QCD LECTURE 2

HCPSS Summer school 2024

Ciaran Williams (SUNY Buffalo)

University at Buffalo The State University of New York

UNDERSTANDING QCD FINAL STATES

PART 1 - Exclusive final states in e^+e^- collisions.

Introduction

At the end of the last section we computed the $\mathcal{O}(\alpha_S)$ corrections to the hadronic cross section at lepton colliders.

This was nice, but we are leaving a lot on the table. We would like to able to look at a wider range of more dynamic variables scattering angles, energies etc. etc.

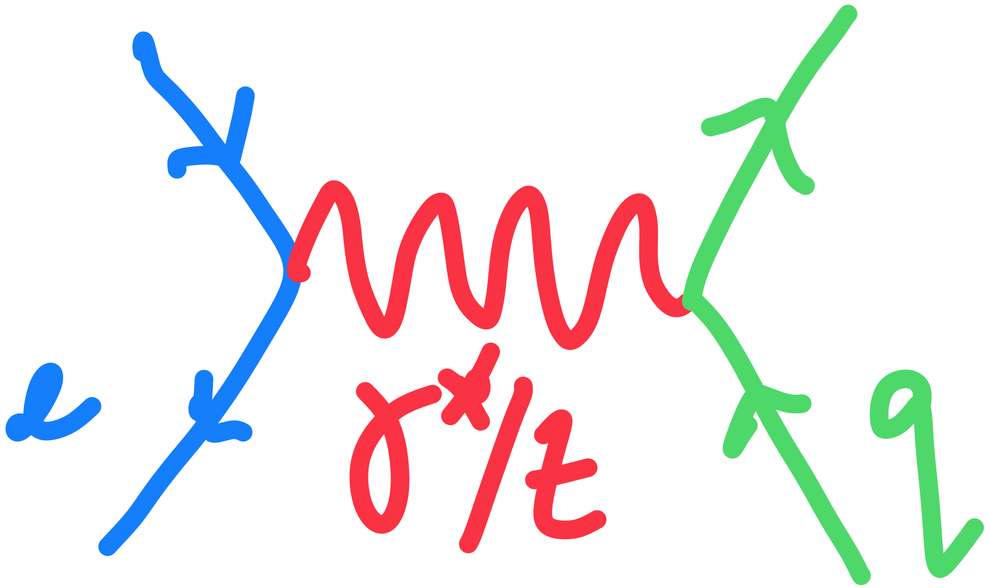
That's the aim of this section.

Since its easier from a pedagogy point of view we'll mostly stick to lepton colliders again today, but all of the ideas we discuss are directly applicable to hadron colliders too.

Jet cross sections

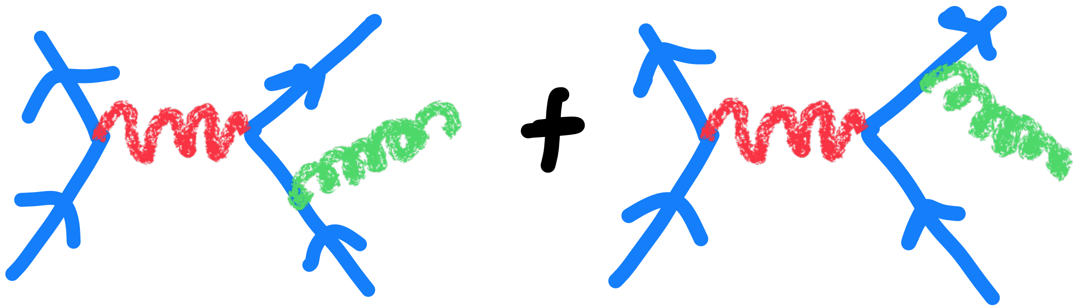
At lowest order, our picture is pretty simple, we produce a quark-antiquark pair which are back-to-back.

We then simply assume that the parton fragments “somehow” into a hadron.



Since the virtual (loop) topology lives in the same phase space, we make the same assumption there too.

Jet cross sections



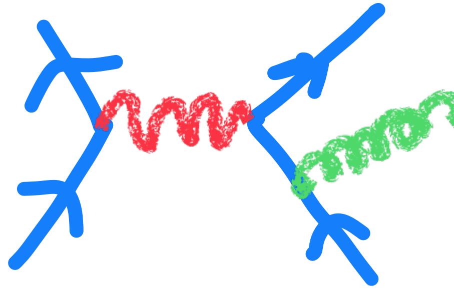
Next we consider the cross section for the production of a quark pair + a gluon at a lepton collider. Recall from last time :

$$\sigma^{q\bar{q}g} = 3\sigma_0 \sum_q Q_q^2 \int dx_1 dx_2 \frac{C_F \alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

Where x_i denotes the energy fraction of each quark i.e. $x_1 = 2E_q/\sqrt{s}$

Jet cross sections

$$\sigma^{q\bar{q}g} = 3\sigma_0 \sum_q Q_q^2 \int dx_1 dx_2 \frac{C_F \alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

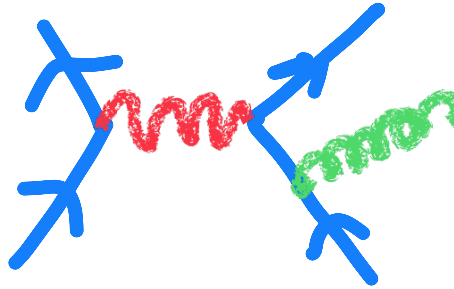


Now we are interested in thinking a little bit more differentially today. So lets re-write our cross section as a doubly differential quantity in the quark-antiquark energy fractions,

$$\frac{1}{\sigma_0} \frac{d^2\sigma^{q\bar{q}g}}{dx_1 dx_2} = \frac{C_F \alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

Jet cross sections

$$\sigma^{q\bar{q}g} = 3\sigma_0 \sum_q Q_q^2 \int dx_1 dx_2 \frac{C_F \alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$



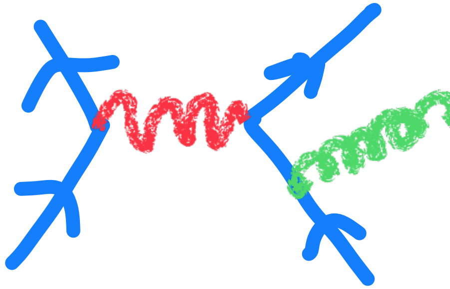
Now we are interested in thinking a little bit more differentially today. So lets re-write our cross section as a doubly differential quantity in the quark-antiquark energy fractions,

$$\frac{1}{\sigma_0} \frac{d^2\sigma^{q\bar{q}g}}{dx_1 dx_2} = \frac{C_F \alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

