under a unit-winding gauge transformation, so $e^{i2\pi S_{CS}}$ invariant

$e^{i2\pi S_{CS}}$

$$\hat{T}_3\ e^{i2\pi \int_{T^3} \mathrm{tr}(\hat{A}d\hat{A}+\ldots)}\ \hat{T}_3^{-1} = e^{i\frac{2\pi}{N}m_3}\ e^{i2\pi \int_{T^3} \mathrm{tr}(\hat{A}d\hat{A}+\ldots)}$$

however, CS action shifts by $\dfrac{m_3}{N}$ under fractional winding

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$\hat{T}_3$, the $Z_N^{(1)}$ generator in the direction orthogonal to the (12) plane of the twist

has winding number Q = $\frac{m_3}{N}(\text{mod } Z)$

$$(*) \quad \hat{T}_3\, e^{i2\pi \int_{T^3} \text{tr}(\hat{A}d\hat{A}+\ldots)}\, \hat{T}_3^{-1} = e^{i\frac{2\pi}{N}m_3}\, e^{i2\pi \int_{T^3} \text{tr}(\hat{A}d\hat{A}+\ldots)}$$

- fractional winding explained by 't Hooft ~ 1980
- as an equation in Hilbert space (*) appears first in unpublished Ch. 3 of van Baal's PhD thesis, 1984

at the time, (*) significance as an anomaly and implications for spectrum, incl. in TD limit, missed!

- Eq. (*): Hilbert space expression of what GKKS ~2014 call θ -periodicity anomaly (GKKS study Euclidean path integral)

$$U_1\Psi[A] = e^{i\theta}\,\Psi[A] \longrightarrow U_1(e^{i2\pi S_{CS}[A]}\Psi[A]) = e^{i(2\pi+\theta)}\,(e^{i2\pi S_{CS}[A]}\Psi[A])$$

- hence $e^{i2\pi S_{CS}}$ is "operator shifting θ by 2π" (U_1 is operator of unit-winding gauge transform)
- Eq. (*) says that when $m_3 \neq 0$ (mod N), shifting θ by 2π and center symmetry do not commute

we care because 2π shifts of θ can be part of physical symmetry (simplest: parity in pure-YM$_{\theta=\pi}$)

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3. examples of mixed anomalies & implications

- 3.0 will not discuss parity in pure-YM$_{\theta=\pi}$, see Cox, Wandler, EP 2106 for Hilbert space + relation to GKKS Euclidean path integral (for lack of time)
- 3.1 invertible symmetry in $SU(N)$ with adjoint quarks (like super-YM)
- 3.2 noninvertible symmetry in $SU(N) \times U(1)$ with two-index S/AS Dirac quarks

3.1 SU(N) with n_f adjoint Weyl quarks, for definiteness take SYM, $\underline{n_f = 1}$ below:

notation: $Q_{top.} = \frac{1}{32\pi^2}\int d^4x \, F^a_{\mu\nu}F^a_{\lambda\sigma}\epsilon^{\mu\nu\lambda\sigma} =: \int d^4x \, \partial_\mu K^\mu$ with $\int d^3x K^0 \equiv S_{CS}$

classical chiral U(1) $\lambda \to e^{i\alpha}\lambda$ “R-symmetry”

$$\partial_\mu \hat{j}^\mu_f = \partial_\mu(\hat{\lambda}^{a\,\dagger}\bar{\sigma}^\mu\hat{\lambda}^a) = 2n_f N \partial_\mu \hat{K}^\mu \longrightarrow$$

R-current not conserved

$$\hat{J}^\mu_5 = \hat{j}^\mu_f - 2n_f N \hat{K}^\mu$$

$$\hat{Q}_5 = \int d^3x \hat{J}^0_5 = \int d^3x \hat{j}^0_f - 2n_f N \int d^3x \hat{K}^0 \longrightarrow$$

$\hat{Q}_5$ conserved but not gauge invariant

$(n_f = 1)$

$$\hat{X}_{2N} = e^{i\frac{2\pi}{2N}\hat{Q}_5} = e^{i\frac{2\pi}{2N}\int d^3x \hat{j}^0_f} \, e^{-i2\pi\int d^3x \hat{K}_0} \longrightarrow$$

gauge invariant operator of $Z^{(0)}_{2N}$ discrete R-symmetry

$$\hat{T}_3 \, e^{i2\pi\int_{T^3}\mathrm{tr}(\hat{A}d\hat{A}+\ldots)} \, \hat{T}_3^{-1} = e^{i\frac{2\pi}{N}} \, e^{i2\pi\int_{T^3}\mathrm{tr}(\hat{A}d\hat{A}+\ldots)} \Longrightarrow \hat{T}_3 \, \hat{X}_{2N} \, \hat{T}_3^{-1} = e^{-i\frac{2\pi}{N}} \, \hat{X}_{2N}$$

$\uparrow$

$\int d^3x \hat{K}_0 = S_{CS}$

on $\mathbb{T}^3$ with $m_3 = 1$: mixed 0-form/1-form anomaly

Ex 3.1: *SYM on twisted T^3 - invertible chiral/center anomaly*

Hilbert space with spatial ‘t Hooft twist $n_{12} = m_3 = 1$; SYM has two global symmetries, $\hat{T}_3$ and $\hat{X}_{2N}$, 1-form and 0-form, invertible (=normal unitary operators on Hilbert space) commute with Hamiltonian, but not with each other:

$$\hat{T}_3 \, \hat{X}_{2N} \, \hat{T}_3^{-1} = e^{-i\frac{2\pi}{N}} \, \hat{X}_{2N} \quad \longrightarrow \quad \hat{X}_{2N} \, | \, E, e_3 \rangle = | \, E, e_3 - 1 \rangle$$

action of chiral symmetry changes e_3 flux of state but not energy

all energy levels on the twisted T^3 are N-fold degenerate,
exact degeneracy at any finite volume, provided $n_{12} = m_3 = 1$!

[Cox, Wandler, EP 2106]

unusual in QFT!
different from topological order (e.g. $\mathbb{Z}_2$ in superconductors)
where degeneracy only in “topological scaling limit”

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<u>exact degeneracy at any finite volume,</u> provided $n_{12} = m_3 = 1$!

[Cox, Wandler, EP 2106]

as volume goes to infinity, if theory confines (center unbroken) clustering ground states are the lowest energy degenerate flux states, related by broken discrete chiral symmetry
here, a consequence of the mixed anomaly, not SUSY!

gaugino bilinear phase in different flux sectors: $\langle E, e_3 | \mathrm{tr}\lambda\lambda | E, e_3 \rangle = e^{i\frac{2\pi}{N}} \langle E, e_3 + 1 | \mathrm{tr}\lambda\lambda | E, e_3 + 1 \rangle$

degeneracy does not require SUSY, similar degeneracies in non-SUSY QCD(adj)
exact degeneracies less severe if gauge group has smaller center… SP, Spin, E6, E7

[Cox, Wandler,
EP 2106]

Ex 3.2: $SU(N) \times U(1)$+ S/AS Dirac - noninvertible chiral/center anomaly

[Anber, EP 2305]

$$(\overset{\text{gauged}}{\overline{SU(N), U(1)}}, \overset{\text{global classical chiral}}{U(1)_\chi})$$

$$\psi_R \sim (\overset{\downarrow}{R}, \overset{\downarrow}{1}, \overset{\swarrow}{1})$$

$$\psi_{\bar{R}} \sim (\bar{R}, -1, 1)$$

+ global $Z_N^{(1)}$ 1-form center: $\hat{T}_i$ now are SU(N) gauge trfs periodic up to center

simultaneous with U(1) gauge transforms periodic up to $\mathbb{Z}_N \subset U(1)$

(to make transformed fermions single valued on the torus)

$d_R = \frac{N(N\pm1)}{2}$ for S/AS

$T_R = N \pm 2$

in the non-gauged U(1) (vector) case, $U(1)_\chi \rightarrow Z^{(0)}_{2T_R}$ by anomaly (like SYM)

with gauged U(1), $Z^{(0)}_{2T_R}$ becomes “noninvertible” $\tilde{Z}^{(0)}_{2T_R}$

to see, integrate anomaly equation to find conserved charge operator:

$$\partial_\mu j^\mu_\chi - \underset{\substack{\uparrow \\ U(1)_\chi SU(N)^2}}{2T_R \partial_\mu K^\mu_{SU(N)}} - \frac{2d_R}{8\pi^2} \underset{\substack{\uparrow \\ U(1)_\chi U(1)^2}}{\epsilon_{\mu\nu\lambda\sigma}} \partial^\mu a^\nu \partial^\lambda a^\sigma = 0$$

in the non-gauged U(1) (vector) case, $U(1)_\chi \to Z^{(0)}_{2T_R}$ by anomaly (~ SYM)

with gauged U(1), $Z^{(0)}_{2T_R}$ becomes “noninvertible” $\tilde{Z}^{(0)}_{2T_R}$

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$$Q_\chi = \int_{\mathbb{T}^3} d^3x \left[j^0_\chi - 2T_R \overset{\text{SU(N) CS}}{\overset{\swarrow}{K^{CS}(A)}} - 2d_R \overset{\text{U(1) CS}}{\overset{\swarrow}{K^0(a)}} \right]$$

