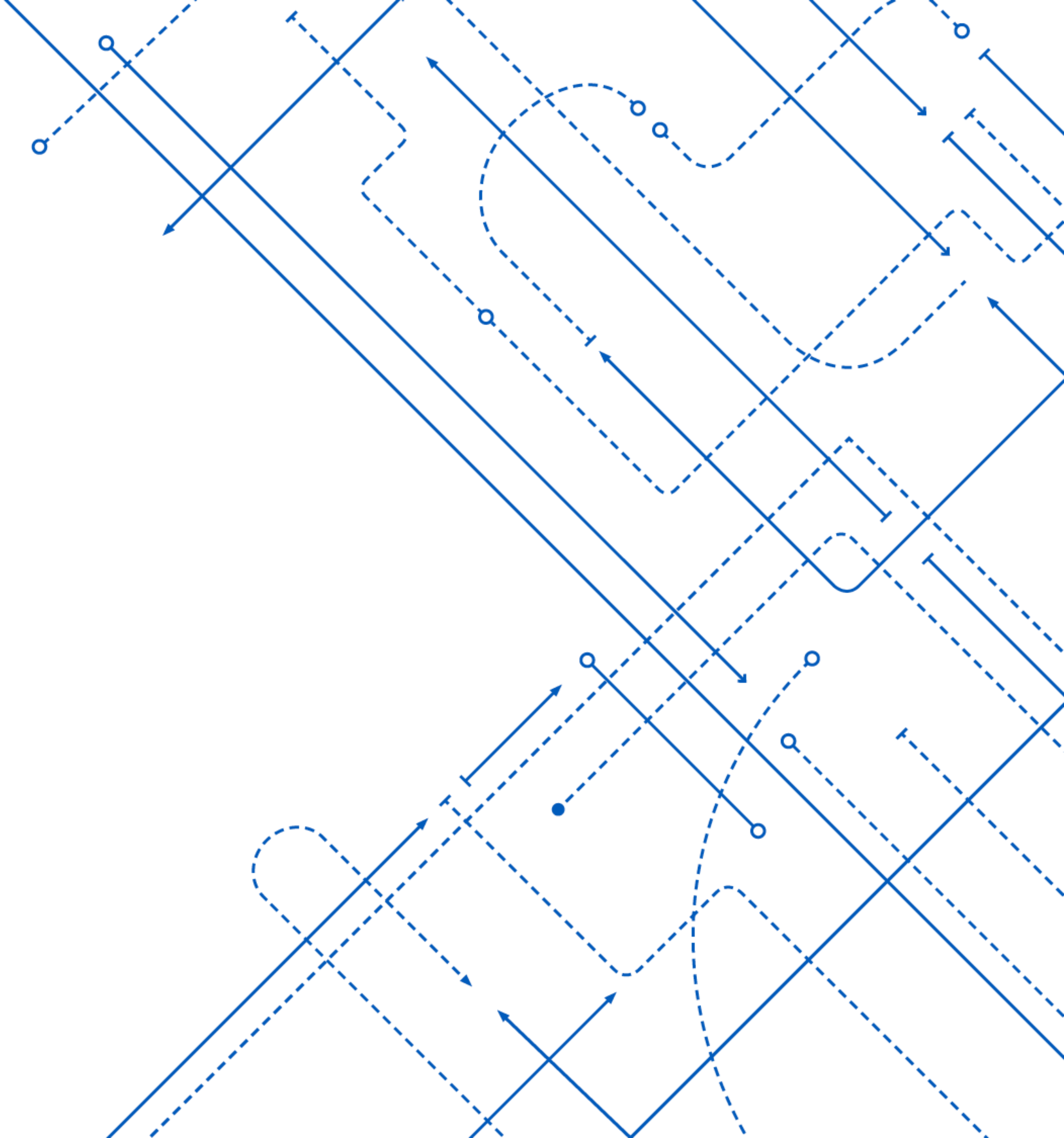
QCD

HCPSS Summer school 2024

Ciaran Williams (SUNY Buffalo)



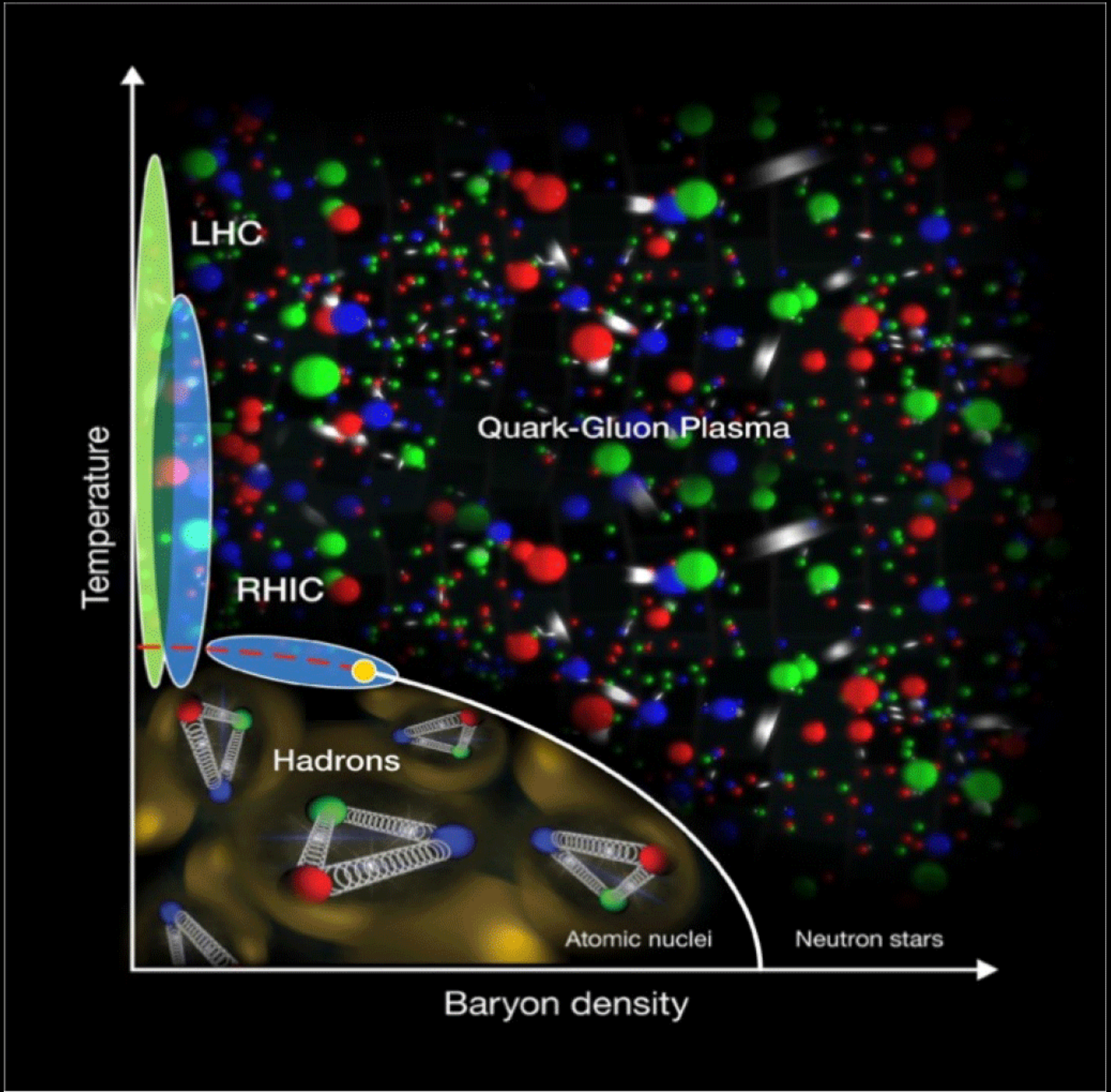
University at Buffalo The State University of New York

QCD - is broad

QCD is a rich theory, which is only partly understood.

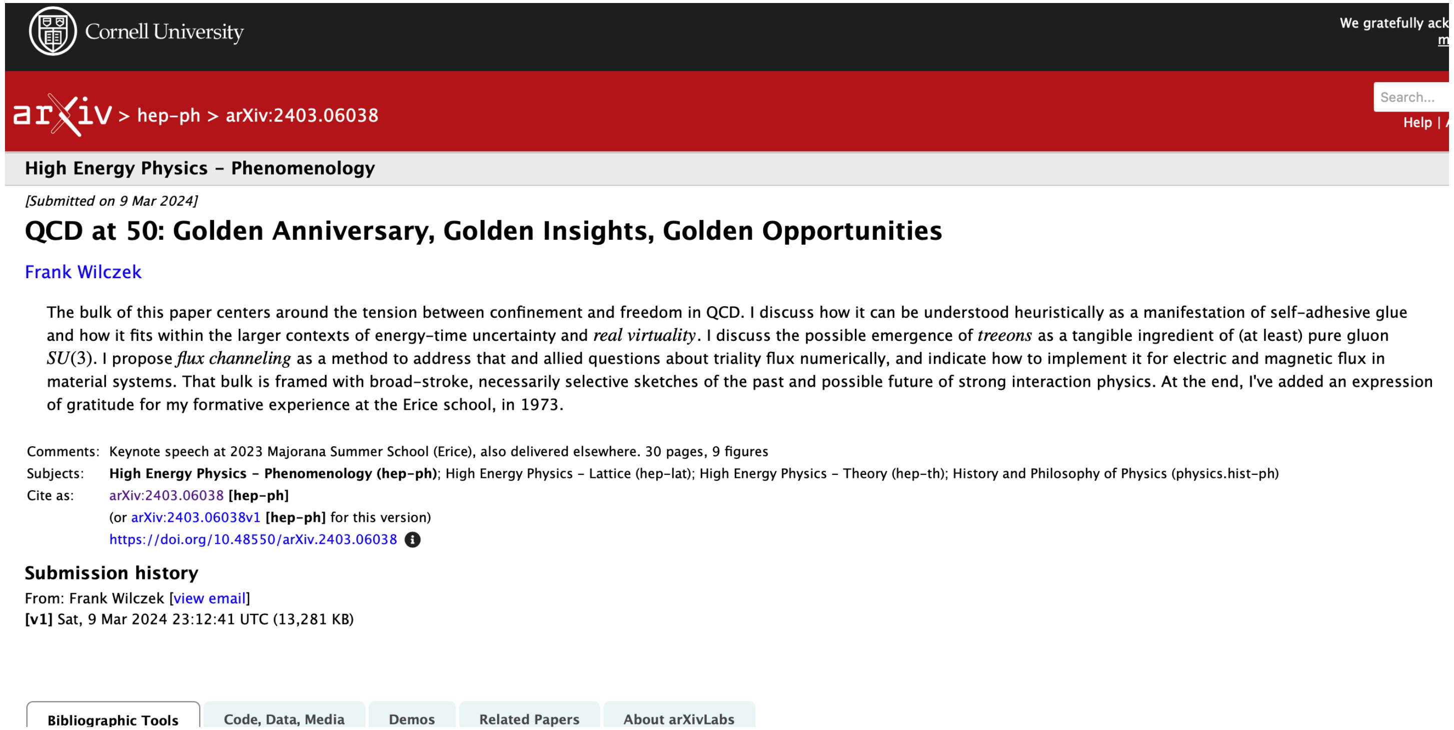
It describes wide ranging phenomena, from nuclear matter, to neutron stars to the properties of the early universe.

These lectures will focus on a small window of QCD, that relevant for collider physics.



https://gdrqcd.in2p3.fr/working-group-2/

QCD is old



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High Energy Physics – Phenomenology

[Submitted on 9 Mar 2024]

QCD at 50: Golden Anniversary, Golden Insights, Golden Opportunities

Frank Wilczek

The bulk of this paper centers around the tension between confinement and freedom in QCD. I discuss how it can be understood heuristically as a manifestation of self-adhesive glue and how it fits within the larger contexts of energy-time uncertainty and *real virtuality*. I discuss the possible emergence of *treeons* as a tangible ingredient of (at least) pure gluon $SU(3)$. I propose *flux channeling* as a method to address that and allied questions about triality flux numerically, and indicate how to implement it for electric and magnetic flux in material systems. That bulk is framed with broad-stroke, necessarily selective sketches of the past and possible future of strong interaction physics. At the end, I've added an expression of gratitude for my formative experience at the Erice school, in 1973.