Let’s think about what happens for various values of x_i. (Recall $0 \leq x_i \leq 1$ and $x_1 + x_2 \geq 1$)

Jet cross sections

$$\sigma^{q\bar{q}g} = 3\sigma_0 \sum_q Q_q^2 \int dx_1 dx_2 \frac{C_F \alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$



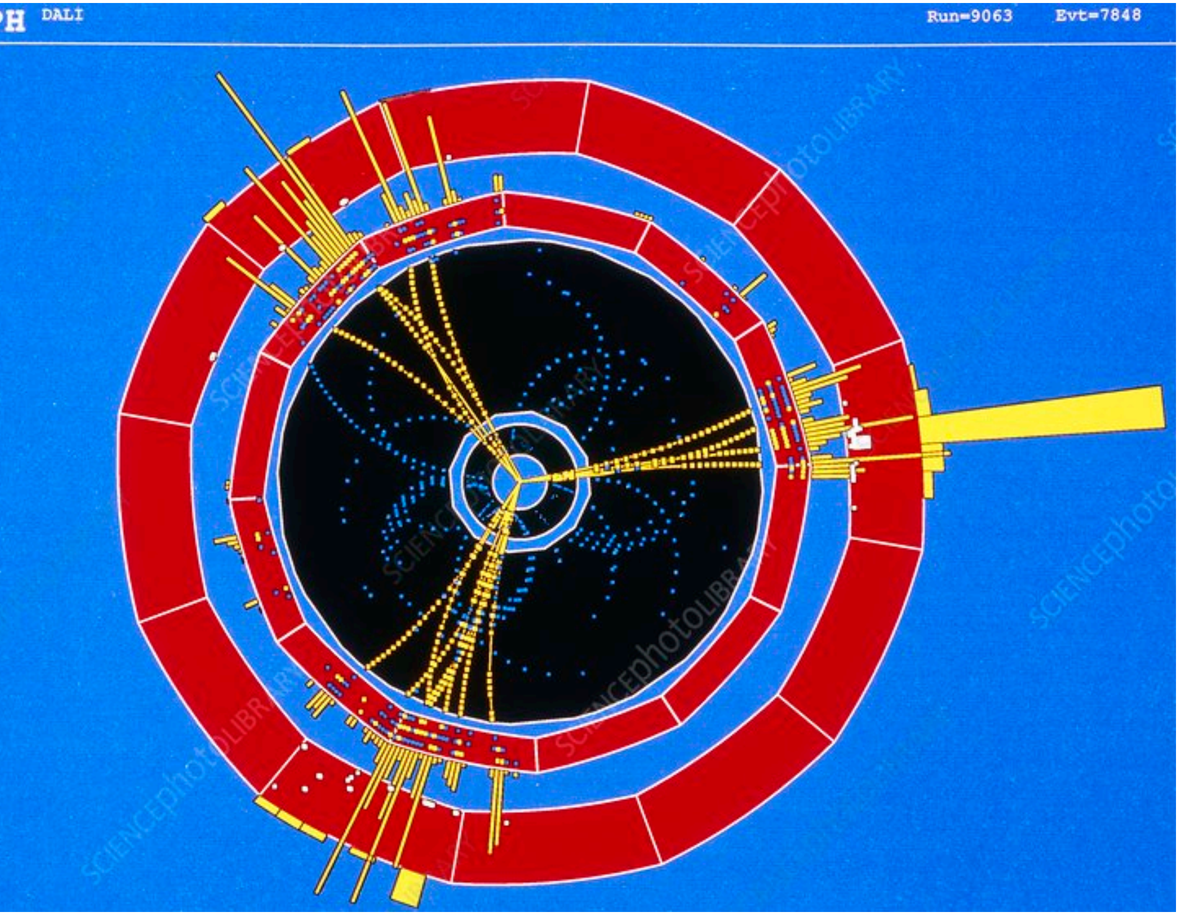
Now we are interested in thinking a little bit more differentially today. So lets re-write our cross section as a doubly differential quantity in the quark-antiquark energy fractions,

$$\frac{1}{\sigma_0} \frac{d^2\sigma^{q\bar{q}g}}{dx_1 dx_2} = \frac{C_F \alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

Let's think about what happens for various values of x_i. (Recall $0 \leq x_i \leq 1$ and $x_1 + x_2 \geq 1$)

If both x_i are not close to 1 (say for instance around 0.5) then the differential cross section for the emission of a gluon is suppressed relative to the no-emission cross section by a power of α_S, i.e. expected perturbative scaling

Jet cross sections



$$\frac{1}{\sigma_0}\frac{d^2\sigma^{q\bar{q}g}}{dx_1 dx_2} = \frac{C_F\alpha_s}{2\pi}\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

Note that in the region just defined the gluon possess a sizable energy fraction (since $x_g = 2 - x_1 - x_2$) and is not emitted close to either quark or anti-quark.

Mostly likely this type of configuration will therefore have 3 distinct “resolved” emissions.

Jet cross sections

$$\frac{1}{\sigma_0}\frac{d^2\sigma^{q\bar{q}g}}{dx_1 dx_2} = \frac{C_F\alpha_s}{2\pi}\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

What about the region where one of the x_i is near 1? Then the differential cross section diverges.

Jet cross sections

$$\frac{1}{\sigma_0}\frac{d^2\sigma^{q\bar{q}g}}{dx_1 dx_2} = \frac{C_F\alpha_s}{2\pi}\frac{x_1^2 + x_2^2}{(1 - x_1)(1 - x_2)}$$

What about the region where one of the x_i is near 1? Then the differential cross section diverges.

Here perturbation theory appears to breaks down, since the scaling compared to the lower order cross section is much greater than α_S.