+ boundary terms [depend on U(1)], due to twists of b.c. (not shown)

$$X_{2T_R} = e^{i\frac{2\pi}{2T_R}Q_\chi}$$

gauge invariant under large SU(N) gauge transforms (exactly as in SYM)

in the absence of dynamical U(1) fields, $\mathbb{Z}_{2T_R} \subset U(1)_\chi$ is anomaly free invertible (i.e. acts as a normal unitary operator) global symmetry in the non-gauged U(1) (vector) case, $U(1)_\chi \to Z^{(0)}_{2T_R}$ by anomaly (~ SYM)

with gauged U(1), $Z^{(0)}_{2T_R}$ becomes “noninvertible” $\tilde{Z}^{(0)}_{2T_R}$

to see, integrate anomaly equation to find conserved charge operator:

$$\partial_\mu j^\mu_\chi - 2T_R \partial_\mu K^\mu_{SU(N)} - \frac{2d_R}{8\pi^2}\epsilon_{\mu\nu\lambda\sigma}\partial^\mu a^\nu \partial^\lambda a^\sigma = 0$$

SU(N) CS

U(1) CS

$Q_\chi = \int_{\mathbb{T}^3} d^3x \left[j^0_\chi - 2T_R K^{CS}(A) - 2d_R K^0(a) \right]$ + boundary terms [depend on U(1)], due to twists of b.c. (not shown)

$X_{2T_R} = e^{i\frac{2\pi}{2T_R}Q_\chi}$ gauge invariant under large SU(N) gauge transforms (exactly as in SYM)

but under large (winding) U(1) trfs $a_i \to a_i - \partial_i \lambda,\ \text{with}\ \lambda(x + \hat{e}_i L_i) = \lambda(x) + 2\pi n_i$ X_{2T_R} changes:

$$X_{2T_R}[A, a - d\lambda] = X_{2T_R}[A, a]\ e^{-i2\pi\left[n_x \frac{d_R}{T_R}(\mu_{yz} + \frac{2}{N}m_x) + \text{cyclic}(\text{x} \to \text{y} \to \text{z})\right]}$$

background for $\mathbb{Z}_N^{(1)}$

$\oint dydzB_{yz} = \frac{2\pi m_x}{N}(\text{mod}2\pi)$

winding number of U(1) trfm

integer (quantized U(1) magnetic flux in yz plane)

$Z^{(0)}_{2T_R}$ operator not gauge invariant for $n_{x,y,z} \neq 0$:

$$X_{2T_R}[A, a - d\lambda] = X_{2T_R}[A, a]\; e^{-i2\pi\left[n_x \frac{d_R}{T_R}(\mu_{yz} + \frac{2}{N} m_x) + \text{cyclic(x} \to \text{y} \to \text{z)}\right]}$$

define noninvertible $\tilde{Z}^{(0)}_{2T_R}$ operator = sum over large U(1) gauge transforms [Karasik; Iqbal, García Extebarria 2022]

∃ other ways…appear equivalent [Choi, Lam, Shao; Cordova, Ohmori 2022]

$$\tilde{X}_{2T_R} = e^{i\frac{2\pi}{2T_R}Q_\chi}\left(\sum_{n_x \in \mathbb{Z}} e^{-i2\pi n_x \underline{\underline{\frac{d_R}{T_R}(\mu_{yz} + \frac{2}{N} m_x)}}}\right) \times \; (\text{cyclic(x} \to \text{y} \to \text{z))}$$

- $\tilde{X}_{2T_R}$ acts as a unitary gauge invariant operator in sectors of Hilbert space $\frac{d_R}{T_R}(\mu_{yz} + \frac{2}{N} m_x) \in \mathbb{Z}$

- all other sectors are annihilated by $\tilde{X}_{2T_R}$; thus it is a kind of projection operator - no inverse

⟶ “noninvertible” discrete $\mathbb{Z}_{2T_R}$ chiral symmetry

summary, Ex.2:

$$(\overbrace{SU(N), U(1)}^{\text{gauged}}, \overset{\text{global}}{U(1)_\chi})$$

$$\psi_R \sim (R, 1, 1)$$

$$\psi_{\bar{R}} \sim (\bar{R}, -1, 1)$$

$T_R = N \pm 2$

$d_R = \frac{N(N\pm 1)}{2}$ for S/AS

for us, on $\mathbb{T}^3$, noninvertible $\tilde{Z}^{(0)}_{2T_R}$ operator is:

$$\tilde{X}_{2T_R} = e^{i\frac{2\pi}{2T_R}Q_\chi}$$, when $l_x \equiv \frac{d_R}{T_R}(\mu_{yz} + \frac{2}{N}m_x) \in \mathbb{Z}$ (+ cyclic)

= 0, otherwise

commutation relation with $Z^{(1)}_N$ (easily computable from all above!):

$$T_x\, \tilde{X}_{2T_R}\, T_x^{-1} = \tilde{X}_{2T_R}\, e^{-i2\pi(\frac{m_x}{N} - \frac{2}{N}l_x)}$$

$l_x \equiv \frac{d_R}{T_R}(\mu_{yz} + \frac{2}{N}m_x) \in \mathbb{Z}$

↑

$Z^{(1)}_N$ in x-direction

↑

<u>mixed anomaly!</u>

as in SYM, if phase not unity

—> degeneracy!

exact on any torus

between $Z^{(1)}_N$ flux sectors

summary, Ex.2:

gauged global

$(SU(N), U(1), U(1)_\chi)$

$$\psi_R \sim (R, 1, 1)$$

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$= 0$, otherwise

commutation relation with $Z_N^{(1)}$ (easily computable from all above!):

$$T_x\, \tilde{X}_{2T_R}\, T_x^{-1} = \tilde{X}_{2T_R}\, e^{-i2\pi(\frac{m_x}{N} - \frac{2}{N}l_x)}$$

$l_x \equiv \frac{d_R}{T_R}(\mu_{yz} + \frac{2}{N}m_x) \in \mathbb{Z}$

↑

$Z_N^{(1)}$ in x-direction

↑

mixed anomaly!
as in SYM, if phase not unity
—> degeneracy!
exact on any torus
between $Z_N^{(1)}$ flux sectors

- degeneracy due to mixed center/noninvertible chiral seen on torus with specific flux sectors (not seen in others!); holds at any size

- infinite volume limit expected independent of b.c.,
- degeneracy should persist: implies symmetry breaking

[Anber, EP 2305]

first (only?) class of 4d examples of anomaly involving noninvertible + dynamical consequence!

obtained using only pedestrian old-fashioned tools…

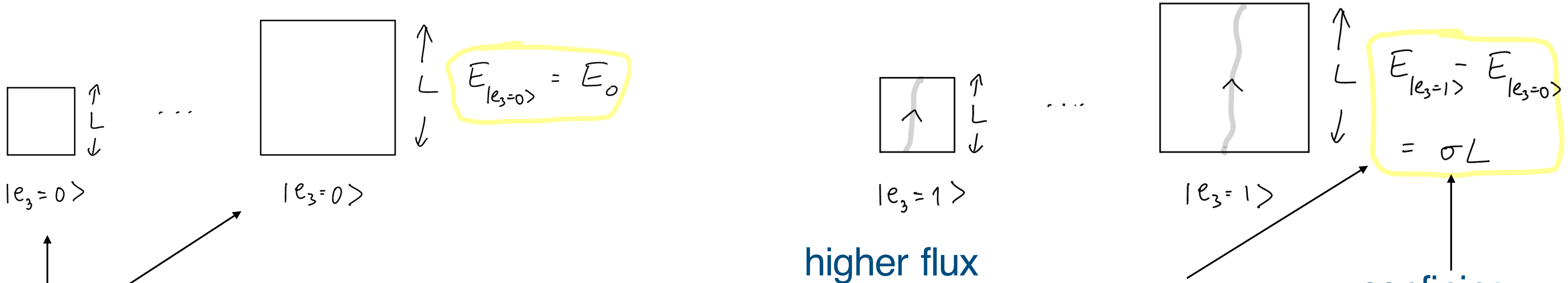
summary:

Studying a gauge theory on torus with twisted b.c.
(=in 2-form background fields for the 1-form symmetry)
is a powerful probe of the dynamics, especially in the presence of anomalies.

Mixed anomaly of invertible or noninvertible chiral symmetries with center symmetry implies exact degeneracy of flux sectors, which remains in infinite volume limit.

Cartoon picture to remember:

A.) no anomaly: lowest energy in $e_3 \neq 0$ flux sector $\rightarrow \sigma L \rightarrow \infty$



torus with 2-form background (any),
upon increasing size to ∞

higher flux
sectors decouple at $L \rightarrow \infty$

confining
string tension
(lattice!)