Comments:	Keynote speech at 2023 Majorana Summer School (Erice), also delivered elsewhere. 30 pages, 9 figures
Subjects:	**High Energy Physics – Phenomenology (hep-ph)**; High Energy Physics – Lattice (hep-lat); High Energy Physics – Theory (hep-th); History and Philosophy of Physics (physics.hist-ph)
Cite as:	arXiv:2403.06038 **[hep-ph]** (or arXiv:2403.06038v1 **[hep-ph]** for this version) https://doi.org/10.48550/arXiv.2403.06038

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From: Frank Wilczek [view email]
[v1] Sat, 9 Mar 2024 23:12:41 UTC (13,281 KB)

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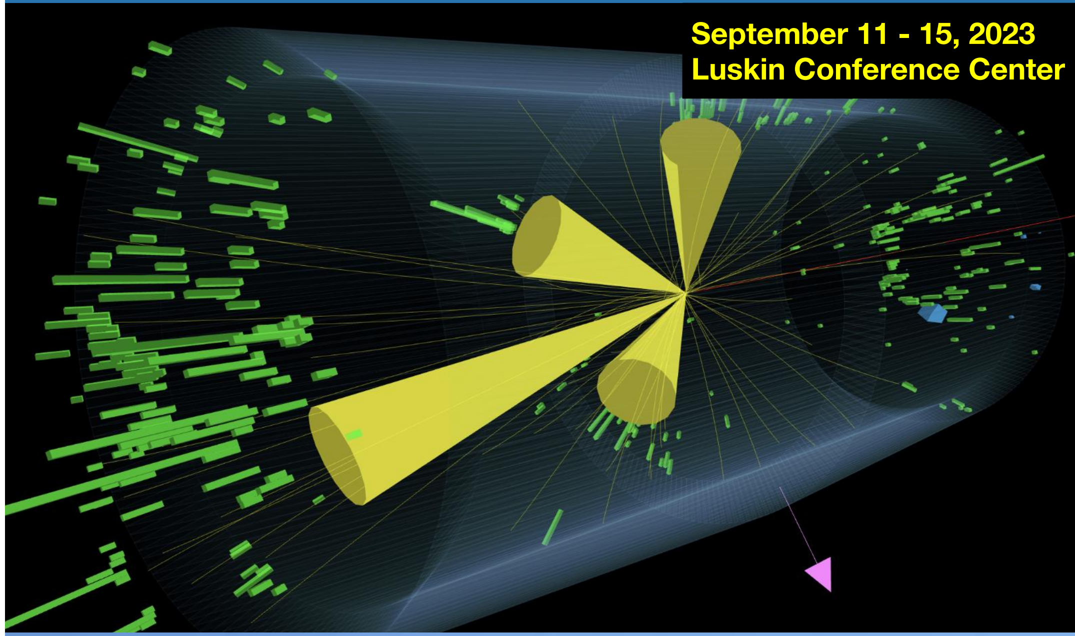
50 Years of QCD



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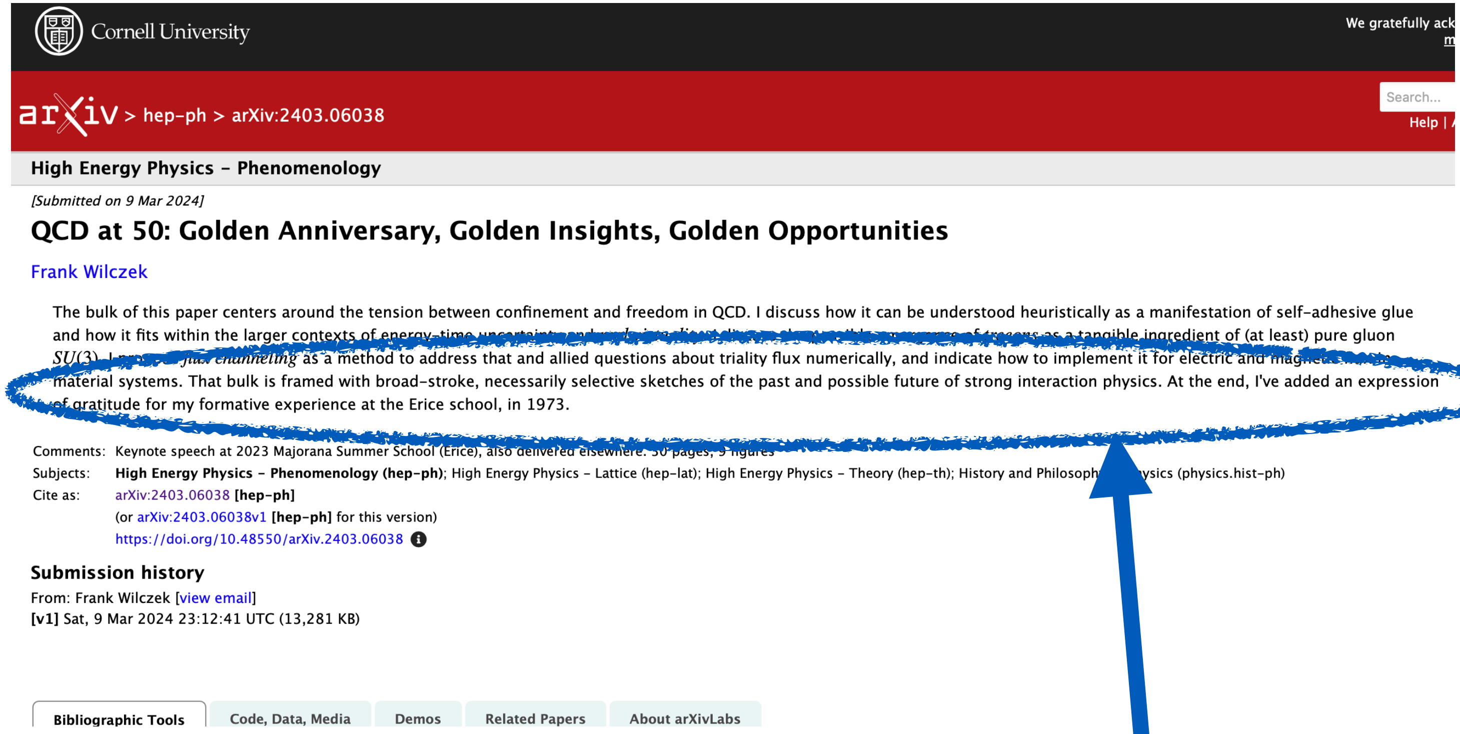
LOCAL CONTACT

Zvi Bern
Zhongbo Kang

UCLA Mani L. Bhaumik Institute for Theoretical Physics

QCD is old



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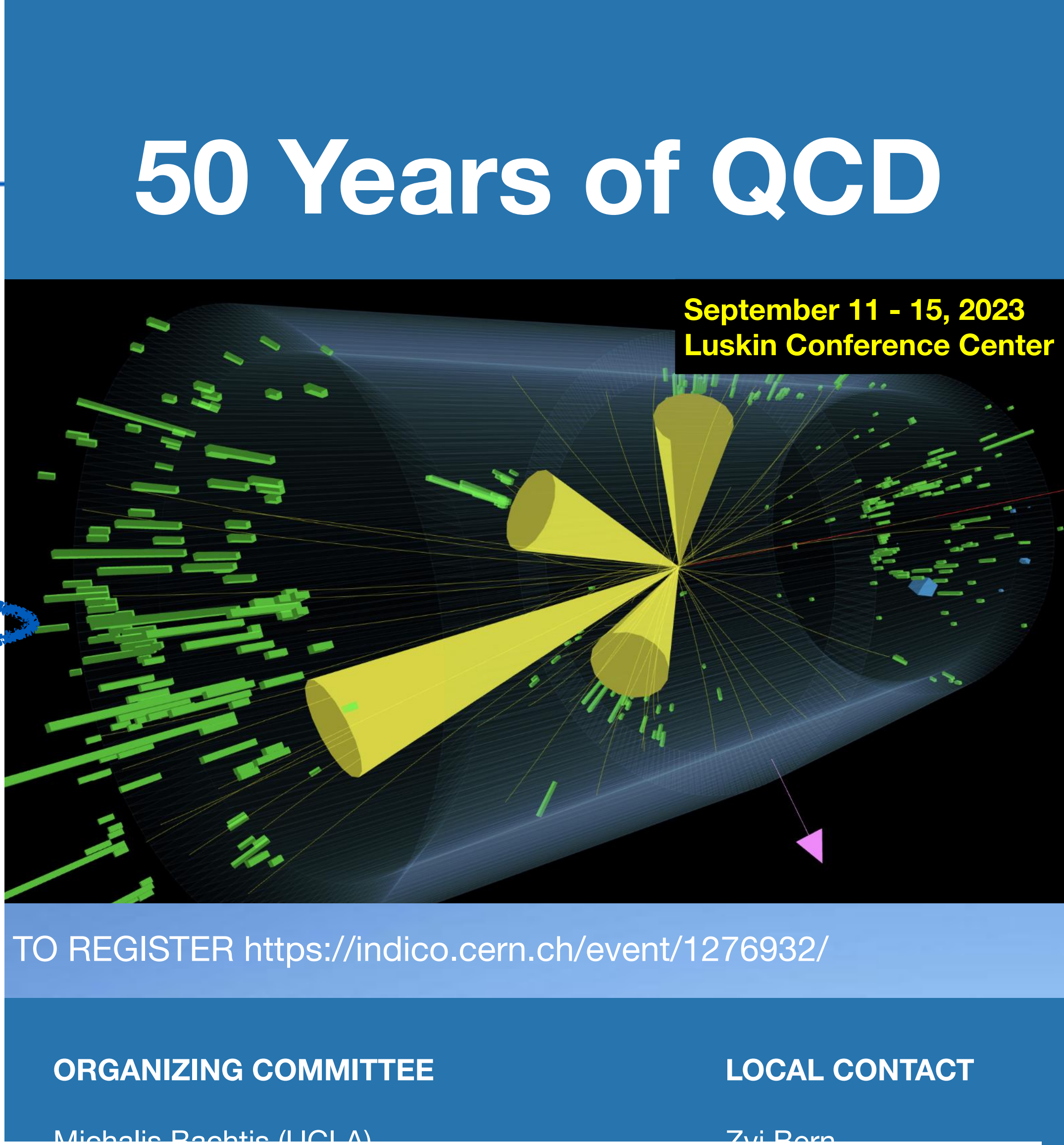

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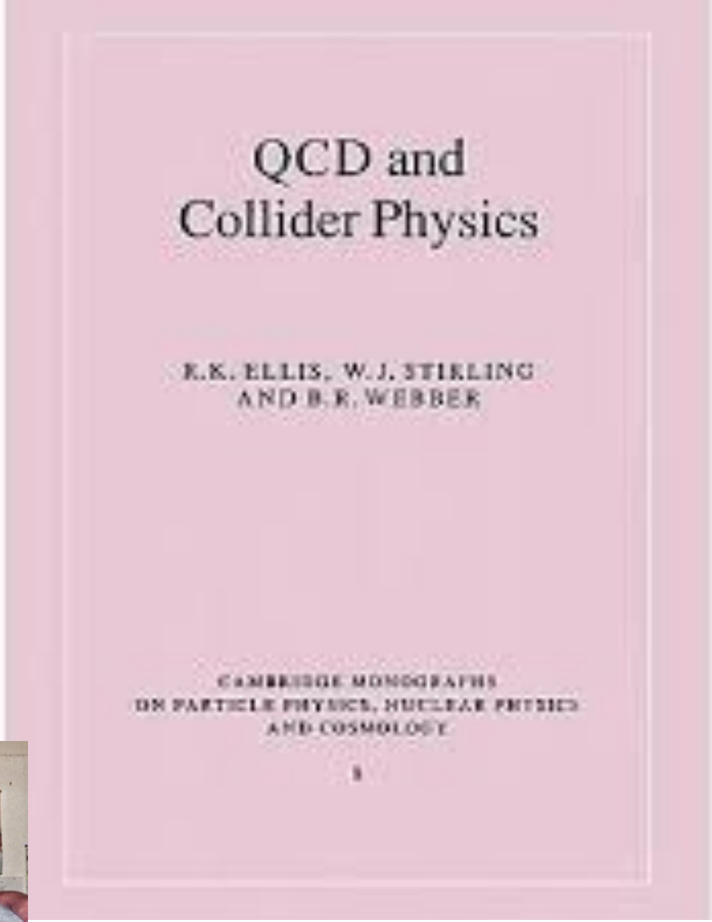
QCD has excellent resources



QCD and
Collider Physics

R.K. ELLIS, W.J. STIRLING
AND B.R. WEBBER

CAMBRIDGE MONOGRAPHS
ON PARTICLE PHYSICS, NUCLEAR PHYSICS
AND COSMOLOGY

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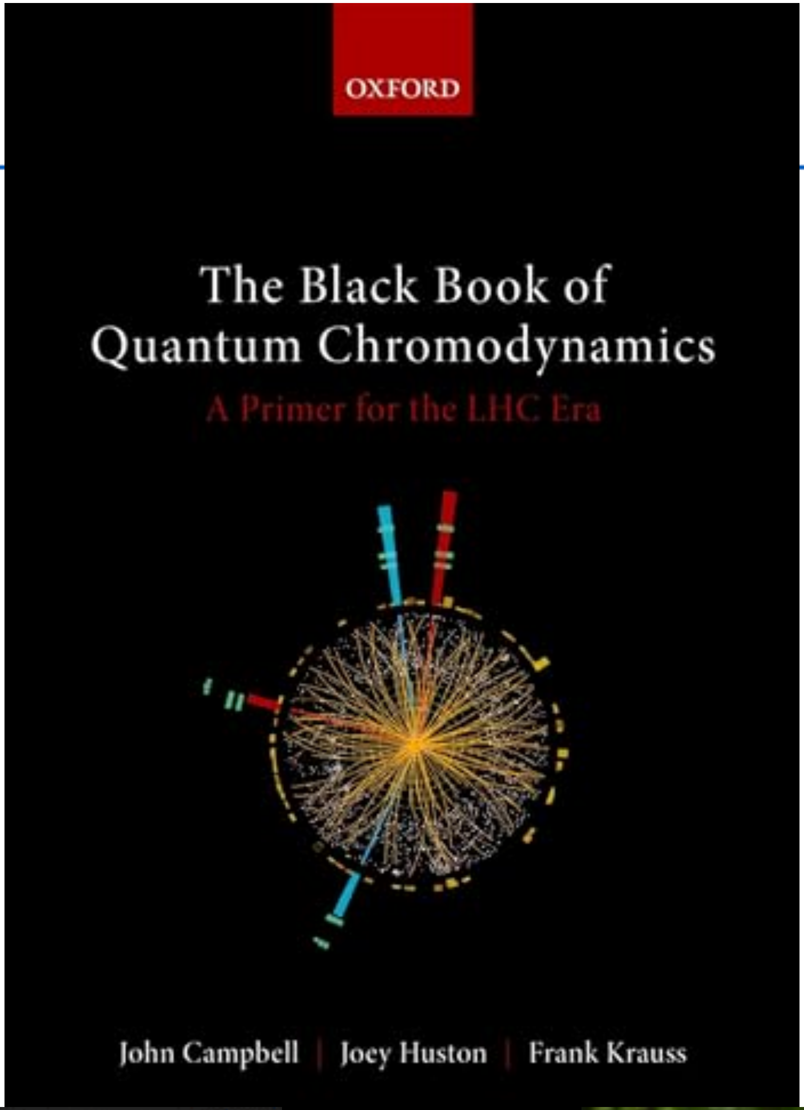

OXFORD

The Black Book of
Quantum Chromodynamics

A Primer for the LHC Era

John Campbell | Joey Huston | Frank Krauss

I've extensively used these references. On nearly every slide!