Jet cross sections

$$\frac{1}{\sigma_0}\frac{d^2\sigma^{q\bar{q}g}}{dx_1 dx_2} = \frac{C_F\alpha_s}{2\pi}\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

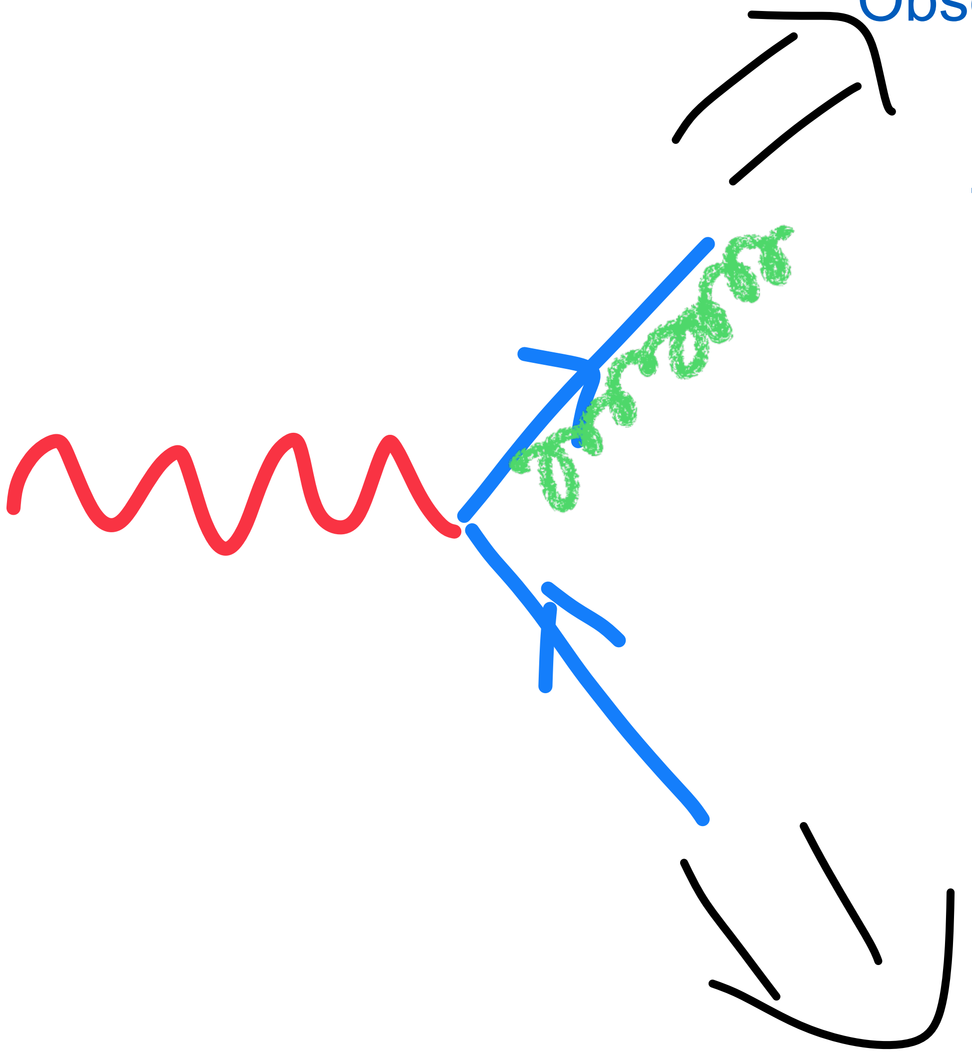
What about the region where one of the x_i is near 1? Then the differential cross section diverges.

Here perturbation theory appears to breaks down, since the scaling compared to the lower order cross section is much greater than α_S.

Thinking back to last time, we recall that in the region where the matrix element blows occurs when the gluon is emitted with either very small energy (soft) or in the same direction as one of the fermions (collinear).

QCD observables

Observable made from hadron(s)

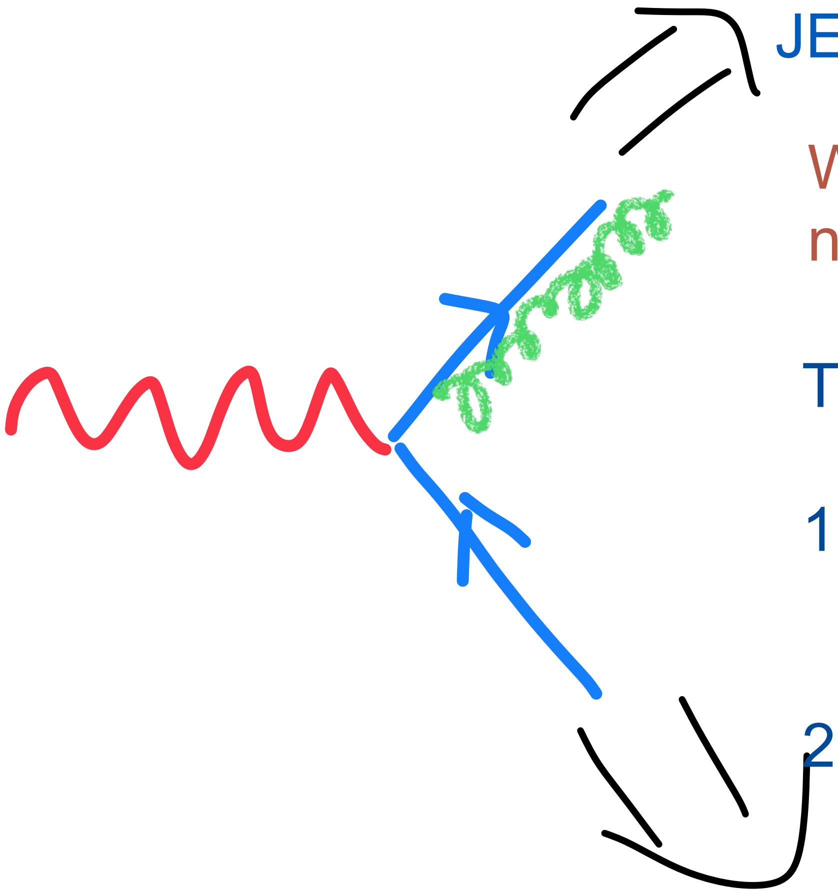


There are two key points:

1) Experiments don’t measure patrons, they detect hadrons
2) When the gluon is emitted close to the parton its impossible to distinguish the two from each other In terms of the hadronic observable.

Observable made from hadron(s)

QCD observables



We need an observable definition which is insensitive to the number of emissions

This observable is called a **jet** it has to do a couple of jobs:

1) Split the differential cross section into individual pieces which are separately free of soft and collinear singularities

2) Be equally applicable to patrons (theory) or hadrons (experiment/theory) to allow for a comparison of the two.

Jet algorithms

Practically jets are defined through a jet algorithm, which takes in a clustering parameter (which will broadly define the size/scale of a jet) and an input event (made of partons, or hadrons) and returns a final state with a given number of jets.

Jet algorithms

Practically jets are defined through a jet algorithm, which takes in a clustering parameter (which will broadly define the size/scale of a jet) and an input event (made of partons, or hadrons) and returns a final state with a given number of jets.