[Teper, Stephenson;
González-Arroyo,…1990s]

summary:

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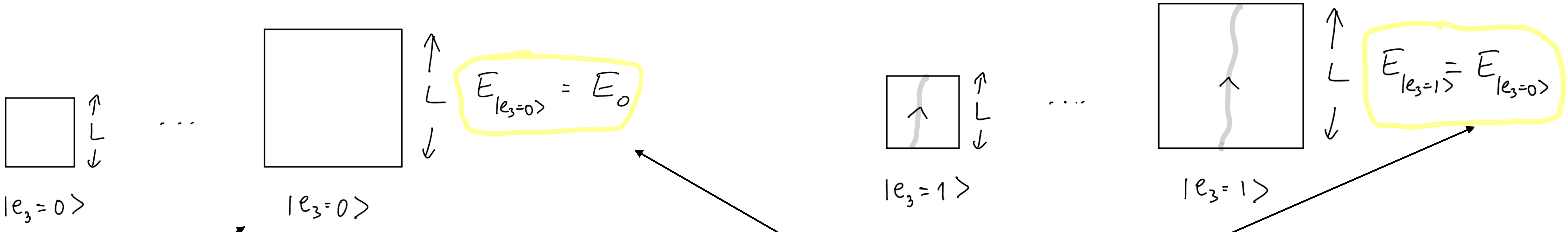
Cartoon picture to remember: for $\mathbb{Z}_2$ valued anomaly

B.) anomaly: lowest energies of $e_3 = 0$, $e_3 = 1$ flux sectors remain equal



torus with 2-form background
(the one revealing anomaly!),
upon increasing size to ∞

interchanged by chiral symmetry $\forall L < \infty$

(or parity, in pure YM at $\theta = \pi$)

summary:

Studying a gauge theory on torus with twisted b.c.
(=in 2-form background fields for the 1-form symmetry)
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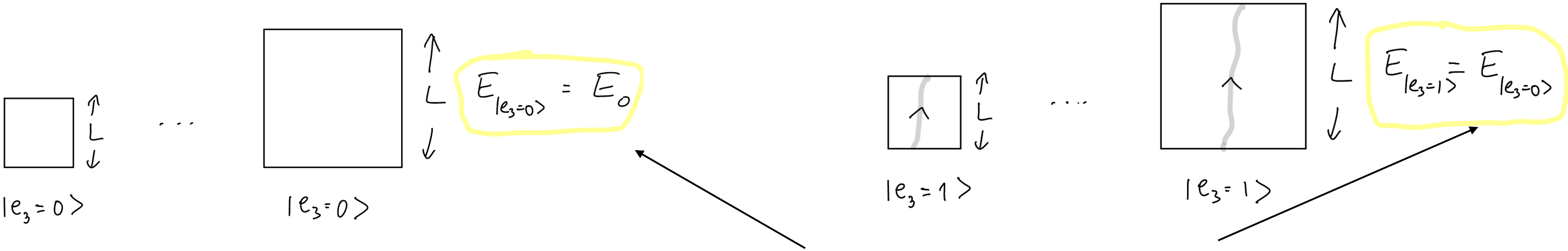

interchanged by chiral symmetry $\forall L < \infty$

infinite L, if center unbroken: these are the clustering vacua, chiral broken

summary:

Studying a gauge theory on torus with twisted b.c.
(=in 2-form background fields for the 1-form symmetry)
is a powerful probe of the dynamics, especially in the presence of anomalies.

Mixed anomaly of invertible or noninvertible chiral symmetries with center symmetry implies exact degeneracy of flux sectors, which remains in infinite volume limit.

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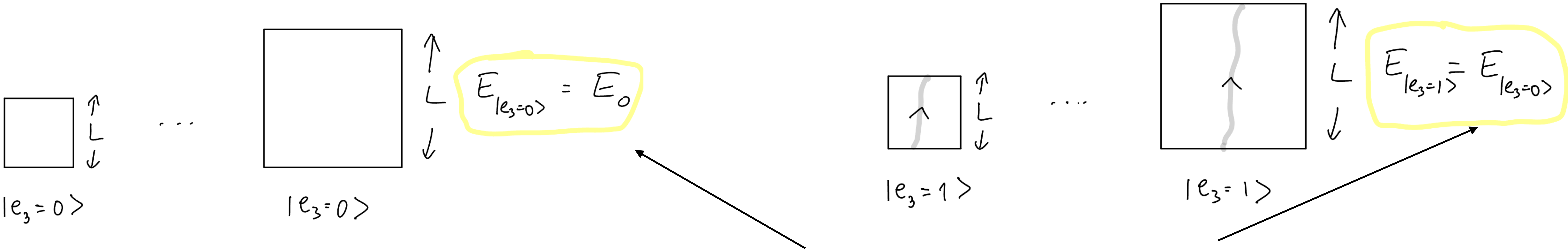
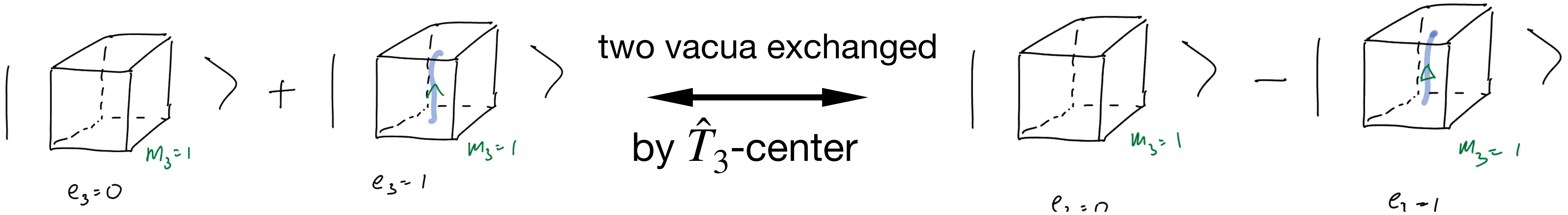


interchanged by chiral symmetry $\forall L < \infty$

infinite L, if center broken:
deconfinement (Z_2 example)


two vacua exchanged

by $\hat{T}_3$-center

outlook:

what I worry about presently…:

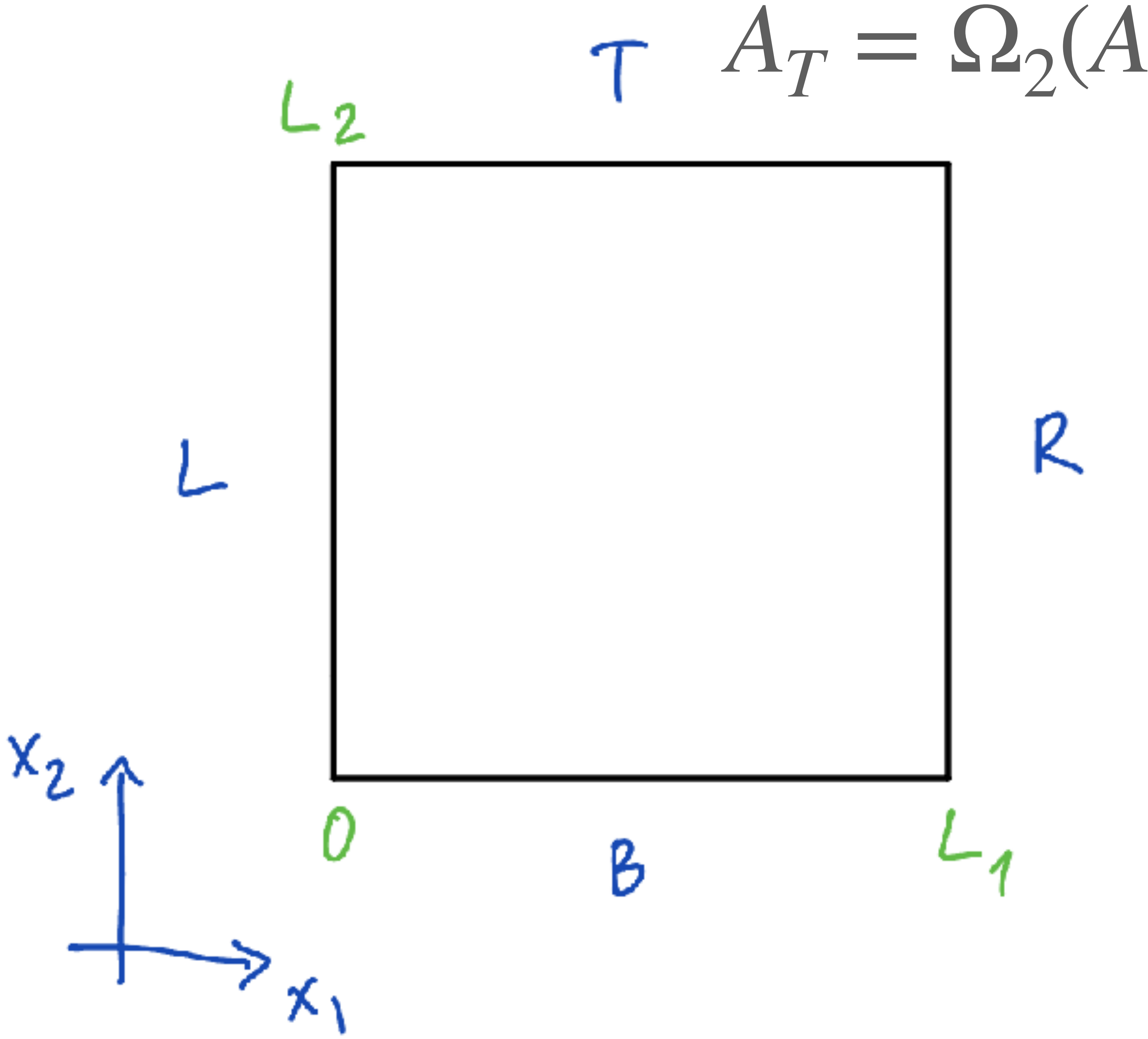
- there is a lot of more mathematically oriented work on noninvertible symmetries, does it also allow more pedestrian ways (so I can look at)?
- some puzzles about infinite volume limit vs finite torus dynamical calculations in supersymmetry, notably the ones presented by the gaugino condensate computed with 't Hooft fluxes (works with Anber, EP 2210, 2307;…)

…beyond me, but interesting:

- the exact degeneracies may be useful for lattice studies (twists are trivially put on lattice) especially if they ever approach $\theta \sim \pi$ regime of pure YM or tackle exact chiral symmetries
- is it useful for constraining/studying real world (real or imagined, e.g. BSM) theories?
- what are the most general consistency conditions following from all possible anomalies in a given theory? (as we saw, things were missed 1980 -> 2014!)