Plan for the lectures here.

It’s clear I cant cover anywhere near ALL of QCD.
Or indeed even much of subset of QCD with 3 hours, so some decisions had to be made.

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I decided to focus on three topics I think are

1) fundamental (and somewhat bespoke) to QCD
2) Important, in that its likely that a collider physicist will come into contact with the physics at some-point in their career.
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1) fundamental (and somewhat bespoke) to QCD
2) Important, in that its likely that a collider physicist will come into contact with the physics at some-point in their career.
3) Sort of hang together as a coherent narrative!

My hope is that these lectures serve as an introduction, and would help you to tackle more detailed topics covered in the bibles of QCD with confidence!

Outline of the lectures

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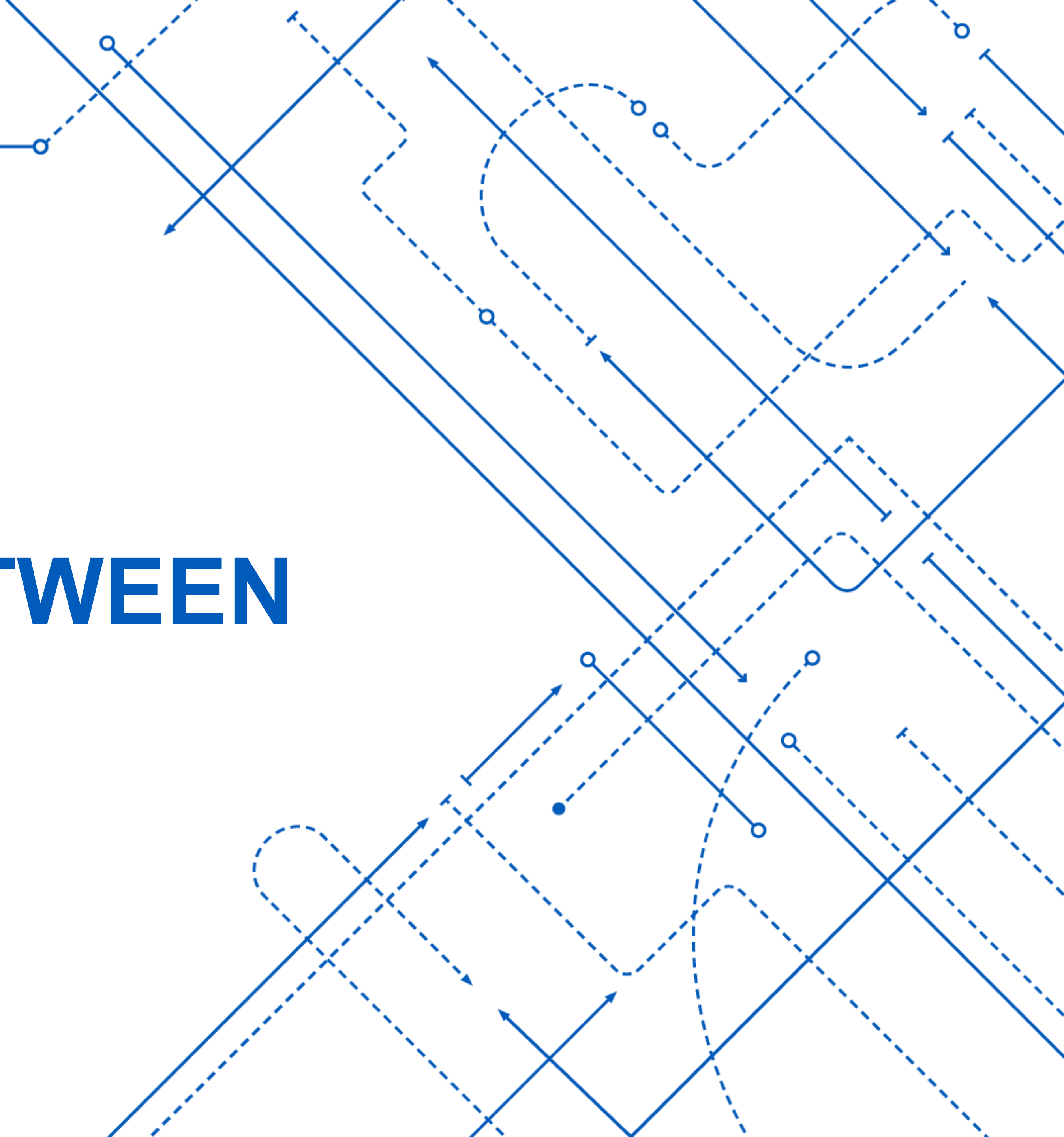
2) How do we evaluate hadron structure? How can we predict scattering processes with initial state hadrons?

3) How can we understand QCD radiation patterns in final states at collider experiments?

WHAT DRIVES THE DIFFERENCES BETWEEN QCD AND QED?



A MINI OVERVIEW OF QCD FIELD THEORY

Fundamentals of QCD

QCD is a quantum field theory, which is described by a Lagrangian (density) with the following form $\mathscr{L}_{\text{classical}} + \mathscr{L}_{\text{gauge-fixing}} + \mathscr{L}_{\text{ghost}}$

The classical Lagrangian is obtained from the following terms

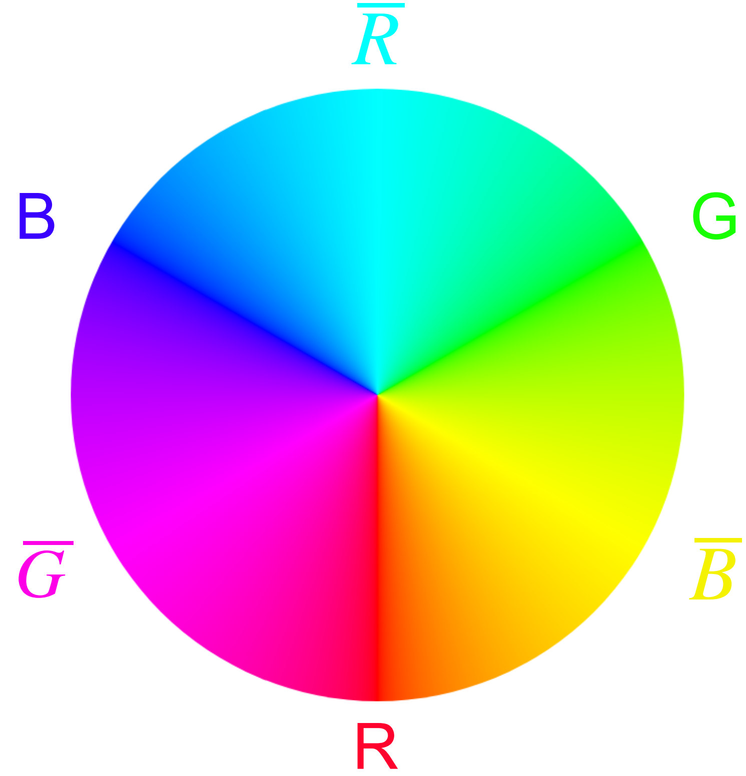
$$\mathscr{L}_{\text{classical}} = -\frac{1}{4} G^A_{\mu\nu} G^{\mu\nu}_A + \sum_{flav} \bar{q}_a (i\gamma_\mu D^\mu - m)_{ab} q_b$$

Where q_a denotes a spin-1/2 quark field and $G^A_{\mu\nu}$ is the field strength tensor constructed from the gluon field $\mathscr{A}^A_\mu$

$$G^A_{\mu\nu} = \partial_\mu \mathscr{A}^A_\nu - \partial_\nu \mathscr{A}^A_\mu - g f^{ABC} \mathscr{A}^A_\nu \mathscr{A}^A_\mu$$

Color SU(3)

$$\mathscr{L}_{\text{classical}} = -\frac{1}{4}G^{A}_{\mu\nu}G^{\mu\nu}_{A} + \sum_{flav}\overline{q}_a(i\gamma_\mu D^\mu - m)_{ab}q_b$$



QCD is an SU(3) gauge theory, which means that its constituents belong to non-trivial representations of the SU(3) group.

Quarks (anti-quarks) $q_a, (\overline{q}_a)$ transform in the fundamental (3) (anti-fundamental, $\overline{3}$) representation

Fundamentals of QCD

$$\mathscr{L}_{\text{classical}} = -\frac{1}{4}G^A_{\mu\nu}G^{\mu\nu}_A + \sum_{flav}\overline{q}_a(i\gamma_\mu D^\mu - m)_{ab}q_b$$

Since quarks transform in the fundamental representation we can define the covariant derivative as follows $(D_\mu)_{ab} = \partial_\mu\delta_{ab} + ig(t^C\mathscr{A}^C_\mu)_{ab}$

Where t^C are generator matrices in the fundamental representation of SU(3), defined by the Lie algebra $[t^A, t^B] = if^{ABC}t^C$, and f^{ABC} define the anti-symmetric structure constants of the group.

A second, useful representation is the adjoint one, T, where $[T^A, T^B] = if^{ABC}T^C$ and $(T^A)_{BC} = -if^{ABC}$

Gluons

$$\mathscr{L}_{\text{classical}} = -\frac{1}{4}G^A_{\mu\nu}G^{\mu\nu}_A + \sum_{flav} \bar{q}_a(i\gamma_\mu D^\mu - m)_{ab}q_b$$

A, B, C denote the eight color degrees of freedom of the gluon field

Aside from the group theory indices, the driving difference between QCD and QED is the final term in the gluon field strength, $gf^{ABC}\mathscr{A}^A_\nu\mathscr{A}^A_\mu$, this non-abelian term couples gluons to each other

$$G^A_{\mu\nu} = \partial_\mu\mathscr{A}^A_\nu - \partial_\nu\mathscr{A}^A_\mu \boxed{- gf^{ABC}\mathscr{A}^A_\nu\mathscr{A}^A_\mu}$$

This term drives the differences between QCD and QED.

Fundamentals of QCD

We can then define two Casimir's of SU(N) by the following relations:

$$\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac} \quad \text{with} \quad C_F = \frac{N^2 - 1}{2N}$$

And

$$\mathrm{Tr} T^C T^D = \sum_{A,B} f^{ABC} f^{ABD} = C_A \delta^{CD} \text{ with } C_A = N$$

Specifically for QCD $N = N_c = 3$ and $C_F = 4/3$ and $C_A = 3$.

Fundamentals of QCD.

$$\sum_A t^A_{ab} t^A_{bc} = C_F \delta_{ac}$$

Key point -

Since we dont observe individually charged color states we always sum over color in our calculations therefore :

QCD amplitudes are nearly always expressed as polynomials in C_F and C_A

Gauge fixing (and ghosts)

The other two pieces of the Lagrangian are more field theoretic in nature and are necessary to define a physical gluon propagator (Gauge fixing) and then cancel any unphysical modes introduced by the gauge fixing (ghost term).

A common choice for the gauge fixing terms is the following

$$\mathscr{L}_{\text{gauge–fixing}} = -\frac{1}{2\lambda}(\partial_\mu \mathscr{A}_a^\mu)^2$$

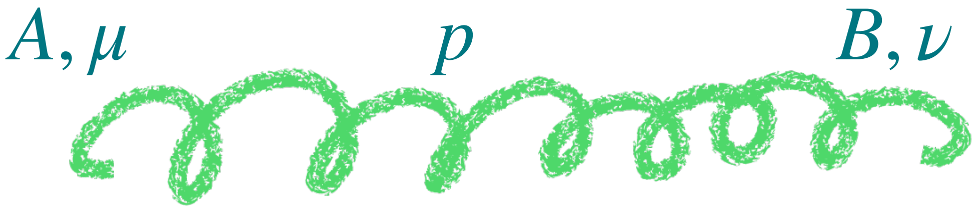
Known as the **covariant** gauge, setting $\lambda = 1$ is the Feynman gauge.

This choice requires the inclusion of the following ghost Lagrangian

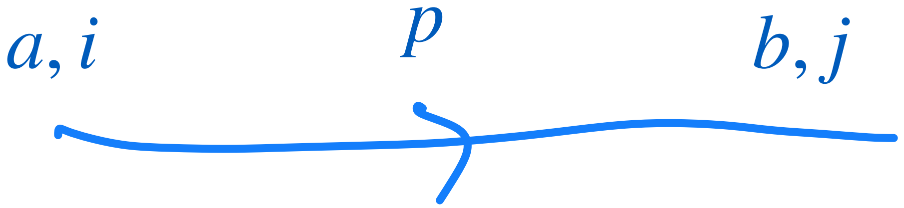
$$\mathscr{L}_{ghost} = (\partial_\mu \eta^A)^\dagger (D^\mu_{AB}\eta^B)$$

Where η is a complex scalar which obeys Fermi-Dirac Statistics.