Note that there is more than one possible jet algorithm (any clustering which is insensitive to the soft/collinear emission problem will provide a well-defined result) and that the same initial event can have a different number of final state jets dependent on the chosen algorithm/clustering parameter chosen.

Jet algorithms

Practically jets are defined through a jet algorithm, which takes in a clustering parameter (which will broadly define the size/scale of a jet) and an input event (made of partons, or hadrons) and returns a final state with a given number of jets.

Note that there is more than one possible jet algorithm (any clustering which is insensitive to the soft/collinear emission problem will provide a well-defined result) and that the same initial event can have a different number of final state jets dependent on the chosen algorithm/clustering parameter chosen.

It's obviously important to compare theory/experiment with the same jet algorithm!

The JADE algorithm

We'll discuss jet algorithms for the LHC in part III. To complete our calculation here we introduce an older jet algorithm which is more appropriate for lepton colliders, the JADE algorithm.

The JADE algorithm

We'll discuss jet algorithms for the LHC in part III. To complete our calculation here we introduce an older jet algorithm which is more appropriate for lepton colliders, the JADE algorithm.

Start with the two partons i and j with the minimum $y_{ij} = (p_i + p_j)^2/s$, where s is the com energy.

The JADE algorithm

We'll discuss jet algorithms for the LHC in part III. To complete our calculation here we introduce an older jet algorithm which is more appropriate for lepton colliders, the JADE algorithm.

Start with the two partons i and j with the minimum $y_{ij} = (p_i + p_j)^2/s$, where s is the com energy.

Compare this to a pre-determined value y_{cut}.

The JADE algorithm

We'll discuss jet algorithms for the LHC in part III. To complete our calculation here we introduce an older jet algorithm which is more appropriate for lepton colliders, the JADE algorithm.

Start with the two partons i and j with the minimum $y_{ij} = (p_i + p_j)^2/s$, where s is the com energy.

Compare this to a pre-determined value y_{cut}.

If $y_{ij} < y_{cut}$ define the clustered object as the combined $p_i^\mu + p_j^\mu$ and remove partons i and j.

The JADE algorithm

We'll discuss jet algorithms for the LHC in part III. To complete our calculation here we introduce an older jet algorithm which is more appropriate for lepton colliders, the JADE algorithm.

Start with the two partons i and j with the minimum $y_{ij} = (p_i + p_j)^2/s$, where s is the com energy.

Compare this to a pre-determined value y_{cut}.

If $y_{ij} < y_{cut}$ define the clustered object as the combined $p_i^\mu + p_j^\mu$ and remove partons i and j.

Repeat until no $y_{ij} < y_{cut}$ the number of remaining objects is the number of jets.

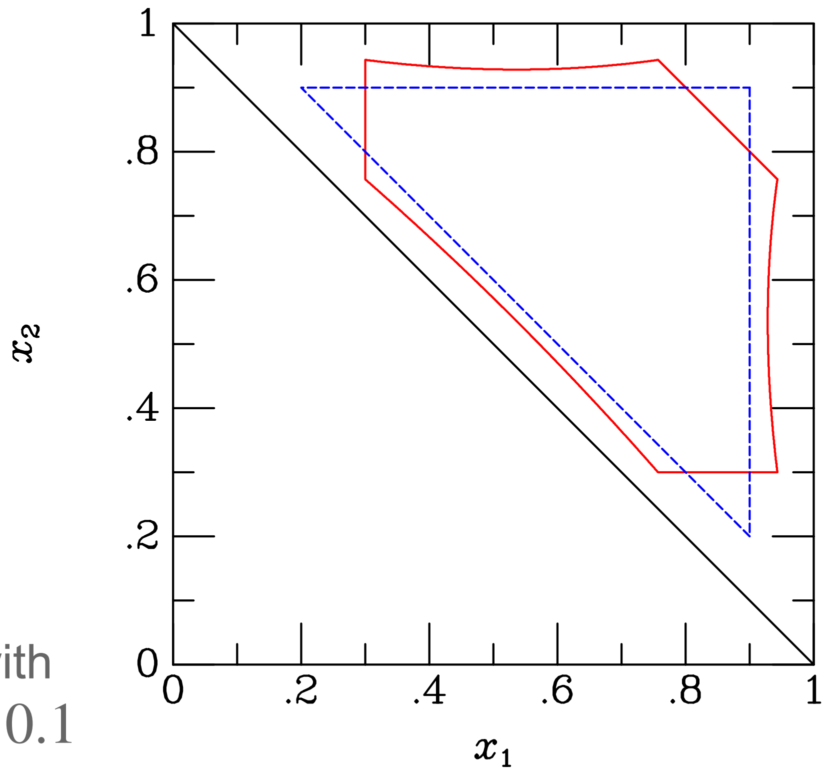
Jet fractions

The nice thing about lepton colliders and the jade algorithm is that we can analytically calculate jet fractions with our cross section, first we define $\sigma = \sigma_2 + \sigma_3$, i.e. the total cross section is made out of either two or three jet events for us.

Then we define jet fractions as $f_i = \sigma_i / \sigma$ (and of course $f_2 = 1 - f_3$ at NLO).

With our definition of JADE algorithm our energy fractions are constrained as follows

$$0 \leq x_i \leq 1 - y_{cut} \text{ and } x_1 + x_2 > 1 + y_{cut}$$



Blue- 3 jet area with JADE and $y_{cut} = 0.1$

Jet fractions at NLO

With our definition of the cross section $\frac{1}{\sigma_0}\frac{d^2\sigma^{q\bar{q}g}}{dx_1 dx_2} = \frac{C_F\alpha_s}{2\pi}\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$