technical back up slides

a quick intro to poor man's twisted bundle

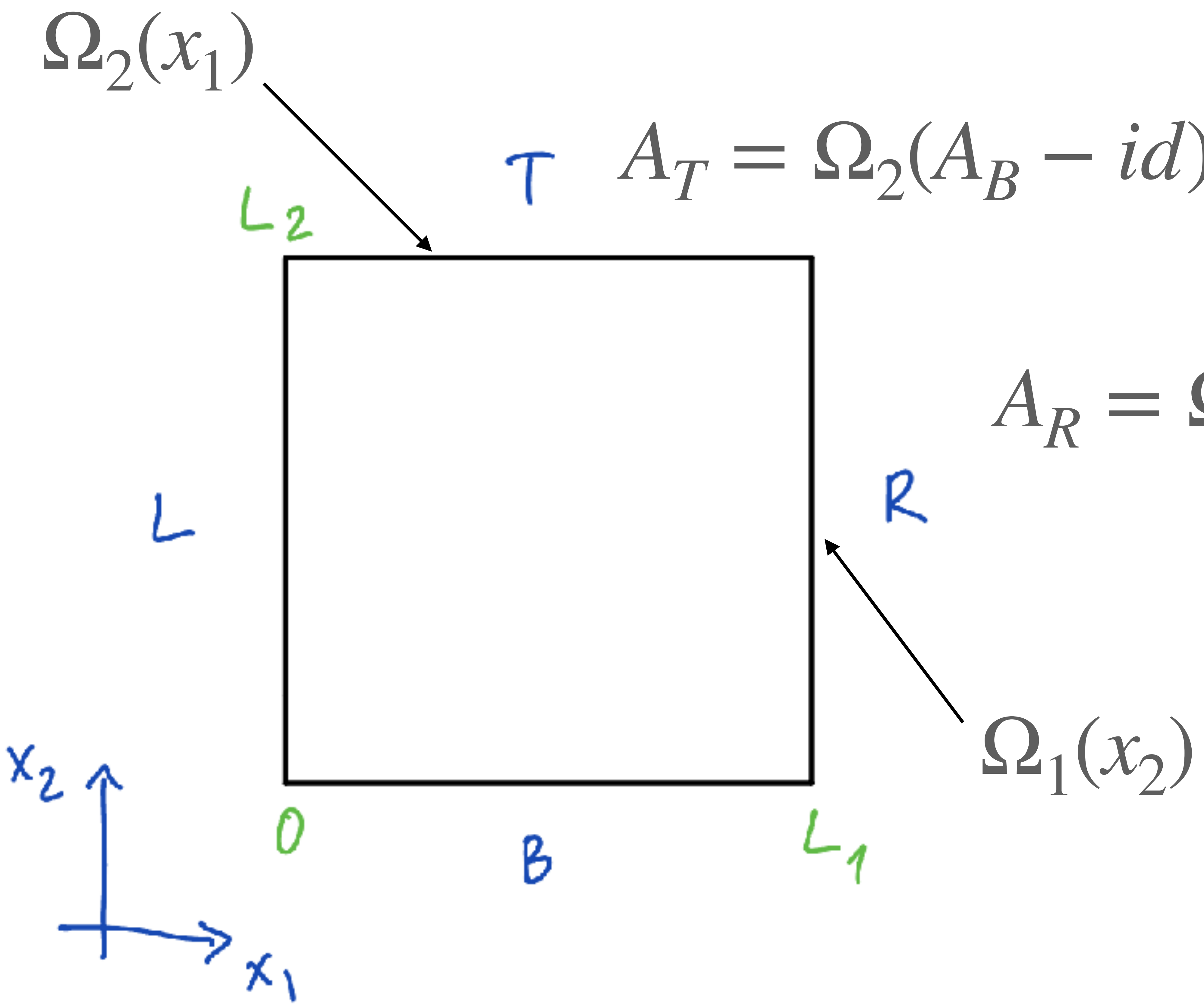


$$A_T = \Omega_2(A_B - id)\Omega_2^{-1}$$

$$A_R = \Omega_1(A_L - id)\Omega_1^{-1}$$

't Hooft ~1980

a quick intro to poor man's twisted bundle



$$A_T = \Omega_2(A_B - id)\Omega_2^{-1}$$

$$A_R = \Omega_1(A_L - id)\Omega_1^{-1}$$

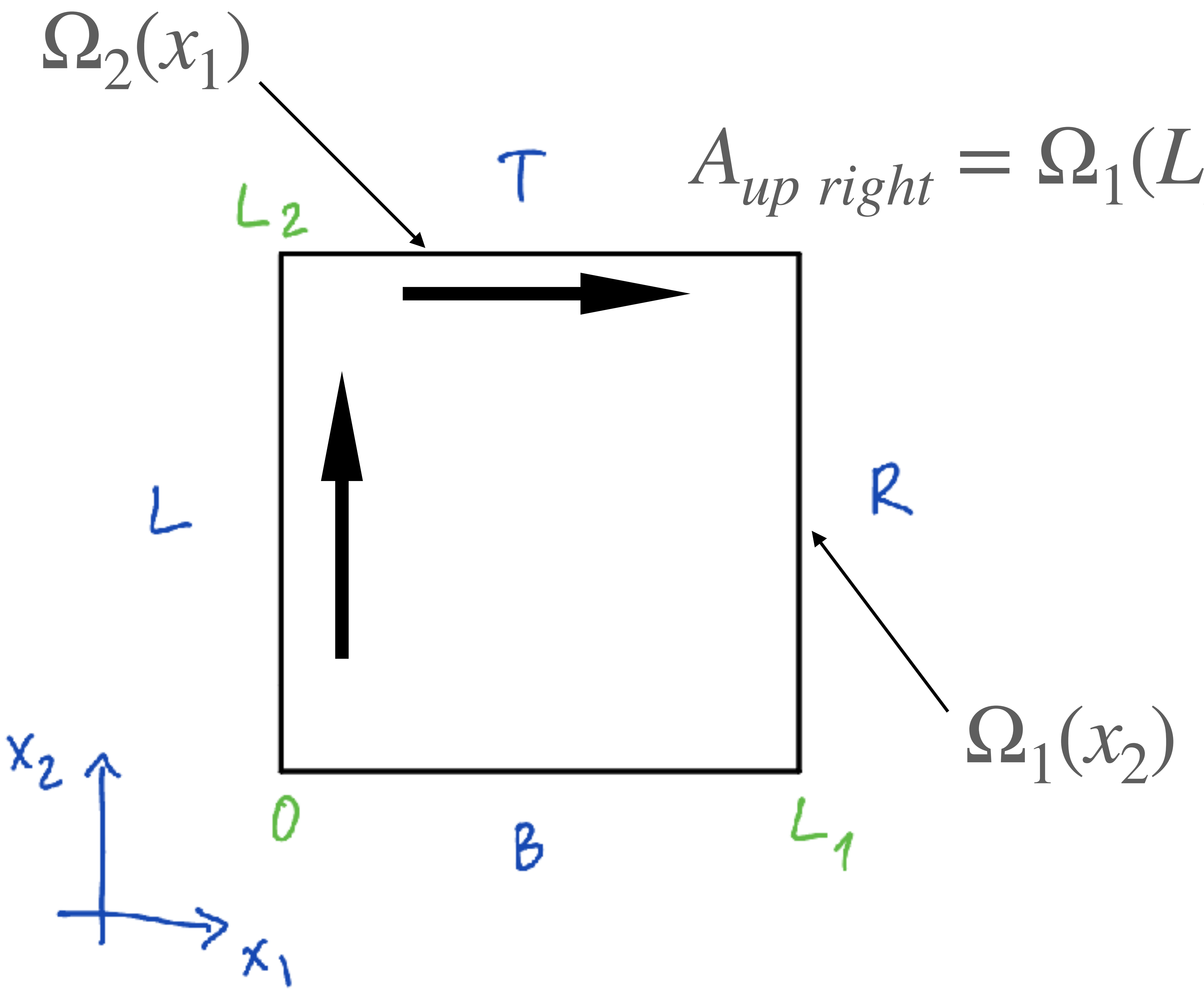
$$A_R = g_R(A_R - id)g_R^{-1}$$

$$A_L = g_L(A_R - id)g_L^{-1}$$

$$\implies \Omega_1 = g_R \Omega_1 g_L^{-1}$$

't Hooft ~1980

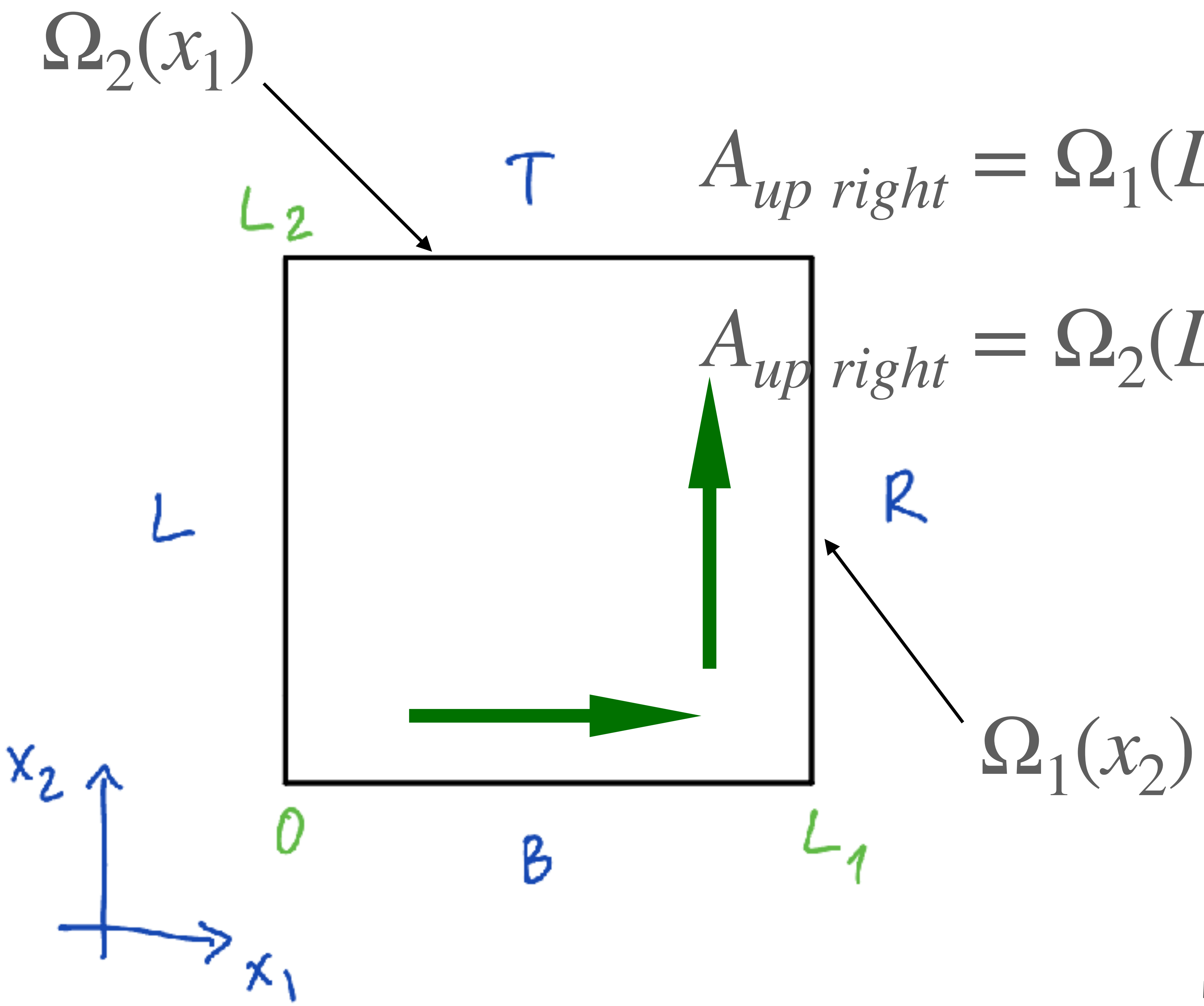
a quick intro to poor man’s twisted bundle



$$A_{up\ right} = \Omega_1(L_2)\Omega_2(0)(A_{bottom\ left} - id)(\Omega_1(L_2)\Omega_2(0))^{-1}$$

’t Hooft ~1980

a quick intro to poor man's twisted bundle



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$$A_{up\ right} = \Omega_2(L_1)\Omega_1(0)(A_{bottom\ left} - id)(\Omega_2(L_1)\Omega_1(0))^{-1}$$