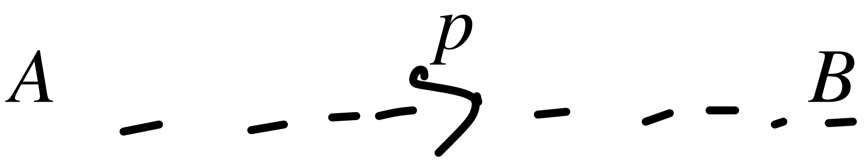
Feynman rules in covariant gauge - Propagators



$$\delta^{AB}[-g^{\mu\nu} + (1-\lambda)\frac{p^\mu p^\nu}{p^2 + i\epsilon}]\frac{i}{p^2 + i\epsilon}$$

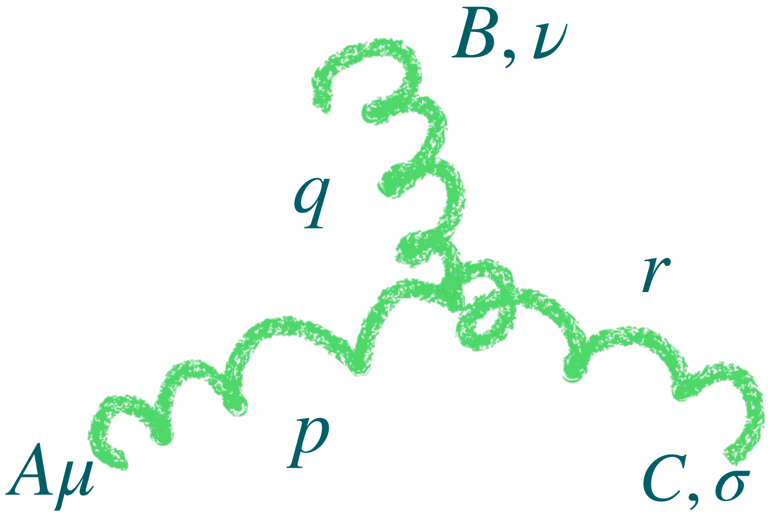


$$\delta^{ab}\frac{i}{(\gamma^\mu p_\mu - m + i\epsilon)_{ji}}$$



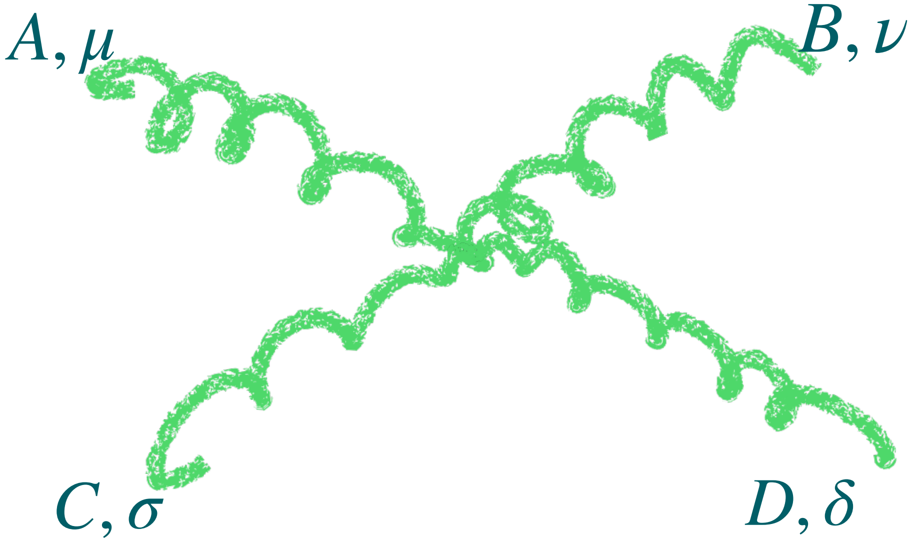
$$\delta^{AB}\frac{i}{p^2 + i\epsilon}$$

Feynman rules - gluon self-interactions Vertices



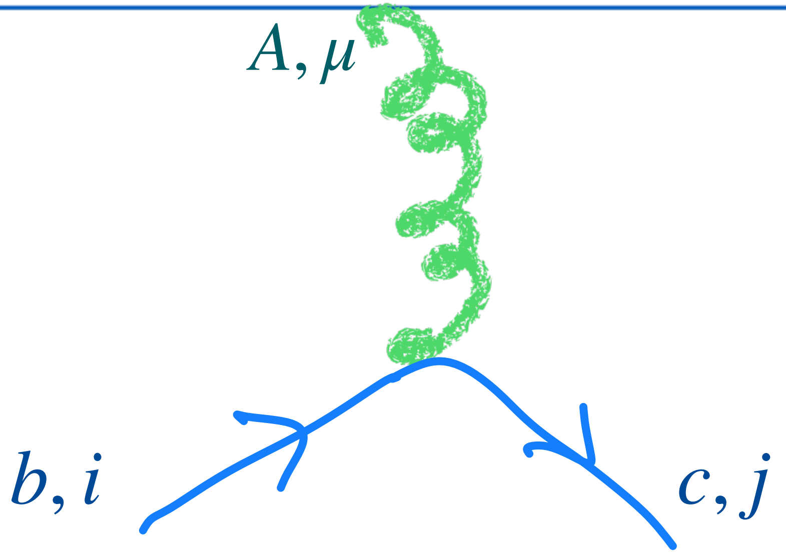
$$-gf^{ABC}[(p-q)^{\sigma}g^{\mu\nu}+(q-r)^{\mu}g^{\nu\sigma}+(r-p)^{\nu}g^{\sigma\mu}]$$

All momentum incoming

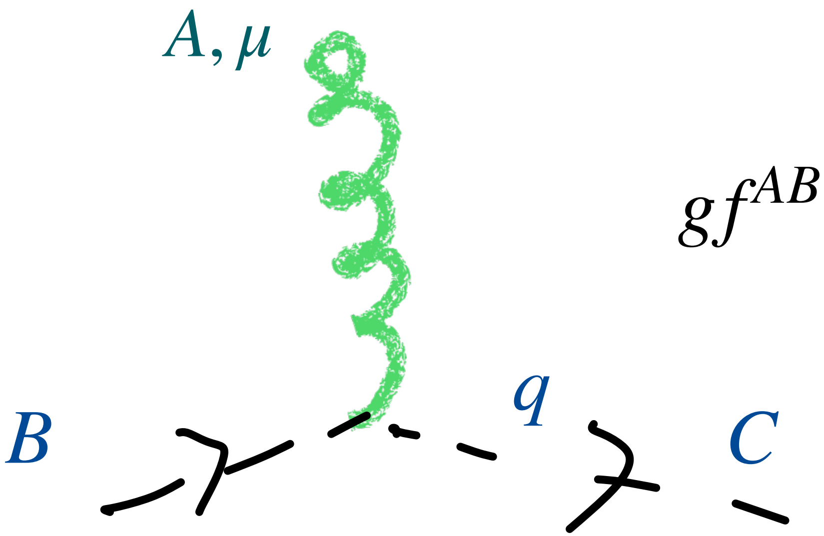


$$\begin{aligned}&-ig^{2}f^{XAC}f^{XBD}(g^{\mu\nu}g^{\sigma\delta}-g^{\mu\delta}g^{\nu\sigma})\\&-ig^{2}f^{XAD}f^{XBC}(g^{\mu\nu}g^{\sigma\delta}-g^{\mu\sigma}g^{\nu\nu})\\&-ig^{2}f^{XAB}f^{XCD}(g^{\mu\sigma}g^{\nu\delta}-g^{\mu\delta}g^{\nu\sigma})\end{aligned}$$

Feynman rules - gluon Fermion/ghost Vertices



$$-ig(t^A)_{cb}(\gamma^\mu)_{ji}$$



$$gf^{ABC}q^\mu$$

THE STRONG COUPLING - α_S

α_s

Let's consider a dimensionless observable R which depends on a single scale Q.

Classically we would expect this to be constant, since there is no other scale to make a dimensionless variable from.

However, in QFT the picture is not so simple. The renormalization of the coupling $\alpha_S = g_s^2/(4\pi)$ introduces a second parameter μ the renormalization scale at which the UV divergences are removed.

So R can depend the ratio of scales Q^2/μ^2 and need not be constant.

Running of α_s

This doesn't seem to make sense though μ is an arbitrary parameter, introduced as part of the renormalization prescription and not a fundamental parameter of the QCD Lagrangian.

The renormalized coupling and dependence on the ratio Q^2/μ^2 must therefore conspire to ensure that R does not depend on μ.

$$\mu^2 \frac{d}{d\mu^2} R(Q^2/\mu^2, \alpha_S) \equiv \left(\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \frac{\partial}{\partial \alpha_S} \right) R(Q^2/\mu^2, \alpha_S) = 0$$

Running of α_s

$$\left(\mu^2\frac{\partial}{\partial\mu^2}+\mu^2\frac{\partial\alpha_S}{\partial\mu^2}\frac{\partial}{\partial\alpha_S}\right)R(Q^2/\mu^2,\alpha_S)=0$$

We can tidy up the equation by defining the following quantities

$$t=\ln\left(\frac{Q^2}{\mu^2}\right) \quad \text{and} \quad \beta(\alpha_S)=\mu^2\frac{\partial\alpha_S}{\partial\mu^2}$$

(Defined at fixed bare coupling)

So that

$$\left(-\frac{\partial}{\partial t}+\beta(\alpha_S)\frac{\partial}{\partial\alpha_S}\right)R(e^t,\alpha_S)=0$$

The running coupling $\alpha_s(Q^2)$

$$\left(-\frac{\partial}{\partial t}+\beta(\alpha_S)\frac{\partial}{\partial \alpha_S}\right)R(e^t,\alpha_S)=0$$

We can solve our equation by introducing the following **running coupling** $\alpha_S(Q^2)$

$$t=\int_{\alpha_S(\mu^2)}^{\alpha_S(Q^2)}\frac{dx}{\beta(x)}$$

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$$t = \int_{\alpha_S(\mu^2)}^{\alpha_S(Q^2)} \frac{dx}{\beta(x)}$$

Differentiating, using the fundamental theorem of calculus we can write

$$\frac{\partial \alpha_S(Q^2)}{\partial t} = \beta(\alpha_S(Q^2))$$ and $$\frac{\partial \alpha_S(Q^2)}{\partial \alpha_S(\mu^2)} = \frac{\beta(\alpha_S(Q^2))}{\beta(\alpha_S(\mu^2))}$$

The running coupling $\alpha_s(Q^2)$

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$$\frac{\partial\alpha_S(Q^2)}{\partial t}=\beta(\alpha_S(Q^2)) \qquad \text{and} \qquad \frac{\partial\alpha_S(Q^2)}{\partial\alpha_S(\mu^2)}=\frac{\beta(\alpha_S(Q^2))}{\beta(\alpha_S(\mu^2))}$$

This means that $R(1,\alpha_S(Q^2))$ solves our μ independence equation,
and **all of the scale dependence in R enters through the running of the coupling** $\alpha_S(Q^2)$.

The β function.

The β function is determined from the renormalization group equation

(RGE) $$Q^2 \frac{\partial \alpha_S(\mu^2)}{\partial Q^2} = \beta(\alpha_S(\mu^2))$$

Expanding as a perturbative series we write

$$\beta(\alpha_S(\mu^2)) = -\alpha_S \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_S(\mu^2)}{4\pi} \right)^{n+1}$$

The β function.

$$\beta(\alpha_S(\mu^2)) = -\alpha_S \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_S(\mu^2)}{4\pi}\right)^{n+1}$$

The first couple of orders in perturbation theory are written as $\beta_0 = 4\pi b$ and $\beta_1 = 16\pi^2 bb'$ with

$$b = \frac{11C_A - 2n_F}{12\pi} \text{ and } b' = \frac{17C_A^2 - 5C_A n_F - 3C_F n_F}{2\pi(11C_A - 2n_F)}$$

This should be contrasted with the beta function from QED

$$\beta_{\text{QED}} = \frac{1}{3\pi}\alpha^2 + \dots$$

Which is **positive**

The β function.

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Crucially if $n_F < 17$ the β function is **negative**

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Which is **positive**

The β function.

$$Q^2 \frac{\partial \alpha_S(\mu^2)}{\partial Q^2} = \beta(\alpha_S(\mu^2))$$

We can gleam more insight into the consequences of this sign difference by deriving a formula which relates the running coupling to the renormalized one.

Our starting point is a perturbative expansion of our differential equation

$$Q^2 \frac{\partial \alpha_S(Q^2)}{\partial Q^2} = -b\alpha_S^2(Q^2)(1 + b'\alpha_S(Q^2) + \ldots)$$

Truncating the RHS at lowest order we can solve the resulting differential equation

$$\boxed{\alpha_S(Q^2) = \frac{\alpha_S(\mu^2)}{1 + b\alpha_S(\mu^2)\ln(Q^2/\mu^2)}}$$