We can write a formula for f_3 as follows,

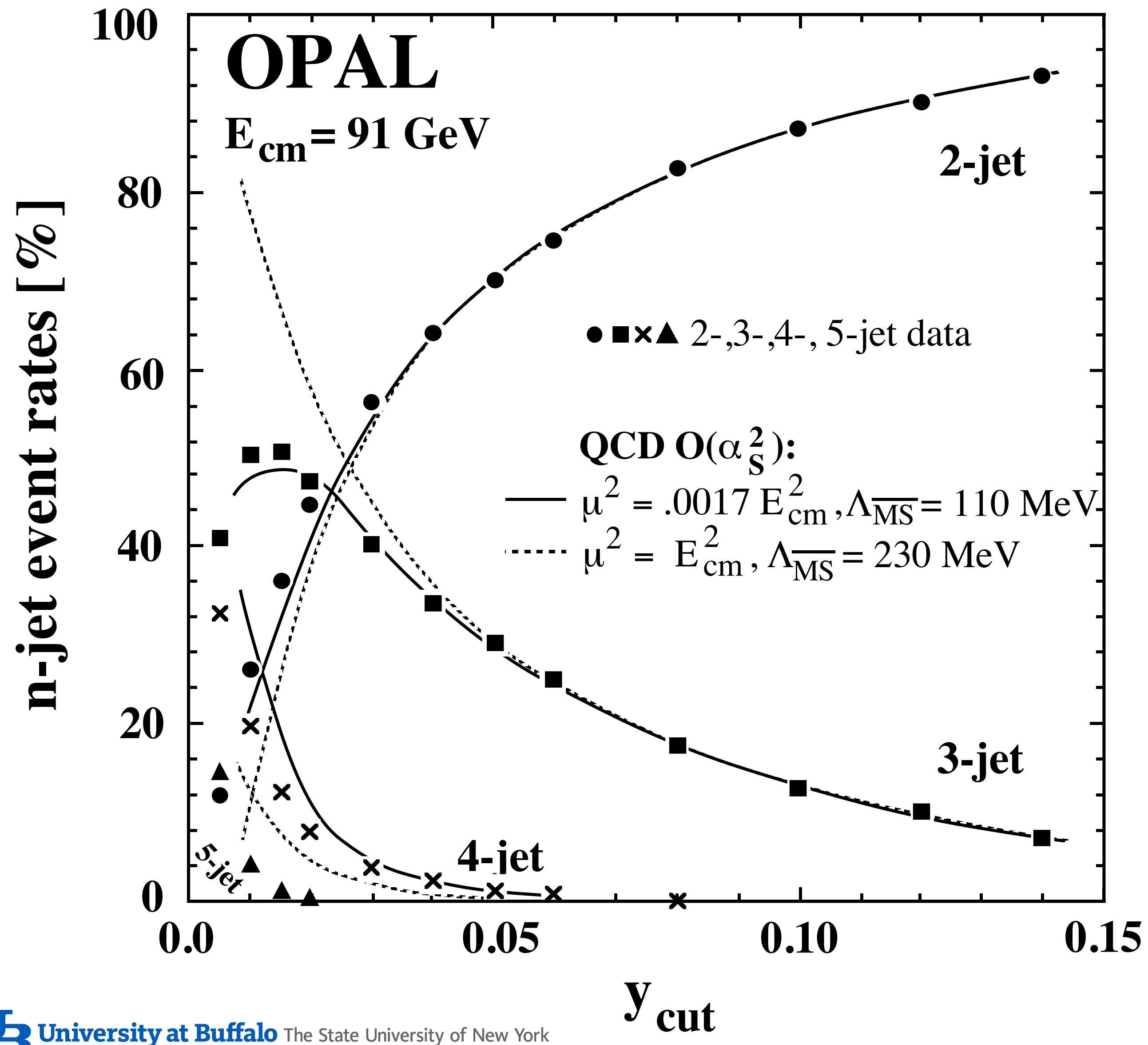
$$f_3 = \frac{C_F\alpha_S}{2\pi}\int_0^{1-y} dx_1 dx_2 \Theta(x_1 + x_2 - 1 - y)\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

Where $\Theta(z) = 1$ if $z > 0$ and 0 otherwise.

So that

$$f_3 = \frac{C_F\alpha_S}{2\pi}\left((3-6y)\ln\frac{y}{1-2y} + 2\ln^2\frac{y}{1-y} + \frac{5}{2} - 6y - \frac{9}{2}y^2 + 4\mathrm{Li}_2\left(\frac{y}{1-y}\right) - \frac{\pi^2}{3}\right)$$

Plot of jet fractions and data.



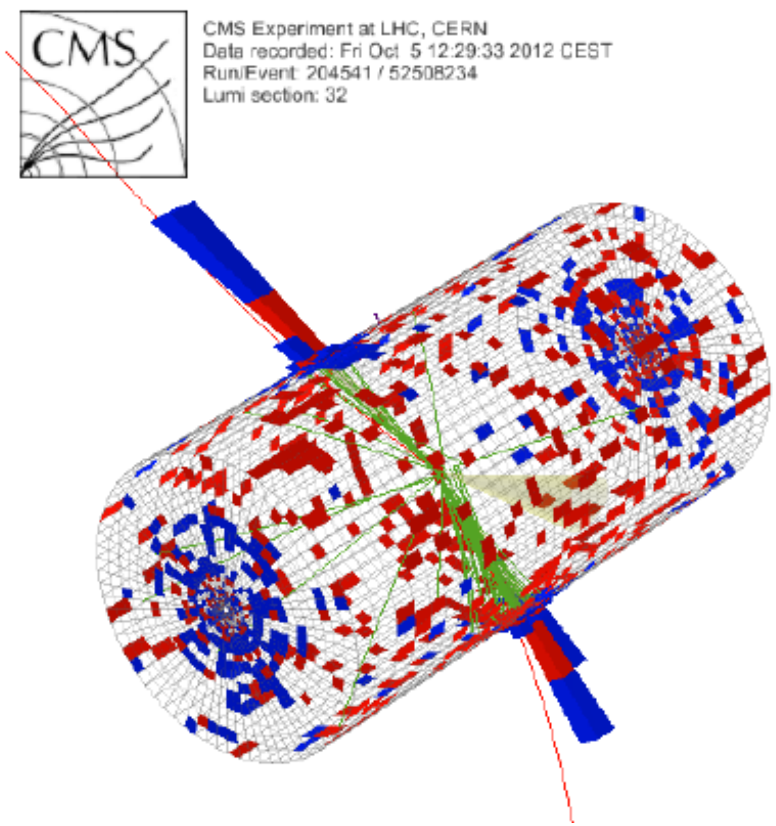
We can then compare our jet fractions with data. (Here at one higher order in pert. theory)