in center of G

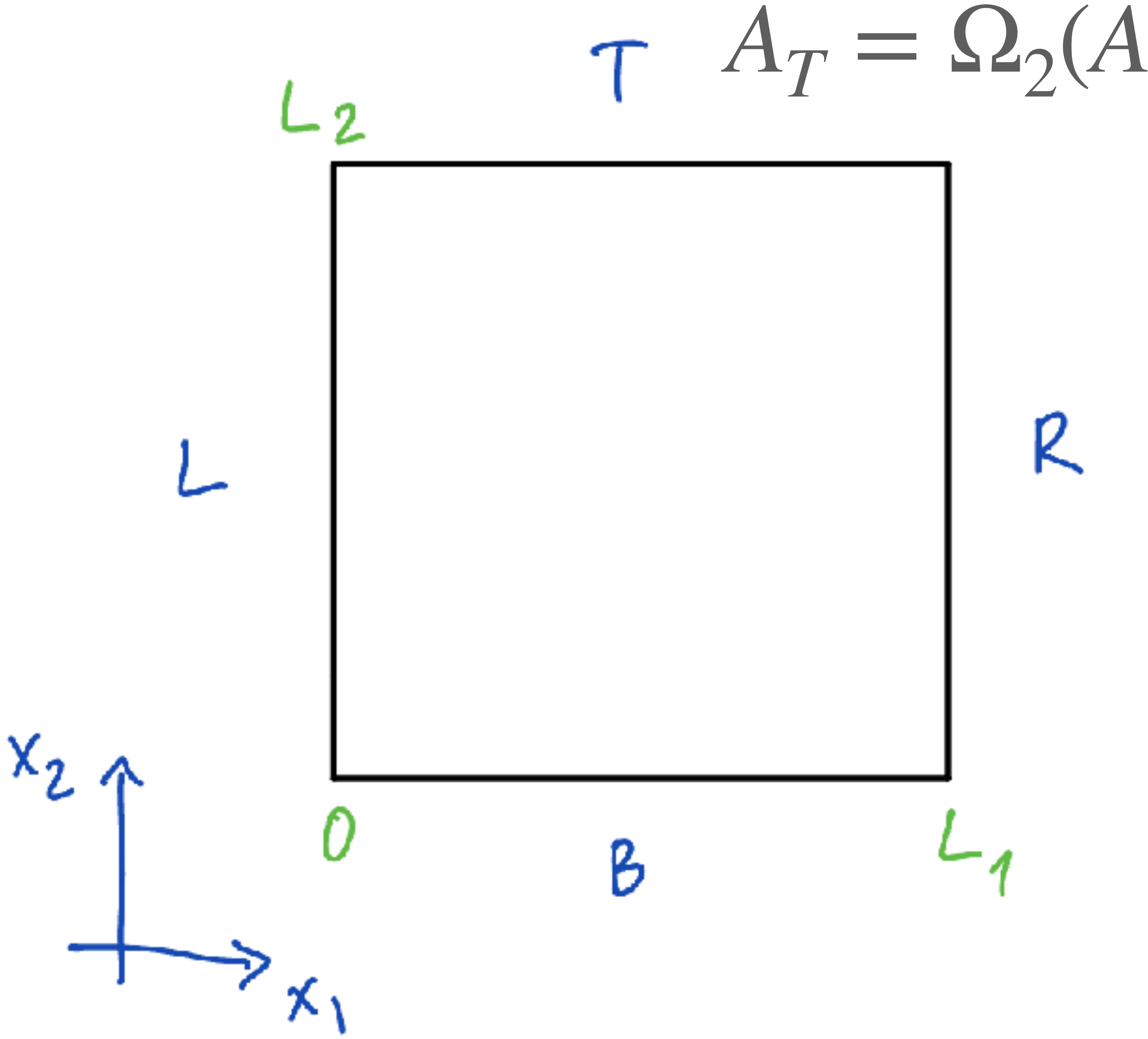
single-valuedness $\Longrightarrow$ $\Omega_1(L_2)\Omega_2(0) = e^{i\alpha}\, \Omega_2(L_1)\Omega_1(0)$

't Hooft ~1980

a quick intro to poor man's twisted bundle

$$\psi_T = \Omega_2^q \, \psi_B$$

$$A_T = \Omega_2 (A_B - id) \Omega_2^{-1}$$



$$A_R = \Omega_1 (A_L - id) \Omega_1^{-1}$$

$$\psi_R = \Omega_1^q \, \psi_L$$

for U(1): q=charge

for SU(N):

- *"q=N" - adjoint(same as A)*
- *q=1 - fund.*
- *q=2 - 2-index S/A*

$$\Longrightarrow \quad \Omega_1(L_2)\Omega_2(0) = e^{i\alpha} \, \Omega_2(L_1)\Omega_1(0)$$

for U(1) and nonzero q: $e^{i\alpha} = e^{i\frac{2\pi}{q}n_{12}}$ **;** *for U(1) and q=0, any* $e^{i\alpha}$

a quick intro to poor man's twisted bundle

$$\psi_T = \Omega_2^q \, \psi_B$$

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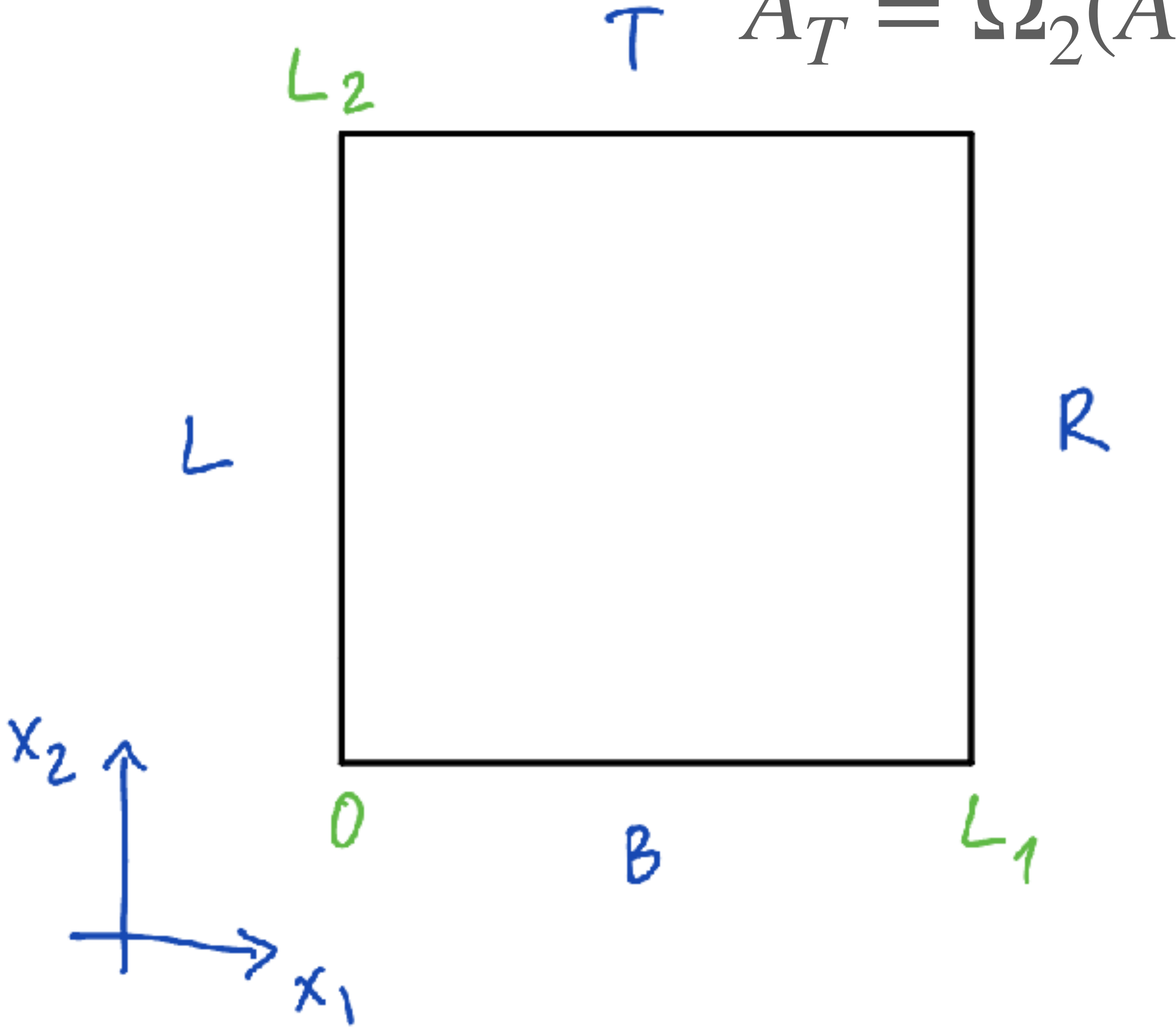