Asymptotic freedom

$$b = \frac{11C_A - 2n_F}{12\pi}$$

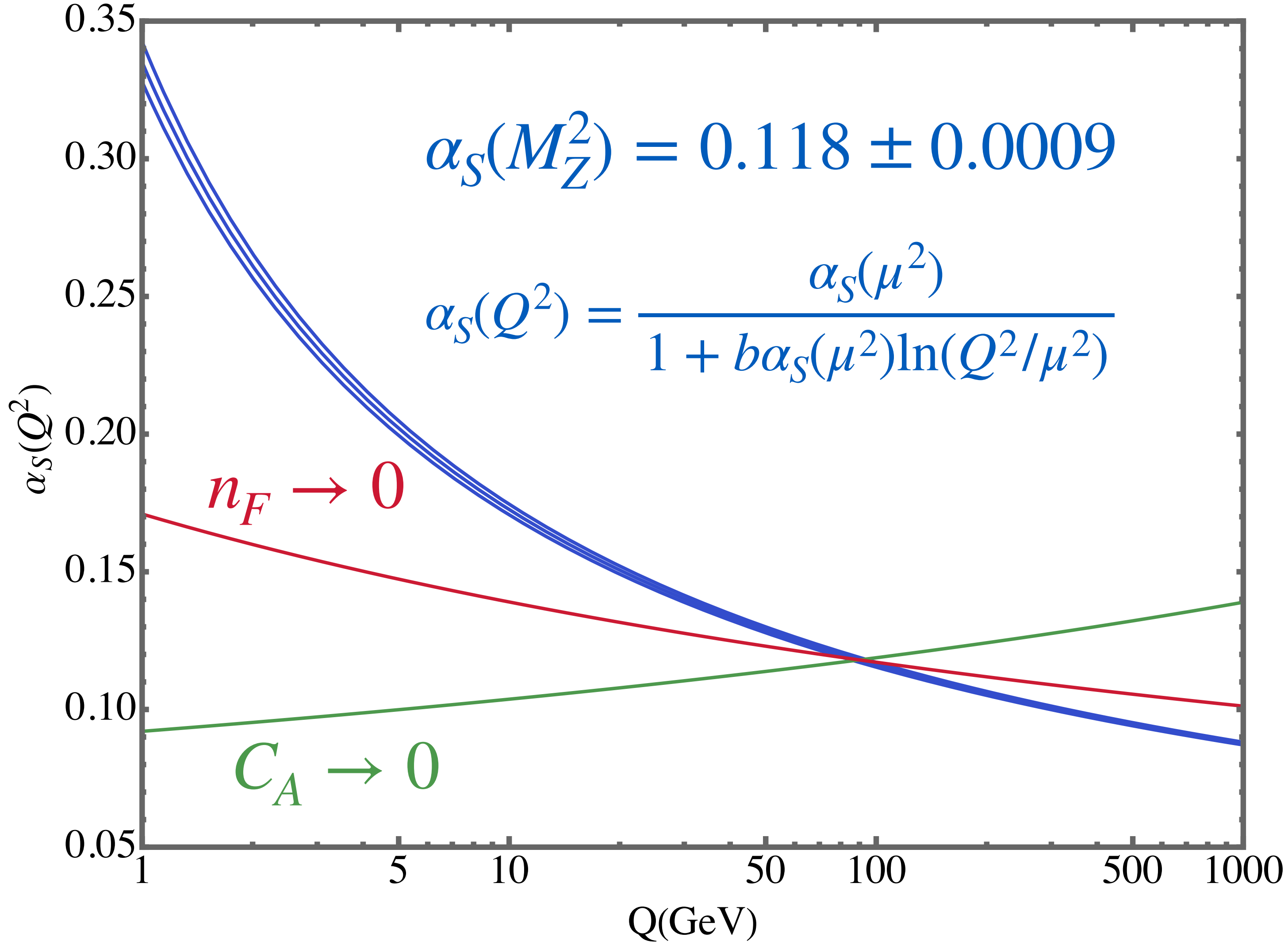
$$\boxed{\alpha_S(Q^2) = \frac{\alpha_S(\mu^2)}{1 + b\alpha_S(\mu^2)\ln(Q^2/\mu^2)}}$$

This equation tells us a lot about the differences between QCD and QED and how they emerge from the different structures of the theory.

As $\ln Q^2/\mu^2$ becomes large (i.e. at very high scales) the coupling is suppressed and becomes small

This phenomenon is known as **asymptotic freedom**

Asymptotic freedom



The blue curve shows us the total 1-loop running for a given input $\alpha_S(M_Z^2)$

The two other curves show us what would happen with gluons only $n_F \to 0$ (red) and quarks only $C_A \to 0$ (green)

Resummation

$$\alpha_S(Q^2) = \frac{\alpha_S(\mu^2)}{1 + b\alpha_S(\mu^2)\ln(Q^2/\mu^2)}$$

Returning to our original observable R, lets expand our theoretical prediction as a perturbative expansion $R = R_1\alpha_S + \ldots$

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We can write $R(1,\alpha_S(Q^2))$ in terms of $\alpha_S(\mu^2)$ using our formula above

$$R(1,\alpha_S(Q^2)) = R_1\alpha_S(\mu^2)\sum_{j=0}^{\infty}\left(-\alpha_S(\mu^2)b\ln\frac{Q^2}{\mu^2}\right)^j = R_1\alpha_S(\mu^2)(1 - \alpha_S(\mu^2)bt + \alpha_S^2(\mu^2)(bt)^2 + \ldots)$$

Resummation

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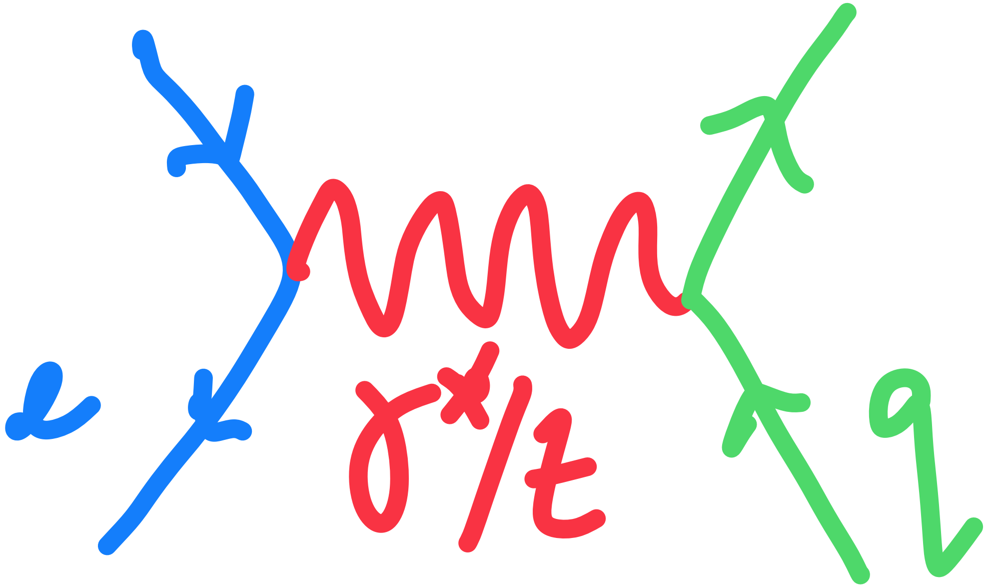
Thus at each order in perturbation theory there are logarithms of Q^2/μ^2 which get **resummed** when we use the running coupling. If the ratio is sizable this can lead to a significant improvement in theoretical accuracy “for free”. This type of idea is widely used at the LHC.

MEASURING - α_S

(ALSO A MINI INTRODUCTION TO HIGHER ORDER CORRECTIONS IN QCD…)

QCD in e^+e^- collisions

As a first example lets look at the QCD equivalent of $e^+e^- \to \mu^+\mu^-$, which is $e^+e^- \to$ hadrons. The Feynman diagram at LO is below.



The R-ratio

It's nice to compare the ratio of the e^+e^- total hadronic cross section to the muonic one. At energies below the Z pole, only the photon exchange diagram contributes and we have:

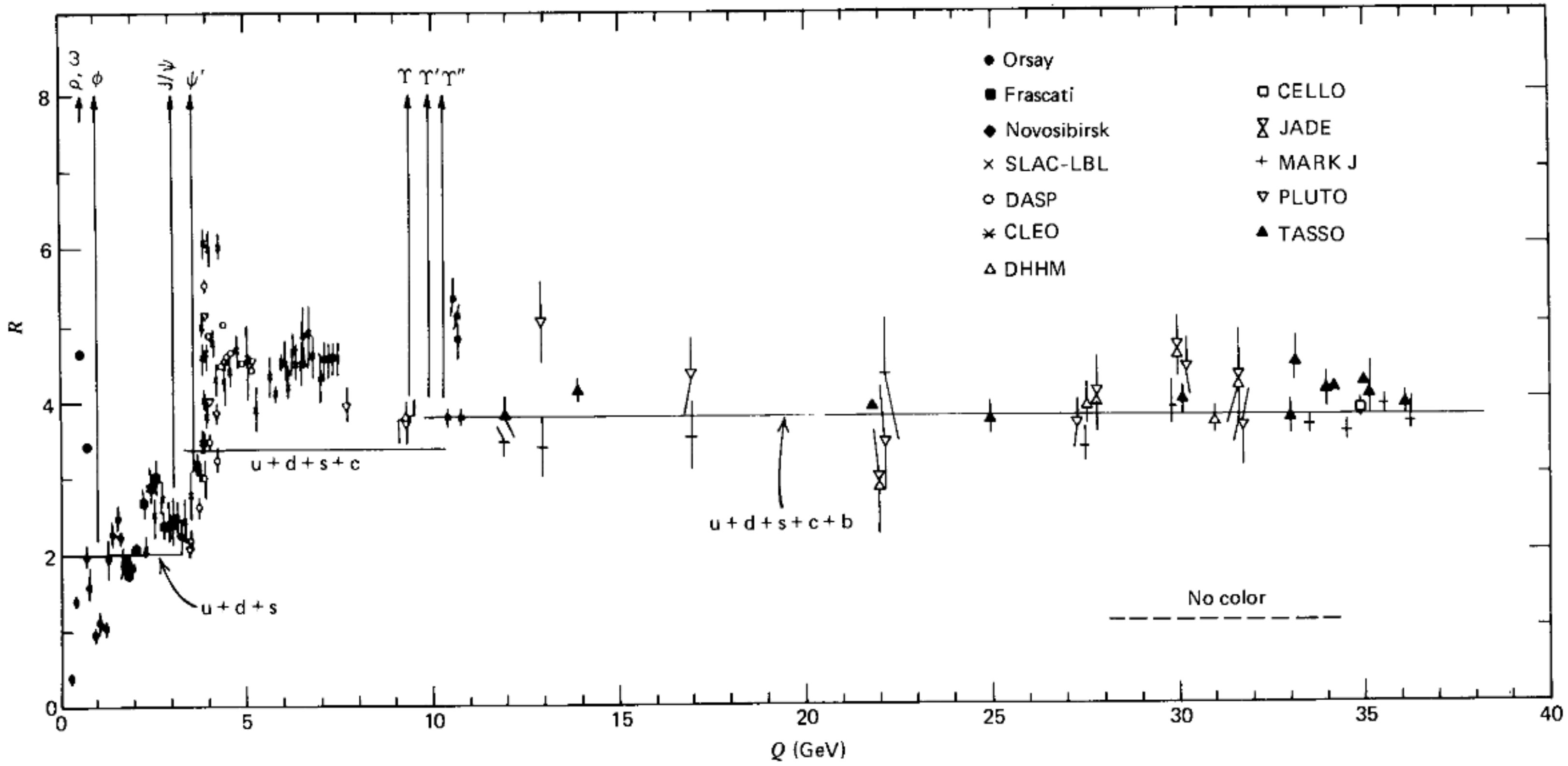
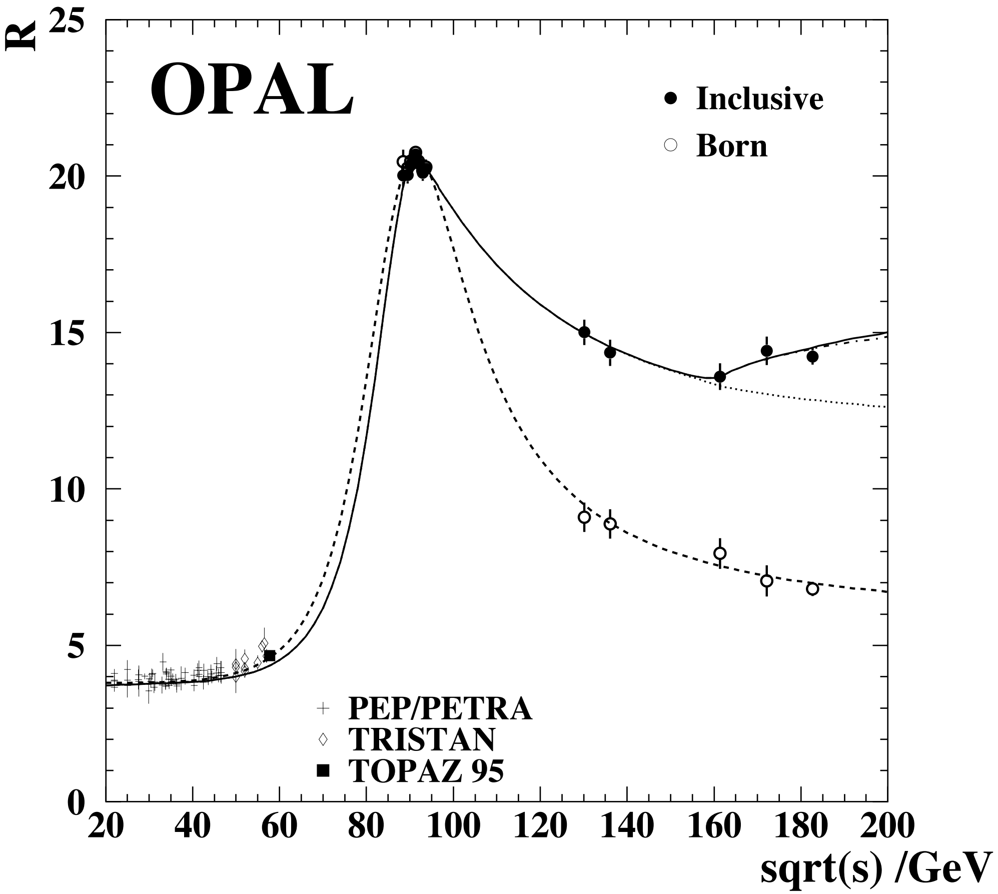


Fig. 11.3 Ratio R of (11.6) as a function of the total e^-e^+ center-of-mass energy. (The sharp peaks correspond to the production of narrov 1^- resonances just below or near the flavor thresholds.)