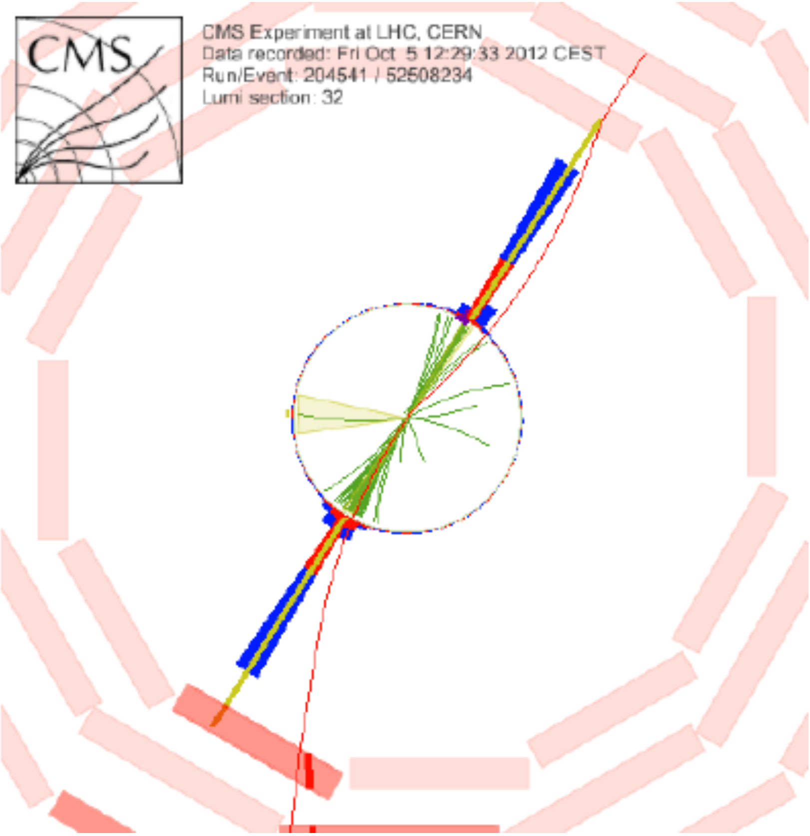
We see that for larger values of y_{cut} the perturbative expansion is doing a good job of describing the the data.

At lower values the perturbation theory is unreliable, since we probe regions which are much more collinear.

Jets at the LHC

Inspecting a typical event illustrates that our initial notion that a “parton becomes a hadron” is a bit naive, a “parton becomes a jet” is a bit better. But its clear just from visually looking at jet displays that an experimental jet contains multiple hadrons which move in an approximately collinear direction.

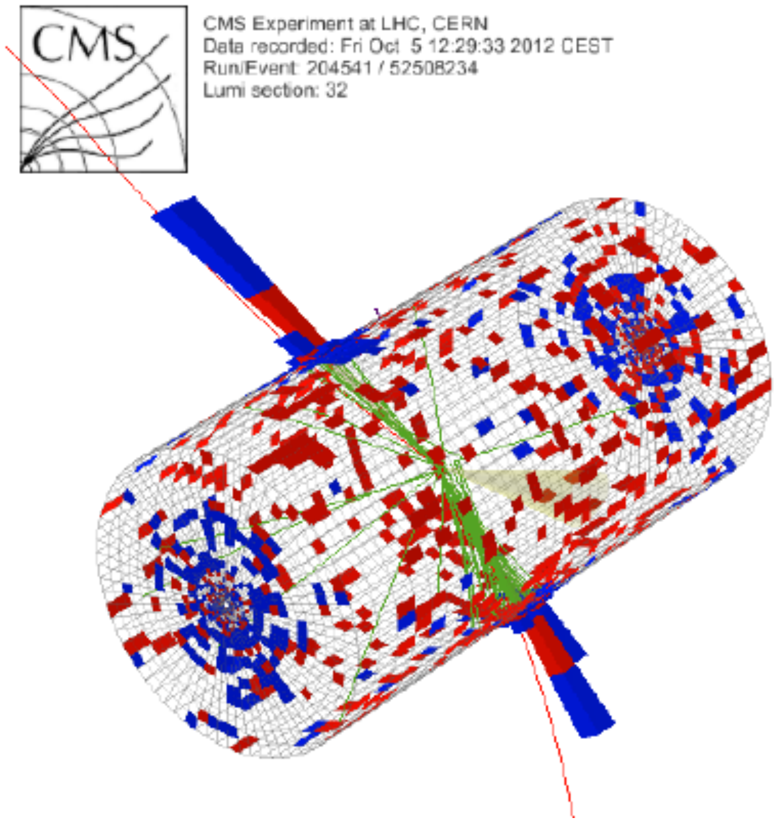


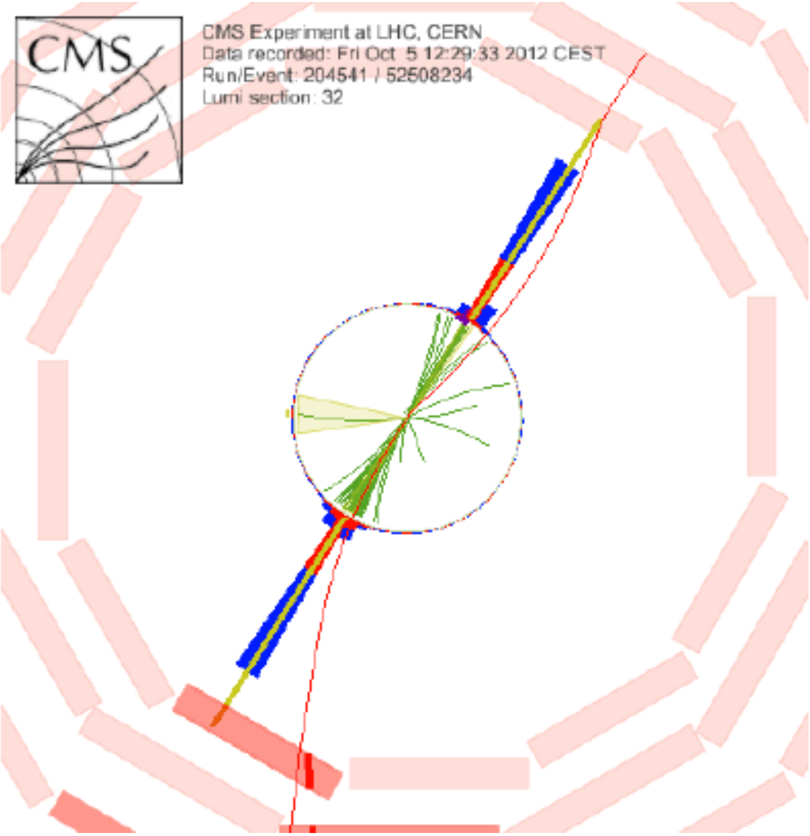


Jets at the LHC

We would like to understand how to make a bridge from the world of partons, which enter the hard scattering process, to jets, which contain multiple hadrons.