$$A_R = \Omega_1 (A_L - id) \Omega_1^{-1}$$

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$$\Longrightarrow \quad \Omega_1(L_2)\Omega_2(0) = e^{i\alpha} \, \Omega_2(L_1)\Omega_1(0)$$

for U(1) and nonzero q: $e^{i\alpha} = e^{i\frac{2\pi}{q} n_{12}}$ **;** *for U(1) and q=0, any* $e^{i\alpha}$

for SU(N)+adjoint: $e^{i\alpha} = e^{i\frac{2\pi}{N} n_{12}}$ **;** *SU(N)+fund.:* $e^{i\alpha} = 1$ **;** *SU(2k)S/AS:* $e^{i\alpha} = e^{i\pi n_{12}}$

a quick intro to poor man's twisted bundle

for U(1): q=charge

for SU(N):

"q=N" - adjoint(same as A)

q=1 - fund.

q=2 - 2-index S/A

$e^{i\alpha}$: 't Hooft twisted b.c., one per 2-plane

$\alpha, n_{12},$ etc. are gauge invariant data

different choices of Ω_i with same α are gauge equivalent

$$A_R = g_R(A_R - id)g_R^{-1}$$

$$A_L = g_L(A_R - id)g_L^{-1}$$

$$\implies \Omega_1 = g_R \Omega_1 g_L^{-1}$$

$$\implies \Omega_1(L_2)\Omega_2(0) = e^{i\alpha}\, \Omega_2(L_1)\Omega_1(0)$$

for U(1) and nonzero q: $e^{i\alpha} = e^{i\frac{2\pi}{q}n_{12}}$; for U(1) and q=0, any $e^{i\alpha}$

for SU(N)+adjoint: $e^{i\alpha} = e^{i\frac{2\pi}{N}n_{12}}$; SU(N)+fund.: $e^{i\alpha} = 1$; SU(2k)S/AS: $e^{i\alpha} = e^{i\pi n_{12}}$

a quick intro to poor man's twisted bundle

$e^{i\alpha}$: 't Hooft twisted b.c., one per 2-plane

$\alpha, n_{12},$ etc. are gauge invariant data

$e^{i\alpha}$ GKKS+: backgrounds for 1-form symmetry

(2-form gauge field background)

$$A_R = g_R(A_R - id)g_R^{-1}$$

$$A_L = g_L(A_R - id)g_L^{-1}$$

$$\Longrightarrow \quad \Omega_1 = g_R \Omega_1 g_L^{-1}$$

$$\Longrightarrow \quad \Omega_1(L_2)\Omega_2(0) = e^{i\alpha}\,\Omega_2(L_1)\Omega_1(0)$$

a quick intro to poor man's twisted bundle

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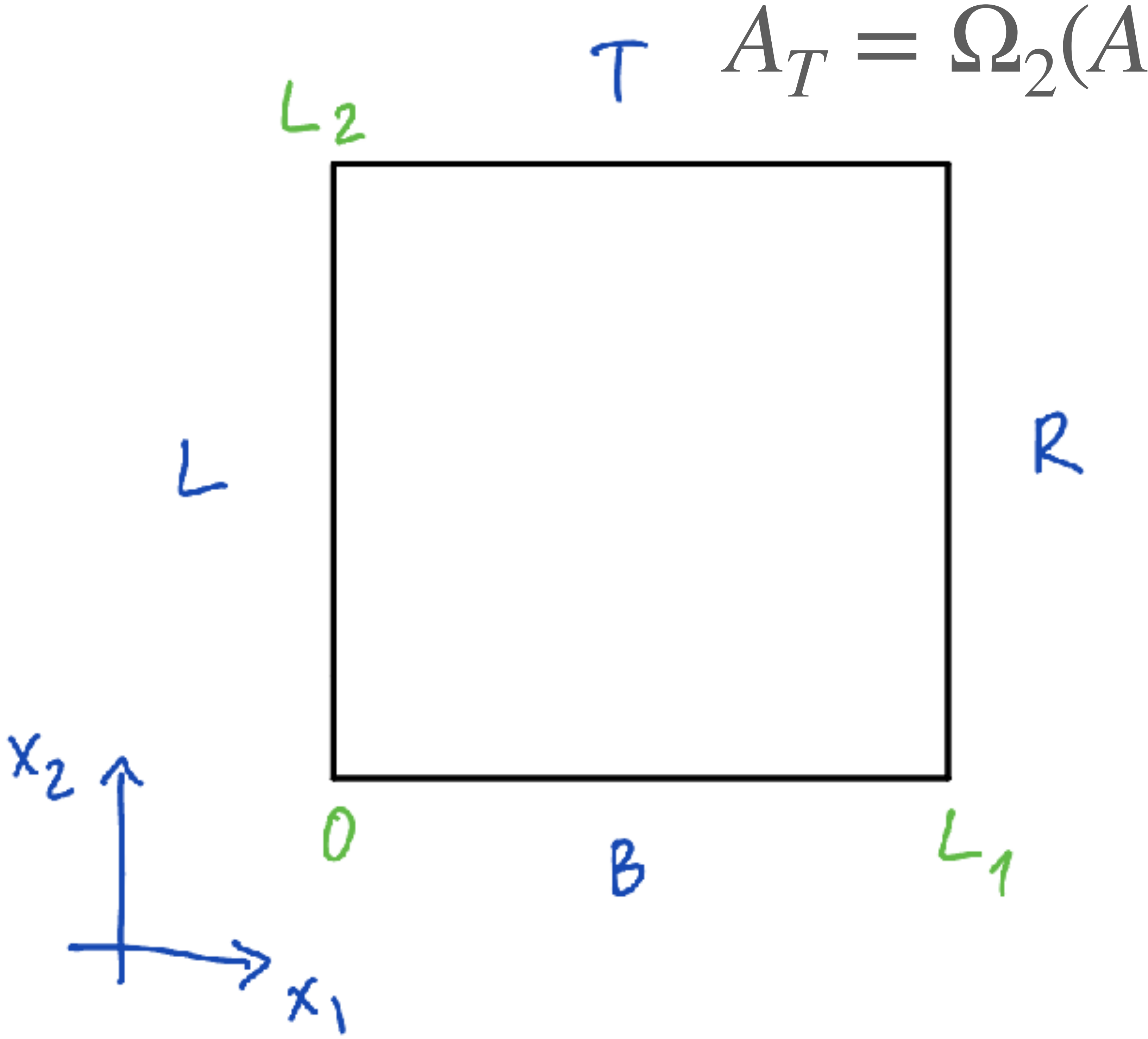