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_q Q_q^2$$

Source : Halzen and Martin "Introductory course in Modern Particle Physics"

Experimental results for the R-ratio



At the Z pole we can neglect the photon contributions and write the ratio from the partial widths

$$R_Z = \frac{\Gamma(Z \to \text{hadrons})}{\Gamma(Z \to \mu^+\mu^-)} = \frac{3\sum_q (A_q^2 + V_q^2)}{A_\mu^2 + V_\mu^2}$$

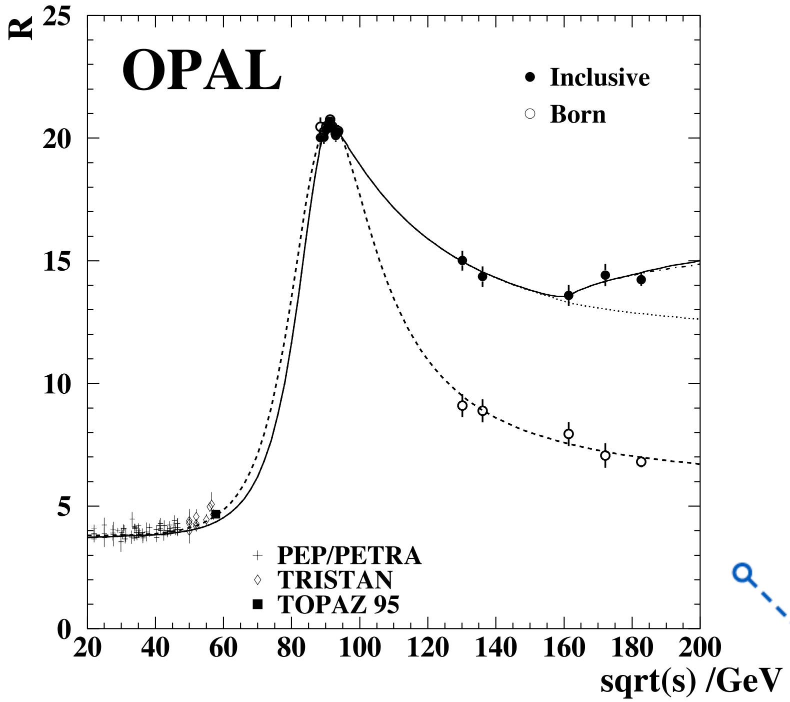
Where A_f and V_f define the axial and vector couplings of the fermion to the Z

Determining R_Z

Taking $\sin^2\theta_W = 0.23$ as an input and 5 active quark flavors we get $R = 11/3$ and $R_Z = 20.09$.

Looking at a measurement from LEP we $R_Z^{\text{exp}} = 20.79 \pm 0.04$

On the one hand, that's quite a good agreement for such a quick calculation, but on the other, its notably off. Can we improve it?



Idea : Assume the difference is driven by higher order corrections, and use these corrections to calculate $\alpha_S(M_z)$

Higher order corrections to R_Z

Our aim is now to compute the $\mathcal{O}(\alpha_S)$ corrections to R_Z

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Spoiler alert: The QCD corrections are independent of the nature of the electroweak boson exchanged. We'll make our notation easier and consider only the corrections to the photon exchange and use this result for the Z case of interest.

Higher order corrections to R_Z

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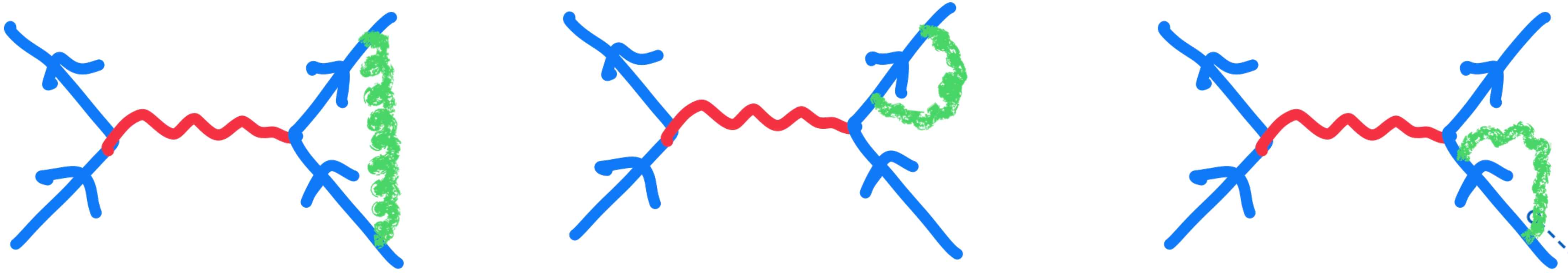
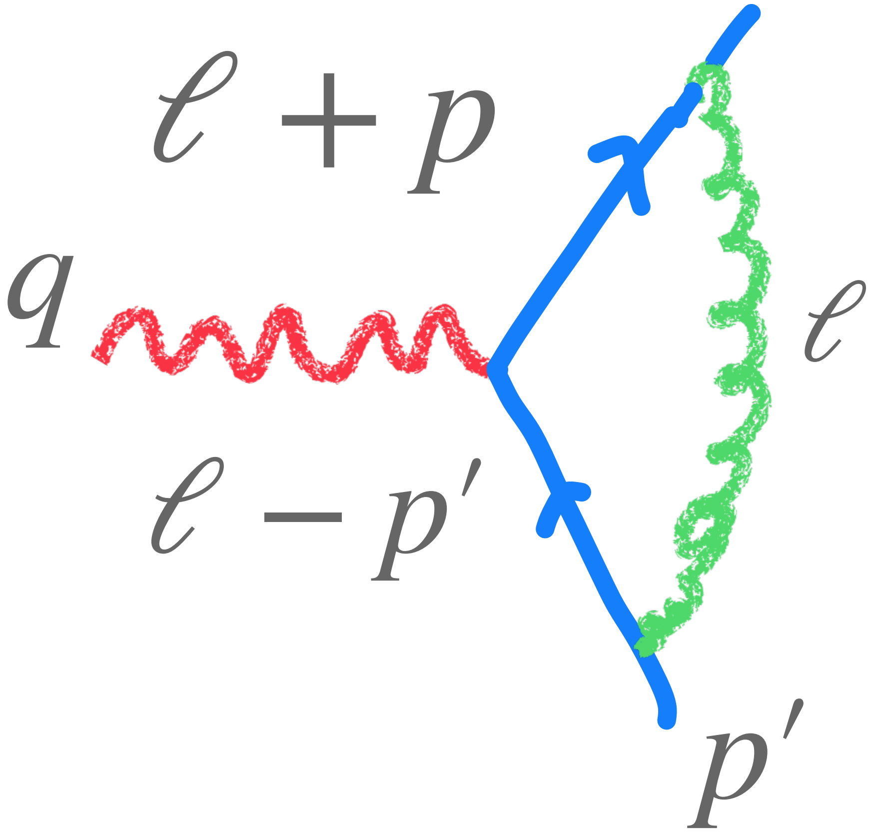
Spoiler alert: The QCD corrections are independent of the nature of the electroweak boson exchanged. We'll make our notation easier and consider only the corrections to the photon exchange and use this result for the Z case of interest.

At one-loop there are three diagrams to consider



Loop corrections to R_Z

For massless quarks the last two diagrams don't contribute, so we can look just at the vertex correction,



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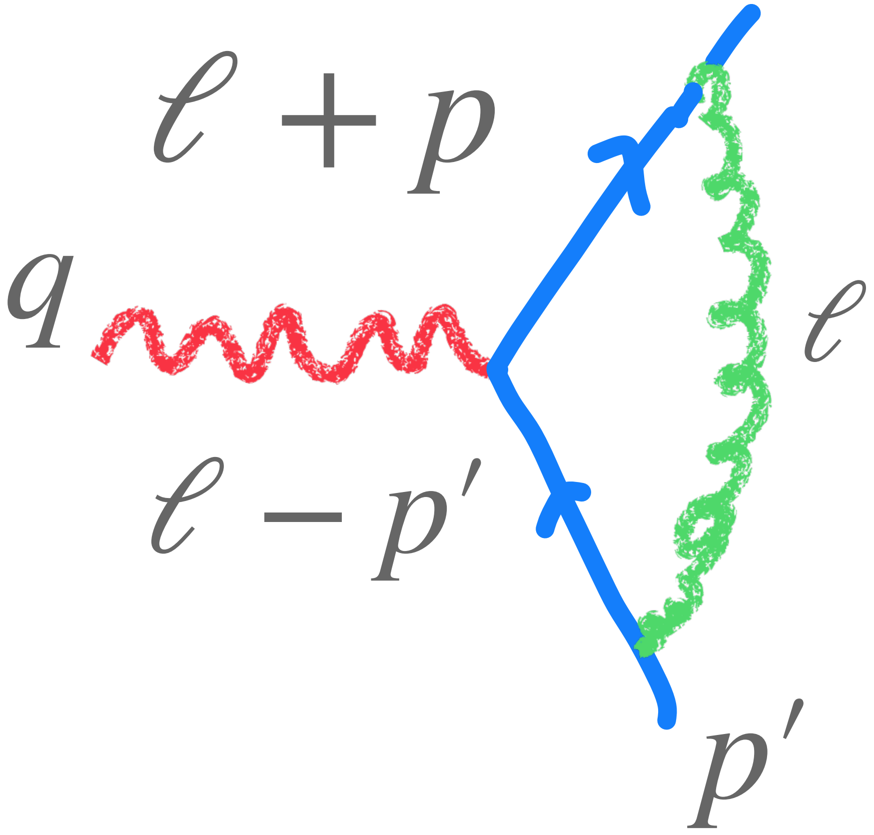

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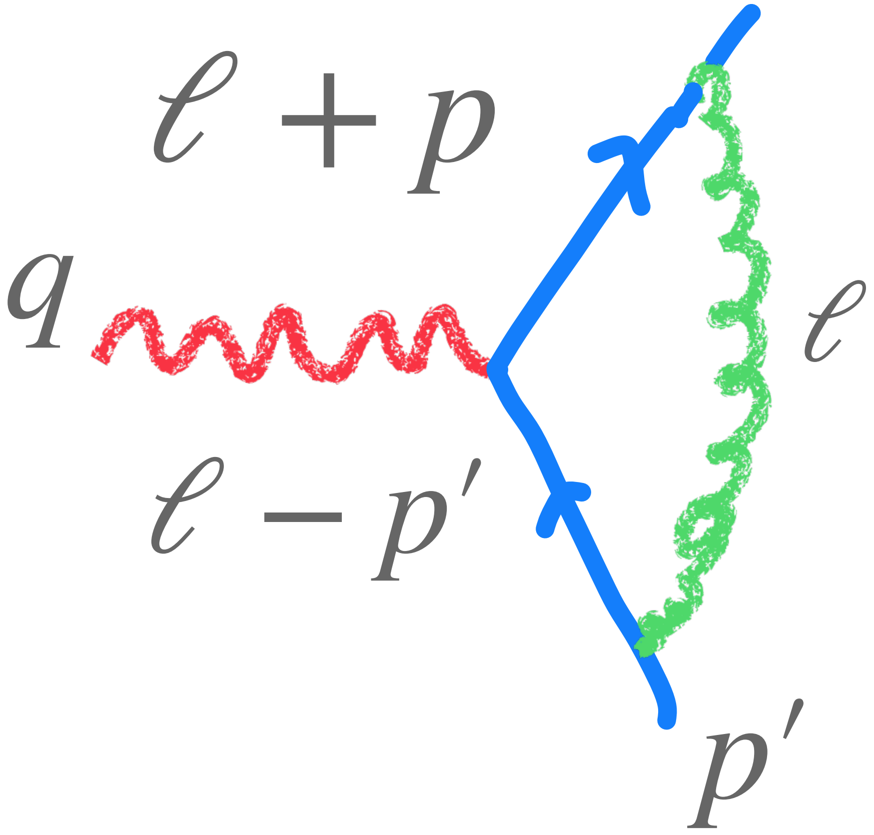


Momentum conservation results in one momentum ℓ, being unconstrained. So we integrate over it.

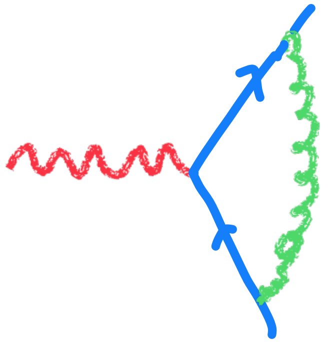
Applying the Feynman rules for QCD we'll find integrals which looks like

$$I_1 = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell+p)^2(\ell-p')^2}$$

And

$$I_1^{\mu\nu} = \int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{\ell^2(\ell+p)^2(\ell-p')^2}$$

Loop corrections to R_Z

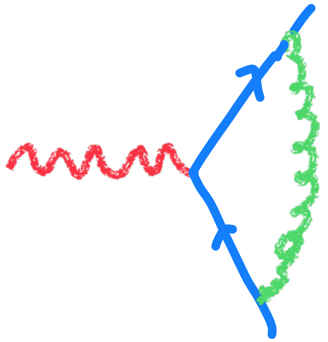


Let consider what happens to the integrals for small values of the loop momentum

$$I_1^{IR} \propto \int d\ell \frac{\ell^3}{\ell^2(\ell \cdot p)(\ell \cdot p')}$$

Where I've used $p^2 = 0$, there's an angular integration here too but we can neglect that for our discussion at present.

Loop corrections to R_Z



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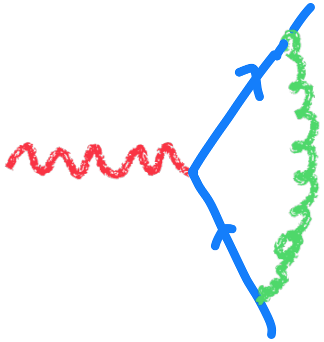

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This scales like $\int dx/x$ and there is a logarithmic singularity present as $\ell \to 0$.

In order of having a chance of understanding this infinity we need to regulate this singularity somehow.

Dimensional regularization

The idea is to regulate the integral by making a change of the form $\int dx/x \to \int dx/x^{1+\epsilon}$, then the singularity will be represented as a term $1/\epsilon$ in our resulting expressions.

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Key point : Loop Feynman integrals are commonly expressed as a series in ϵ, with singularities appearing as poles of the form ϵ^{-n}.