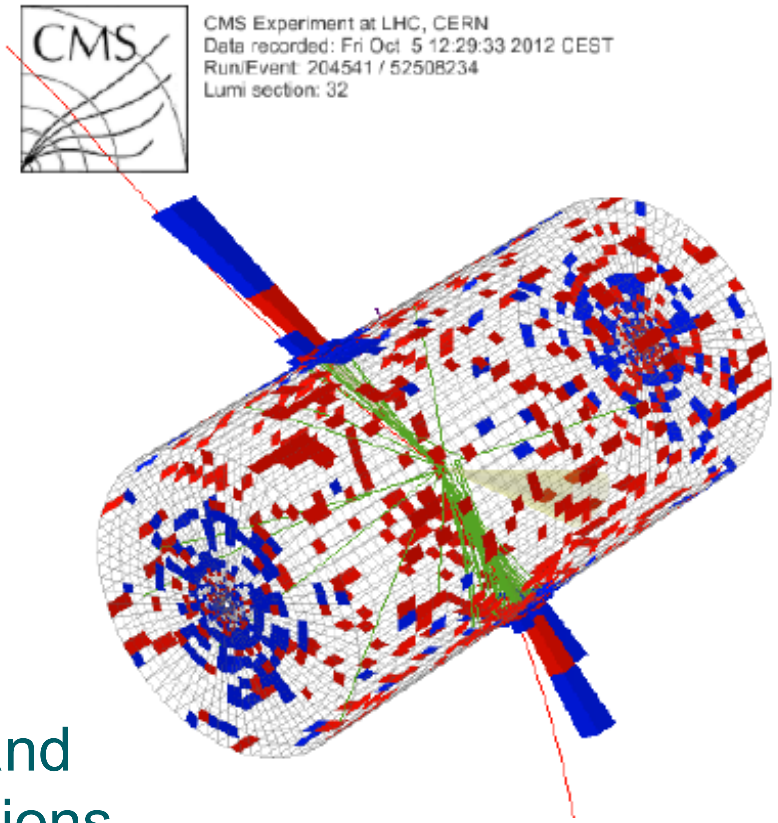
In order to do so we'll need to understand the collinear behavior of QCD a little better.

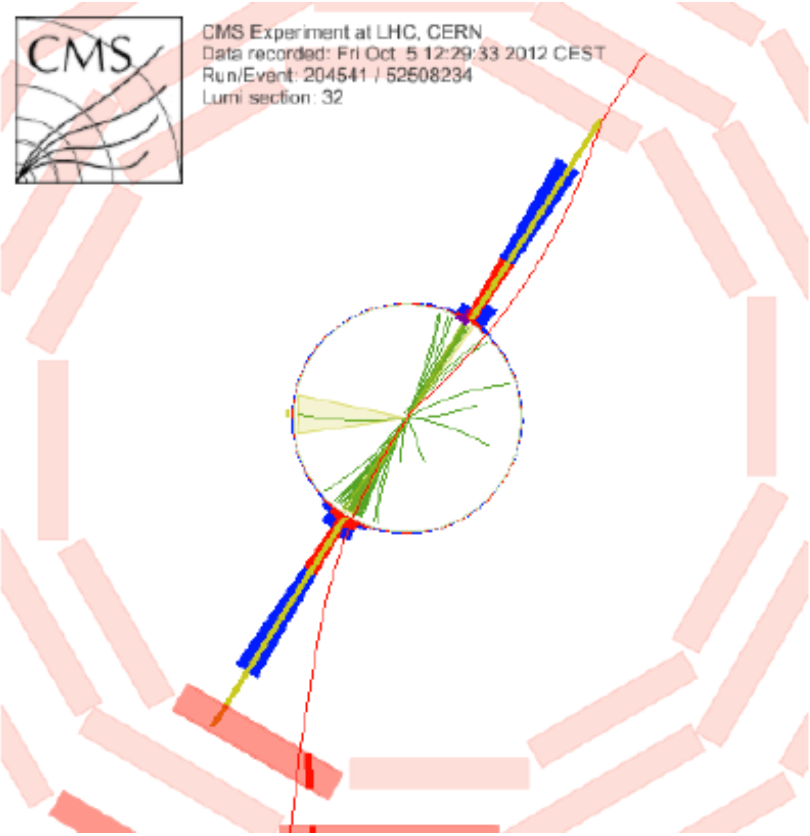




Jets at the LHC

However, to understand the QCD radiation pattern in the final state we will first have to address a more pressing question.





How can we understand and make predictions for collisions involving hadrons?