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$$\Longrightarrow \;\; \Omega_1(L_2)\Omega_2(0) = e^{i\alpha} \, \Omega_2(L_1)\Omega_1(0)$$

for U(1),nonzero q:

$$e^{i\alpha} = e^{i\frac{2\pi}{q}n_{12}}$$

for U(1) and q=0, any $e^{i\alpha}$

for SU(N)+adjoint:

$$e^{i\alpha} = e^{i\frac{2\pi}{N}n_{12}}$$

for SU(N)+fund: $e^{i\alpha} = 1$

$$A_R = g_R (A_R - id) g_R^{-1}$$

$$A_L = g_L (A_R - id) g_L^{-1}$$

$$\Longrightarrow \;\; \Omega_1 = g_R \Omega_1 g_L^{-1}$$

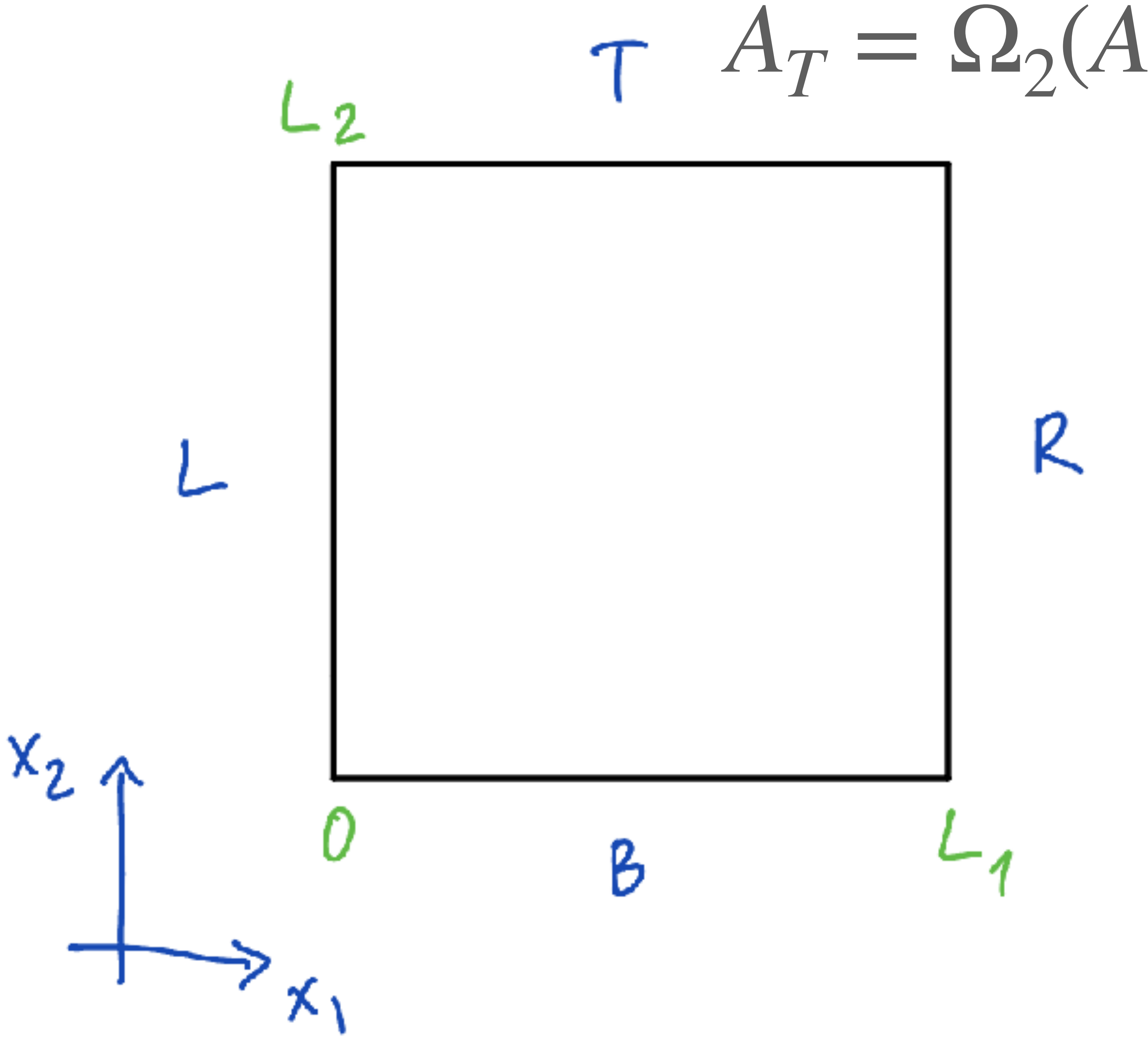
For q's allowing $e^{i\alpha} \neq 1$, all b.c. invariant under: $\Omega_i \to e^{i\frac{2\pi}{q}l_i} \, \Omega_i$, $l_i \in Z(mod \; q)$

(regardless whether we take nontrivial α)

a quick intro to poor man's twisted bundle

$$\psi_T = \Omega_2^q \, \psi_B$$

$$A_T = \Omega_2 (A_B - id) \Omega_2^{-1}$$



$$A_R = \Omega_1 (A_L - id) \Omega_1^{-1}$$

$$\psi_R = \Omega_1^q \, \psi_L$$

$$\Longrightarrow \quad \Omega_1(L_2)\Omega_2(0) = e^{i\alpha} \, \Omega_2(L_1)\Omega_1(0)$$

for U(1),nonzero q:

$$e^{i\alpha} = e^{i\frac{2\pi}{q} n_{12}}$$

for U(1) and q=0, any $e^{i\alpha}$

for SU(N)+adjoint:

$$e^{i\alpha} = e^{i\frac{2\pi}{N} n_{12}}$$

for SU(N)+fund: $e^{i\alpha} = 1$

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For q's allowing $e^{i\alpha} \neq 1$, all b.c. invariant under: $\Omega_i \to e^{i\frac{2\pi}{q} l_i} \, \Omega_i$, $l_i \in Z(\mathrm{mod}\ q)$

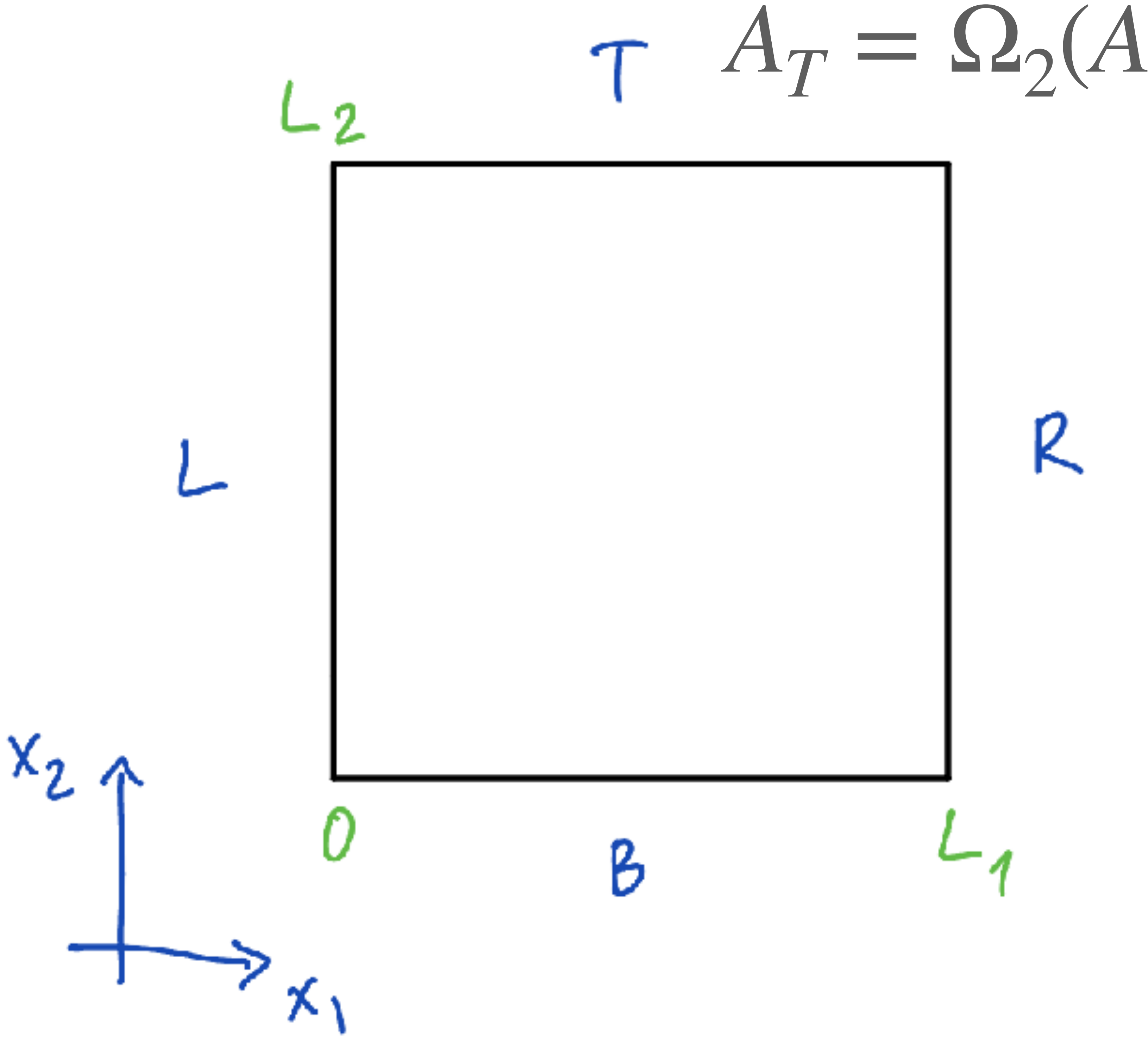
$\Longrightarrow$ Z_q global symmetry, but only acting on transition functions in i-th direction

… strange… what more physical does it act on?