Loop corrections to R_Z

Getting back to our task at hand we can evaluate our Feynman integrals in d -dimensions and find that the contribution to the cross section is (up to an overall normalizing factor which is 1as $\epsilon \to 0$)

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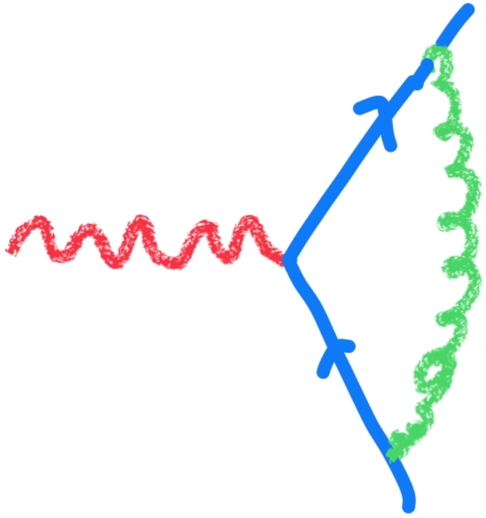
$$\sigma^{q\bar{q}}_{virt} = 3\sigma_0 \sum_q Q^2 \frac{C_F \alpha_S}{2\pi} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon) \right)$$



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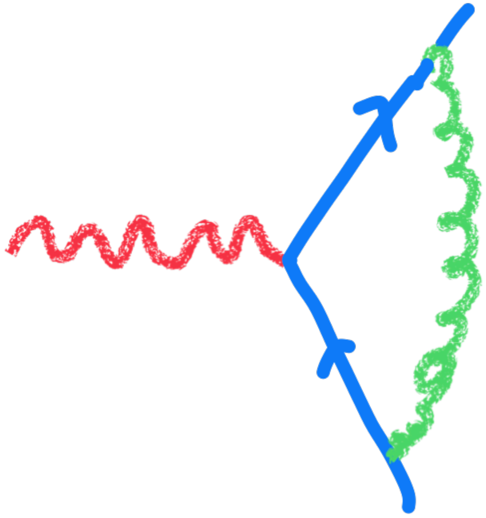


Here σ_0 denotes the lowest order prediction for $e^+e^- \to f\bar{f}$.

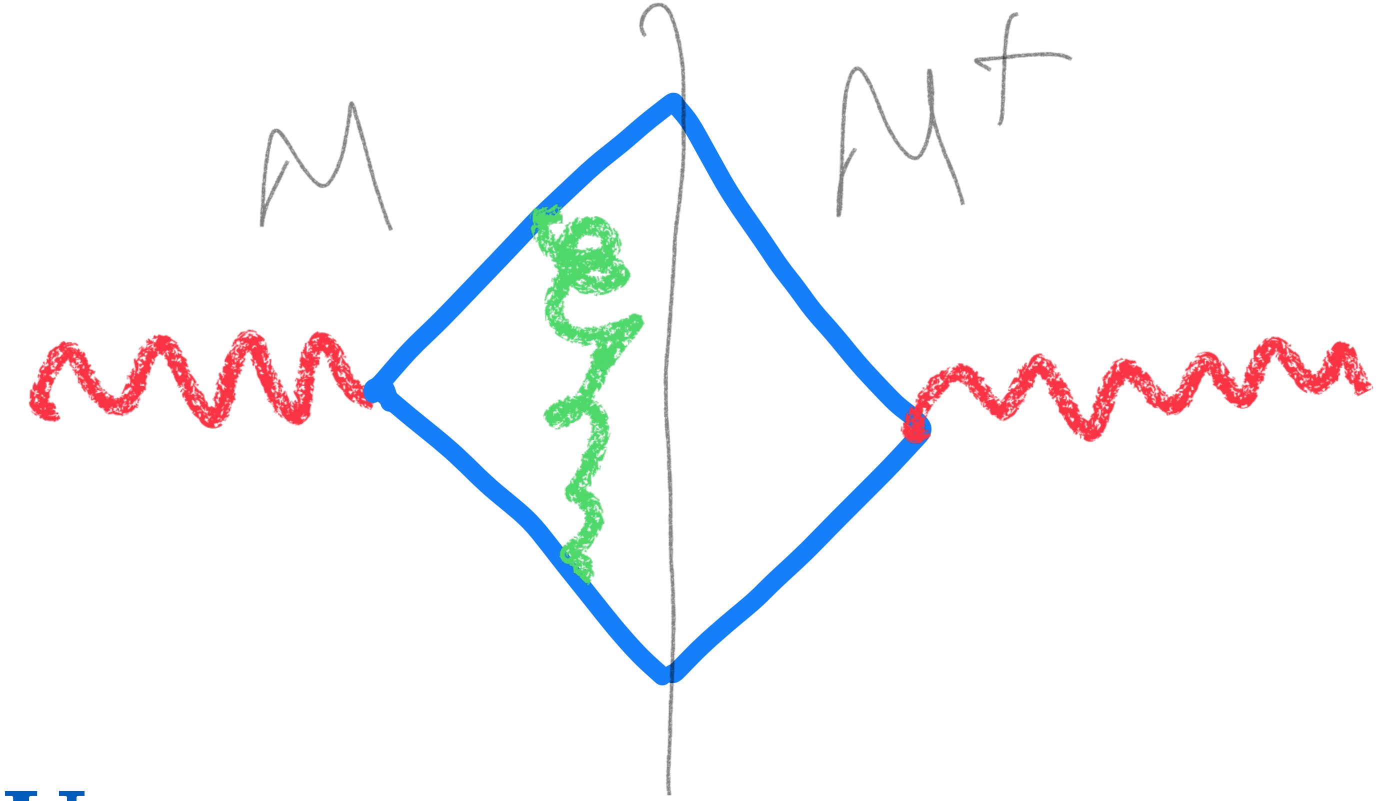
Our cross section is badly divergent, and the divergences are not of a UV origin (i.e. not fixed by renormalization) and also appear to be deeper than we just reasoned.

What gives?

Loop corrections to R_Z

These singularities are associated with low energy limits of the loop momentum, so we call them Infra-red or IR singularities.

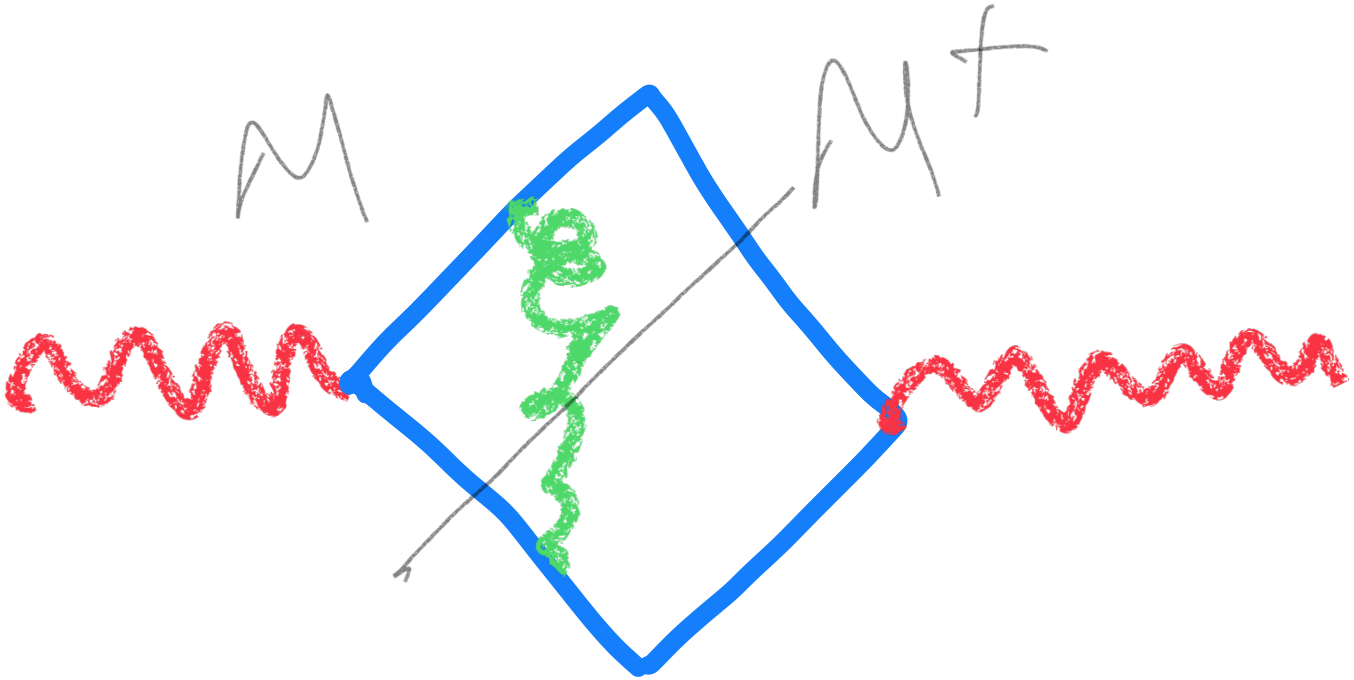
In order to fix the cross section we have to consider other $\mathcal{O}(\alpha_S)$ corrections



We considered virtual corrections, where we interfered a one-loop amplitude with a tree-level one.

Real corrections to R_Z

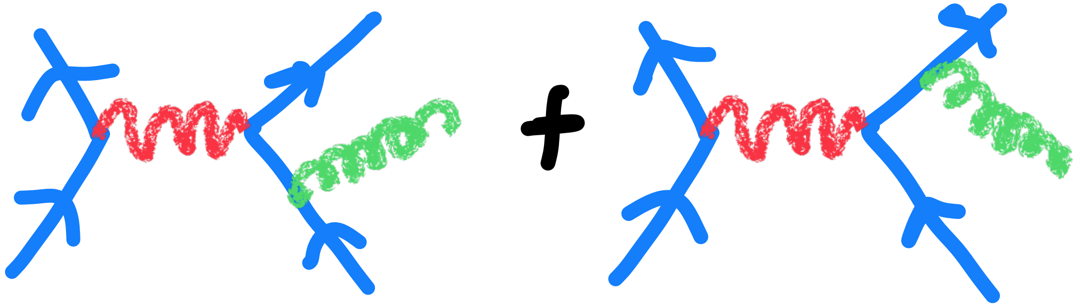
We can construct the cut a different way, in which we cut across the gluon line too and interfere two tree-level amplitudes for $e^+e^- \to q\bar{q}g$



Since the gluon is in the final state, it can be resolved and we call these terms **real** corrections

Real corrections to R_Z

$$\mathscr{M}^{q\bar{q}g} =$$



There are two diagrams for the amplitude for $e^+(q_1) + e^-(q_2) \to q(p_1) + \overline{q}(p_2) + g(k)$ shown above.

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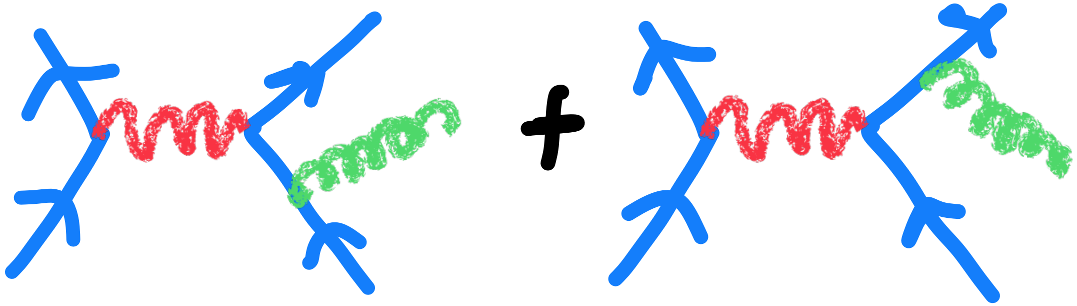

There are two diagrams for the amplitude for $e^+(q_1) + e^-(q_2) \to q(p_1) + \bar{q}(p_2) + g(k)$ shown above.

Applying the Feynman rules, and summing over spins/pols and colors we find the spin averaged matrix element squared to be

$$\frac{1}{4}|\overline{\mathscr{M}}|^2 = 6C_F e^4 Q_q^2 g_s^2 \frac{(p_1 \cdot q_1)^2 + (p_1 \cdot q_2)^2 + (p_2 \cdot q_1)^2 + (p_2 \cdot q_2)^2}{(q_1 \cdot q_2)(p_1 \cdot k)(p_2 \cdot k)}$$

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We introduce the variables $x_1 = 2E_q/\sqrt{s}$ and $x_2 = 2E_{\bar{q}}/\sqrt{s}$.

Inspecting the denominators shows us where the singular regions are.

Real corrections to R_Z

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$p_1 \cdot k = E_q E_g(1 - \cos\theta_{qg})$ can vanish when **either** $E_g \to 0$ **or** $\cos\theta_{qg} \to 1$.
Note that both can occur simultaneously.

We call the first a **soft** singularity (gluon is emitted with very low energy) and the second a **collinear** singularity (gluon of any energy emitted parallel to quark (or anti-quark)

Real corrections to R_Z

We can write the cross section in more transparent way using our x_i variables

$$\sigma^{q\bar{q}g} = 3\sigma_0 \sum_q Q_q^2 \int dx_1 dx_2 \frac{C_F \alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

The bounds of integration are $0 \leq x_i \leq 1$ and $x_1 + x_2 \geq 1$. We see the singular points at $x_i = 1$ (collinear) and soft at $x_1 = x_2 = 1$.

Obviously to get a sensible answer we'll need to regulate the real corrections too. A smart strategy would be to use the same regulator for the real as we did for the virtual (loop) corrections. This means we should reevaluate the integrals in d-dimensions.