a quick intro to poor man's twisted bundle

$$\psi_T = \Omega_2^q \, \psi_B$$

$$A_T = \Omega_2(A_B - id)\Omega_2^{-1}$$



$$A_R = \Omega_1(A_L - id)\Omega_1^{-1}$$

$$\psi_R = \Omega_1^q \, \psi_L$$

$$\Longrightarrow \quad \Omega_1(L_2)\Omega_2(0) = e^{i\alpha} \, \Omega_2(L_1)\Omega_1(0)$$

for U(1),nonzero q:

$$e^{i\alpha} = e^{i\frac{2\pi}{q}n_{12}}$$

for U(1) and q=0, any $e^{i\alpha}$

for SU(N)+adjoint:

$$e^{i\alpha} = e^{i\frac{2\pi}{N}n_{12}}$$

for SU(N)+fund: $e^{i\alpha} = 1$

$$A_R = g_R(A_R - id)g_R^{-1}$$

$$A_L = g_L(A_R - id)g_L^{-1}$$

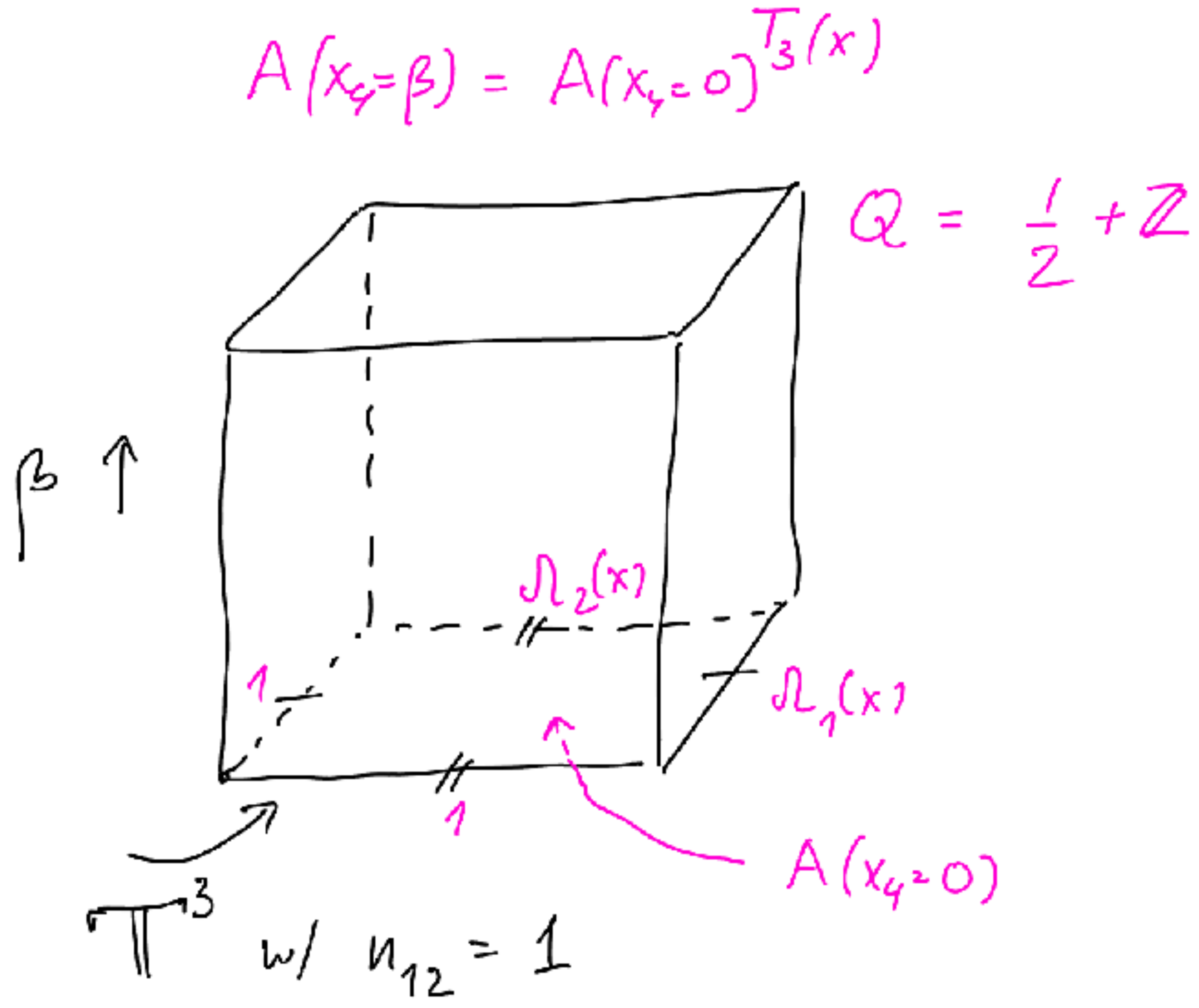
$$\Longrightarrow \quad \Omega_1 = g_R \Omega_1 g_L^{-1}$$

For q's allowing $e^{i\alpha} \neq 1$, all b.c. invariant under: $\Omega_i \to e^{i\frac{2\pi}{q}l_i} \, \Omega_i$, $l_i \in Z(mod \ q)$

winding Wilson loops! $\underline{\mathrm{e.g.}, W_1 = \mathrm{tr}(e^{i\int_L^R A} \, \Omega_1) \ : \ W_1 \to e^{i\frac{2\pi}{q}l_1} W_1}$

Crucial observation ('t Hooft)

$\hat{T}_3$, the $Z_N^{(1)}$ generator in the direction orthogonal to the (12) plane of the twist has winding number Q = $\frac{n_{12}}{N}(\mathrm{mod}\ Z)$



idea only (details are plentiful… see eg appx of 2106 paper w/ Cox, Wandler)

$$Q = \frac{1}{8\pi^2}\int \mathrm{tr}\, F\wedge F = \frac{1}{64\pi^2}\int d^4x F^a_{\mu\nu}F^a_{\lambda\sigma}\epsilon^{\mu\nu\lambda\sigma} = \int d^4x \partial_\mu K^\mu$$

integrand a total derivative, Q only depends on transition functions

for a 4d field configuration twisted by T_3 (denoted C) in time and n_{12} in space:

$$Q[C] = \frac{1}{24\pi^2}\int_{\mathbb{T}^3} \mathrm{tr}\,(CdC^{-1})^3$$

a direct calculation (only requires cocycle conditions, good gauge choice, not explicit form of C=T_3), then gives

Q = $\frac{n_{12}}{N}(\mathrm{mod}\ Z)$ = winding of $\hat{T}_3(\vec{x})$, as map $T^3 \to SU(N)$

(considering 4d field configuration is a clutch ('t Hooft); equiv., can explicitly construct $\hat{T}_3(\vec{x})$ and compute winding…)

[García Pérez, González-Arroyo '92; Selivanov-Smilga '00; Wandler-EP 2211]

’t Hooft: center-symmetry generator “along” $\vec{m}$ has fractional $T^3 \to G$ winding

a picture (*J. Greensite’s demand*) to illustrate fractional winding (holds in our “good” constant-Γ_i gauge)

$x \sim x+1,\ y \sim y+1,\ z \sim z+1$

$SU(2)$, $\vec{m} = (0,0,1)$, $\hat{T}_3(x,y,z): \mathbb{T}^3 \longrightarrow S^3 = SU(2)$

$S^3 \equiv y^\mu \in \mathbb{R}^4$

angle φ, only $\in (0,\pi)$

$S^2 \in S^3$ (const. z)

$y_1 = \cos \pi z$

$y_2 = \sin \pi z \times \sin \pi x \times 4 f(y) f(1+y)$

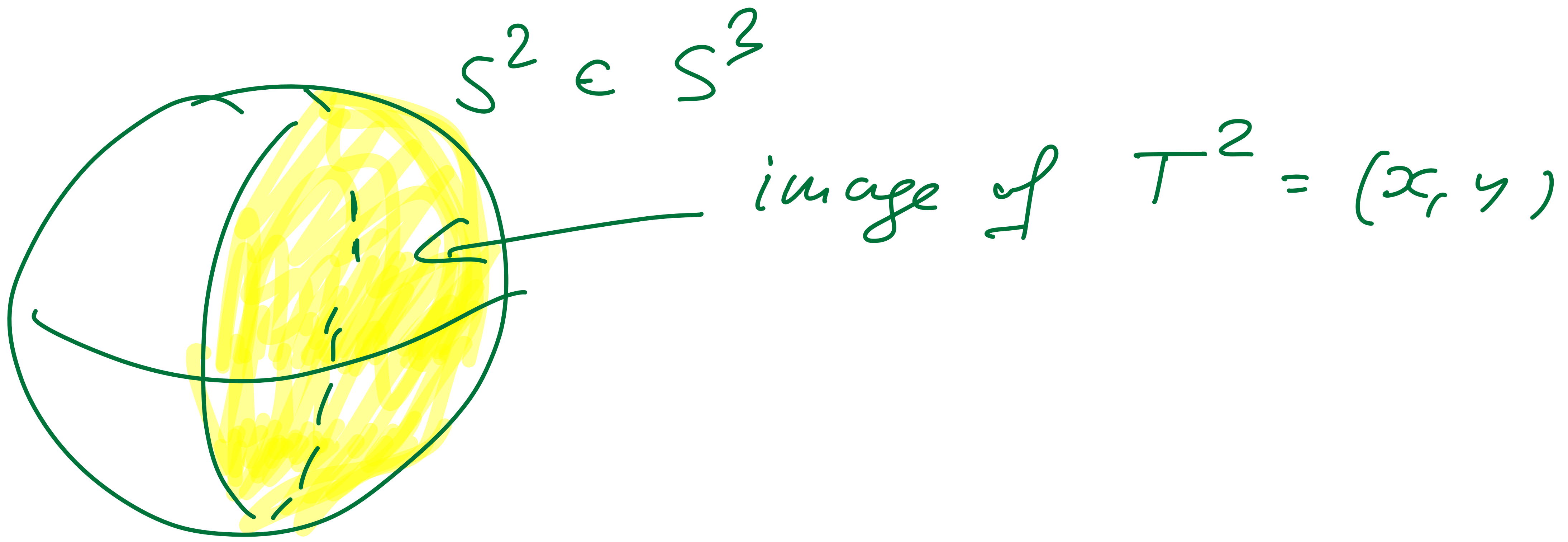
$y_3 = \sin \pi z \times \cos \pi x \times 4 f(y) f(1+y)$

$y_4 = \cos \pi z \times 2(f^2(1+y) - f^2(y))$

$\sim \sin\theta$ (full range)

as $y \in 0,1 \sim \cos\theta$

angle ψ, full range



(explicit form of $\hat{T}_3(x,y,z)$ from Wandler, EP ’22)