Real corrections to R_Z

$$\sigma^{q\bar{q}g} = 3\sigma_0 \sum_q Q_q^2 \int dx_1 dx_2 \frac{C_F\alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

Shifting to d-dimensions (both in the phase space and in the Dirac traces) we get

$$\sigma^{q\bar{q}g} = 3\sigma_0 \sum_q Q_q^2 \int \frac{dx_1 dx_2}{(1-x_1-x_2)^\epsilon} \frac{C_F\alpha_s}{2\pi} \frac{(1-\epsilon)(x_1^2+x_2^2) - 2\epsilon((1-x_1)(1-x_2) - (1-x_1-x_2))}{(1-x_1)^{1+\epsilon}(1-x_2)^{1+\epsilon}}$$

I've normalized this by the same factor as the virtual (but suppressed it for readability)

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This integral can now be performed and the result is

$$\sigma^{q\bar{q}g} = 3\sigma_0 \sum_q Q_q^2 \frac{C_F\alpha_S}{2\pi} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} + \pi^2 + \mathcal{O}(\epsilon) \right)$$

$\mathcal{O}(\alpha_s)$ corrections to the hadronic cross section

We can now combine both type of corrections, real :

$$\sigma^{q\bar{q}g} = 3\sigma_0 \sum_q Q_q^2 \frac{C_F \alpha_S}{2\pi} \left(\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \pi^2 + \mathcal{O}(\epsilon) \right)$$

And virtual

$$\sigma^{q\bar{q}}_{virt} = 3\sigma_0 \sum_q Q^2 \frac{C_F \alpha_S}{2\pi} \left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \pi^2 + \mathcal{O}(\epsilon) \right)$$

Obtaining

$$\boxed{\sigma^{had}_{NLO} = 3\sigma_0 \sum_q Q_q^2 \left(1 + \frac{3}{2} \frac{C_F \alpha_S}{2\pi} \right)}$$

$\mathcal{O}(\alpha_s)$ corrections to the hadronic cross section

We see that the combined virtual (loop) + real contributions are **finite**

$$\sigma_{NLO}^{had} = 3\sigma_0 \sum_q Q_q^2 \left(1 + \frac{3}{2}\frac{C_F\alpha_S}{2\pi}\right)$$

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This is no accident, and is a manifestation of the Bloch Nordsieck and Kinoshita Lee and Nauenberg theorems which state that suitably defined **inclusive** quantities (where we integrate over emissions) in QFT are free from IR singularities.

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The take home message : The total hadronic cross section is a well defined and finite quantity in QFT, whereas the **exclusive** final state $\sigma(e^+e^- \rightarrow q\bar{q})$ (with no emission) is **not Infra-red safe**.

$\mathcal{O}(\alpha_s)$ corrections to R_Z

We can now write down an expression for R_Z (the ratio of the total hadronic cross section to the muonic one at the Z pole) as

$$R_Z^{NLO} = R_Z^{LO}\left(1 + \frac{\alpha_S(M_Z)}{\pi}\right)$$

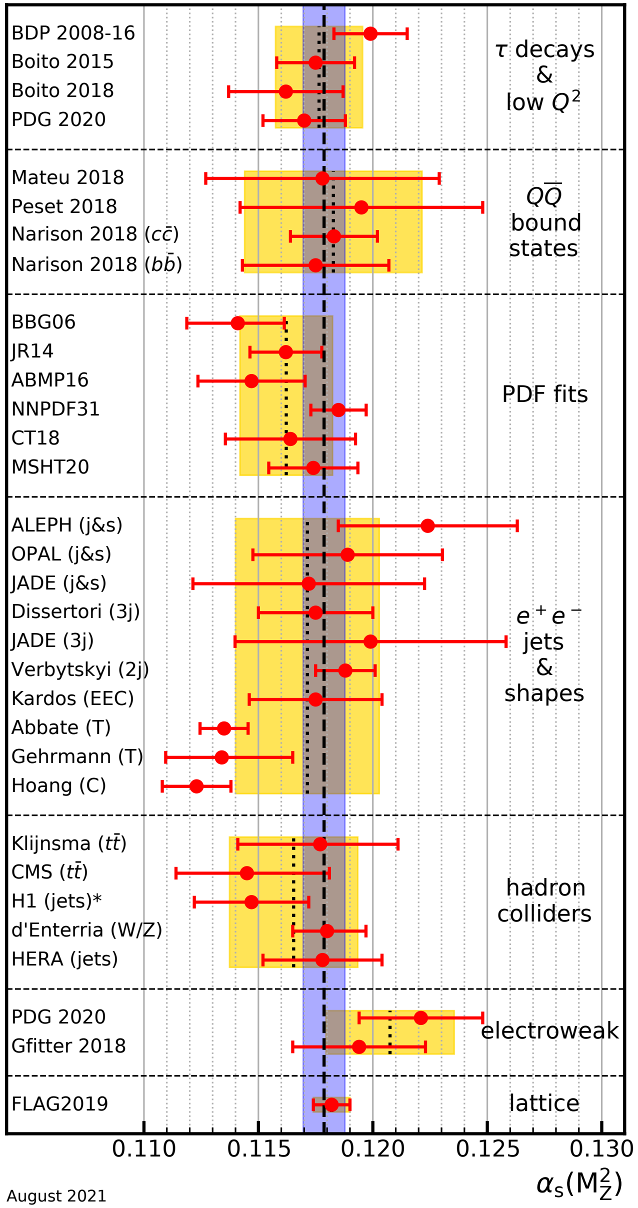
Where we used $C_F = 4/3$. Recall that since $R_Z^{LO} = 20.09$ and we have a measurement from LEP for R_Z^{exp} we can extract a value of $\alpha_s(M_Z)$ from our calculation.

$$\boxed{\alpha_S(M_Z) = \pi\left(\frac{R_Z^{exp}}{R_Z^{LO}} - 1\right) = 0.11}$$

More comprehensive extractions of $\alpha_S(M_z)$

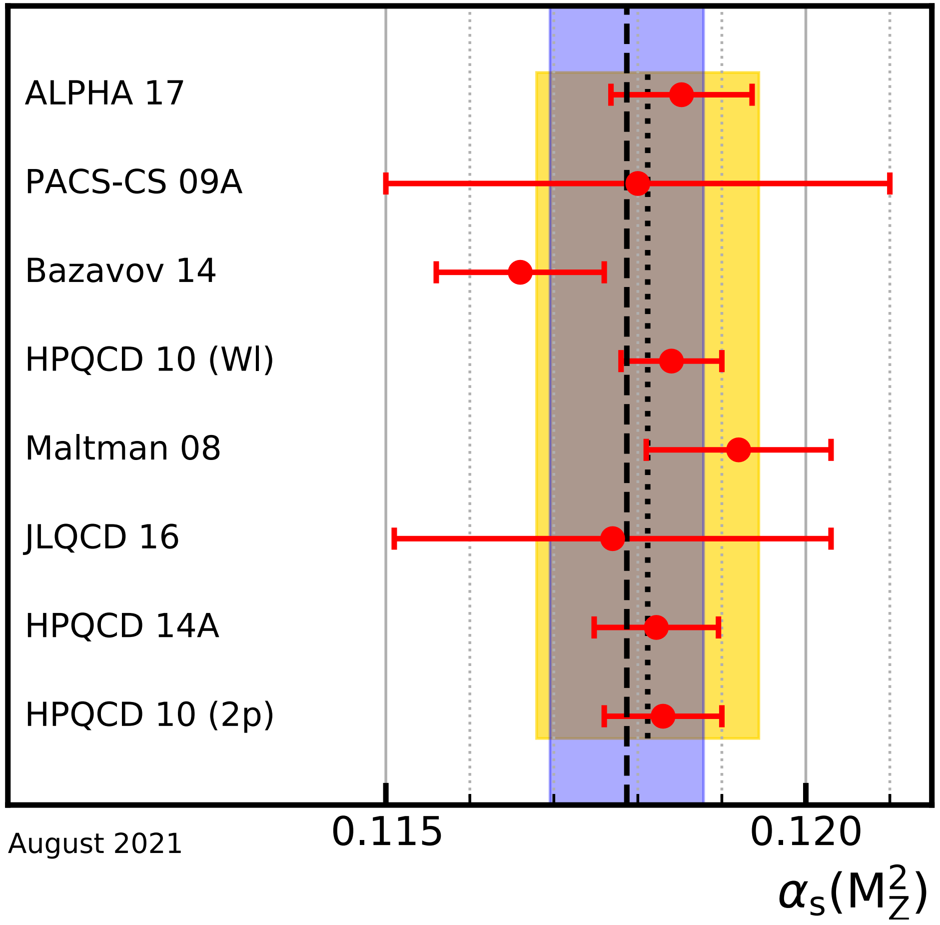
Not bad for a short calculation over a morning together. Here's our (lowest order) extraction and running, compared to the leading world average from the Particle Data group (PDG) (https://pdg.lbl.gov/). The current world average is

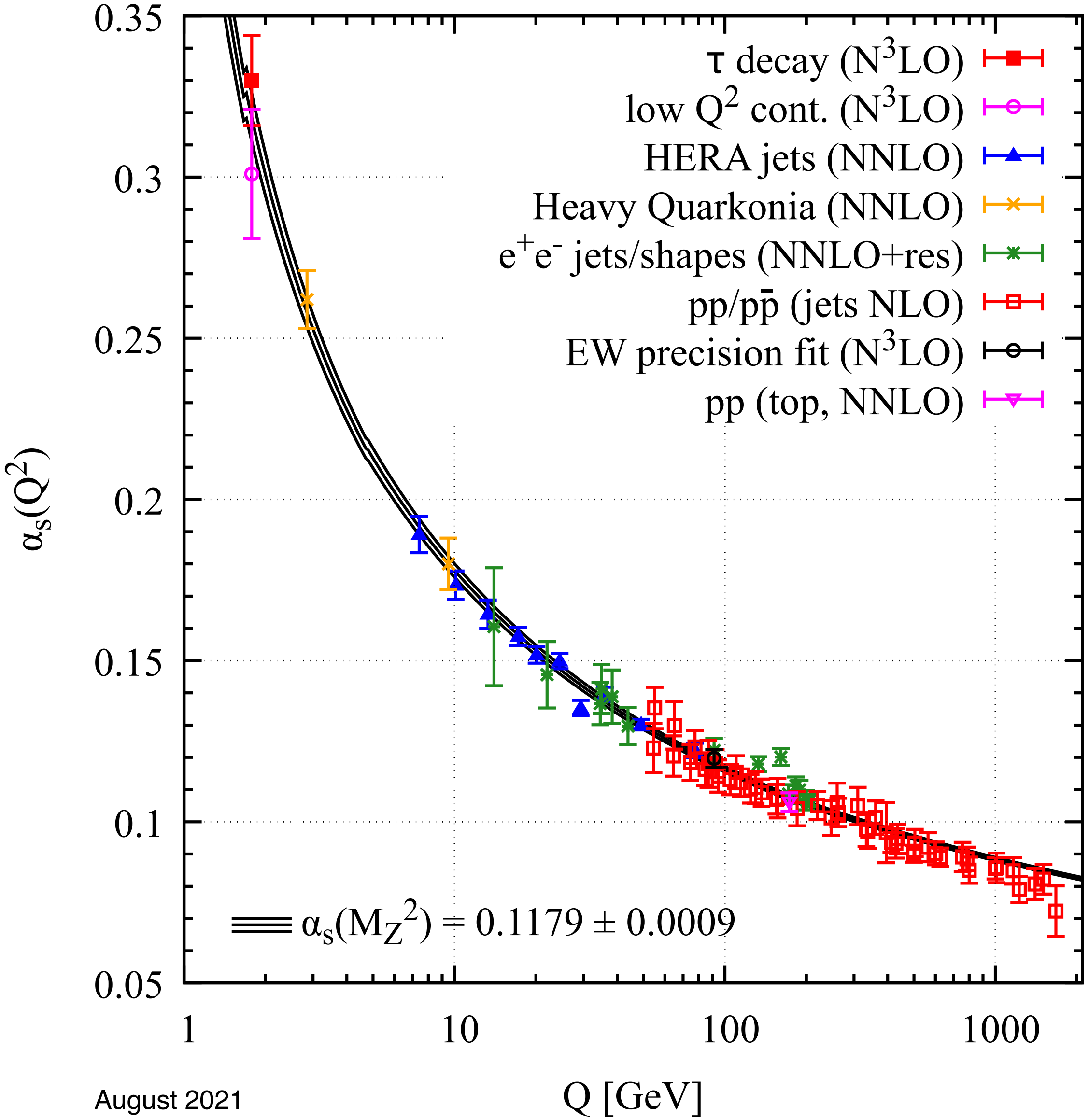
$$\alpha_S(M_Z) = 0.1179 \pm 0.0009$$



August 2021

Lattice QCD vs collider data extractions





Part 1 summary

The aim of this section has been to introduce you to the fundamentals of QCD, starting from the definition of the Lagrangian, and ultimately doing our first NLO calculation to extract $\alpha_S(M_Z)$, here's a summary of this part:

- Determined the underlying nature of QCD as an SU(N_c) gauge theory, with $N_c = 3$ colors.
- The non-abelian nature of the theory results in self-interactions among the gluons which mediate the strong force. This is completely different to QED in which the photons do not carry electric charge.
- As a result of the new diagrams which involve gluon loops the β-function of QCD as the opposite sign from that of QED. This broadly explains the properties of QCD. At high energies the coupling is reduced, a phenomenon known as asymptotic freedom. This explains how with very high energy DIS we can scatter off of individual quarks inside of a proton.
- On the other hand at lower energy's $\sim M_{proton}$ the coupling is large (non-perturbative) and QCD confines the quarks to bound states (baryons $\epsilon_{abc}q^a q^b q^c$ or mesons $q_a\overline{q}^a$)

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- At high energies, far away from the scale of hadronization, perturbation theory can be used to calculate higher order corrections in QCD.
- We performed an example calculation, computing the next-to-leading order corrections to the production of hadrons at an e^+e^- collider.
- We saw that higher corrections entered in two ways, as loop corrections to the underlying LO topology $e^+e^- \to q\bar{q}$ (virtual corrections) and real corrections which involved the emission of a gluon $e^+e^- \to q\bar{q}g$. Both had IR divergences associated with soft and collinear singularities.
- When both types of emission are regulated in the same way, and combined together to make the total cross section, these singularities cancel.