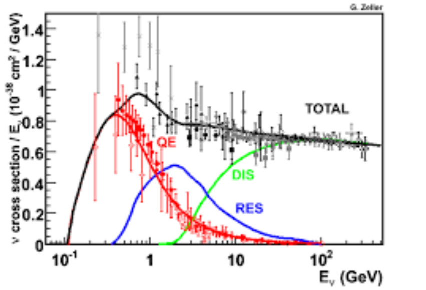
UNDERSTANDING HADRON STRUCTURE

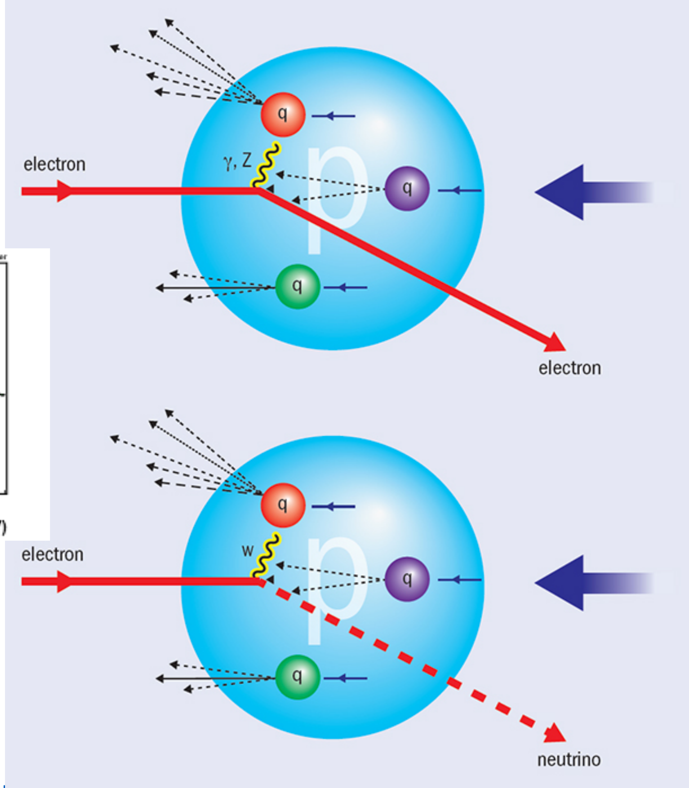
Deep Inelastic Scattering

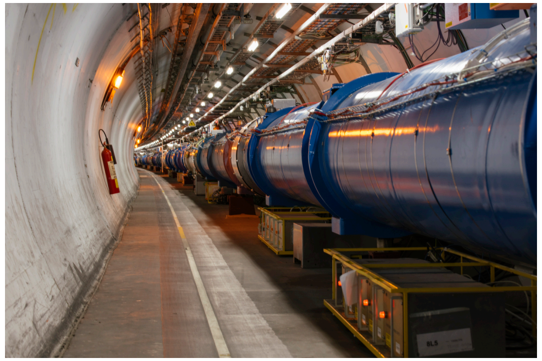
Motivation

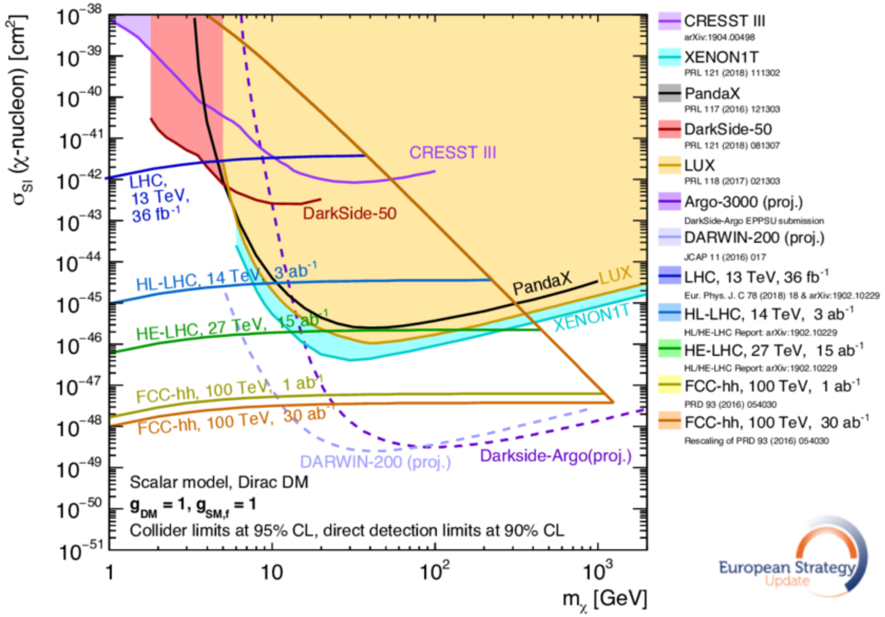
Scattering of some kind of target off a hadron or nucleus makes up a HUGE amount of particle and nuclear physics.

We are either trying to learn more about the hadrons/QCD itself, or are using the hadron as a source of energy/recoil to observe some other hard scattering event.





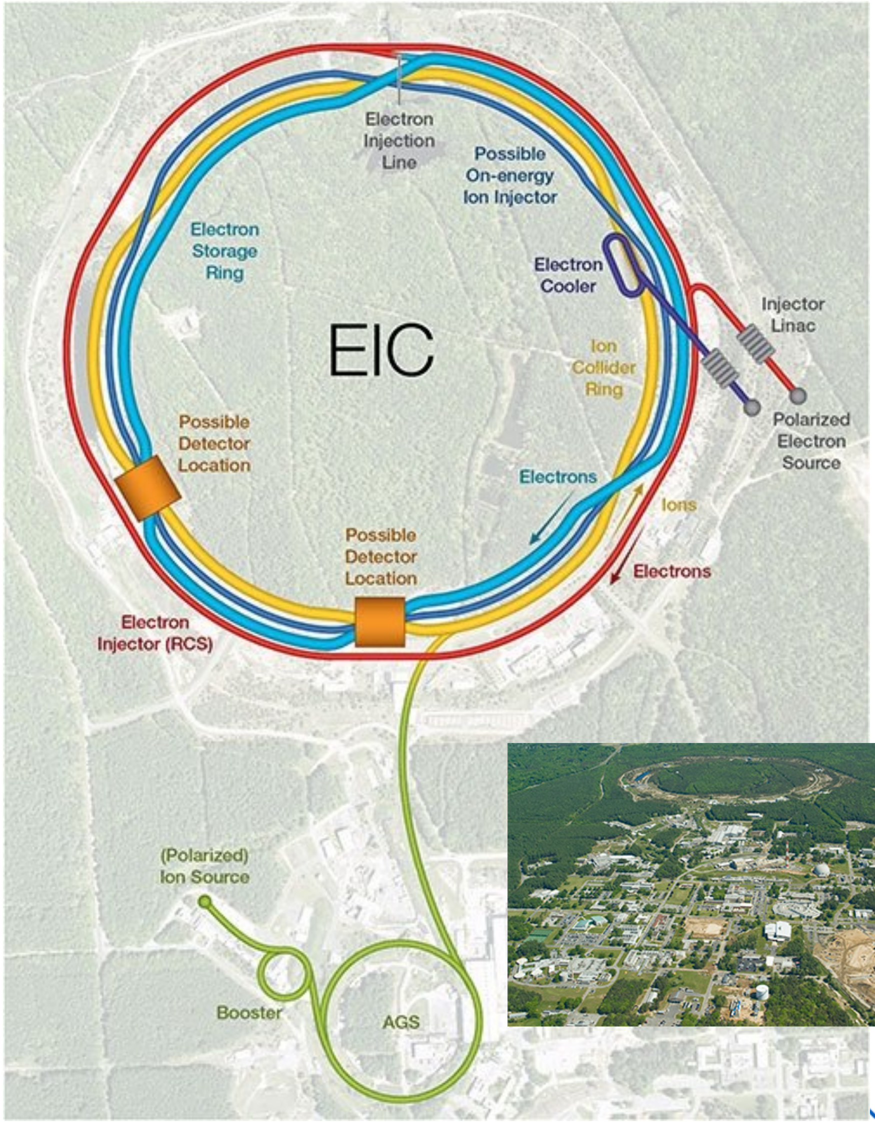




ep colliders

For modern particle physics, much of our most useful data comes from the HERA experiment, which ran at DESY in Hamburg from 1992 to 2007

A next generation machine (primarily focussed on nuclear physics, but with great potential for overlap) is the Electron Ion collider which will run at Brookhaven National Lab

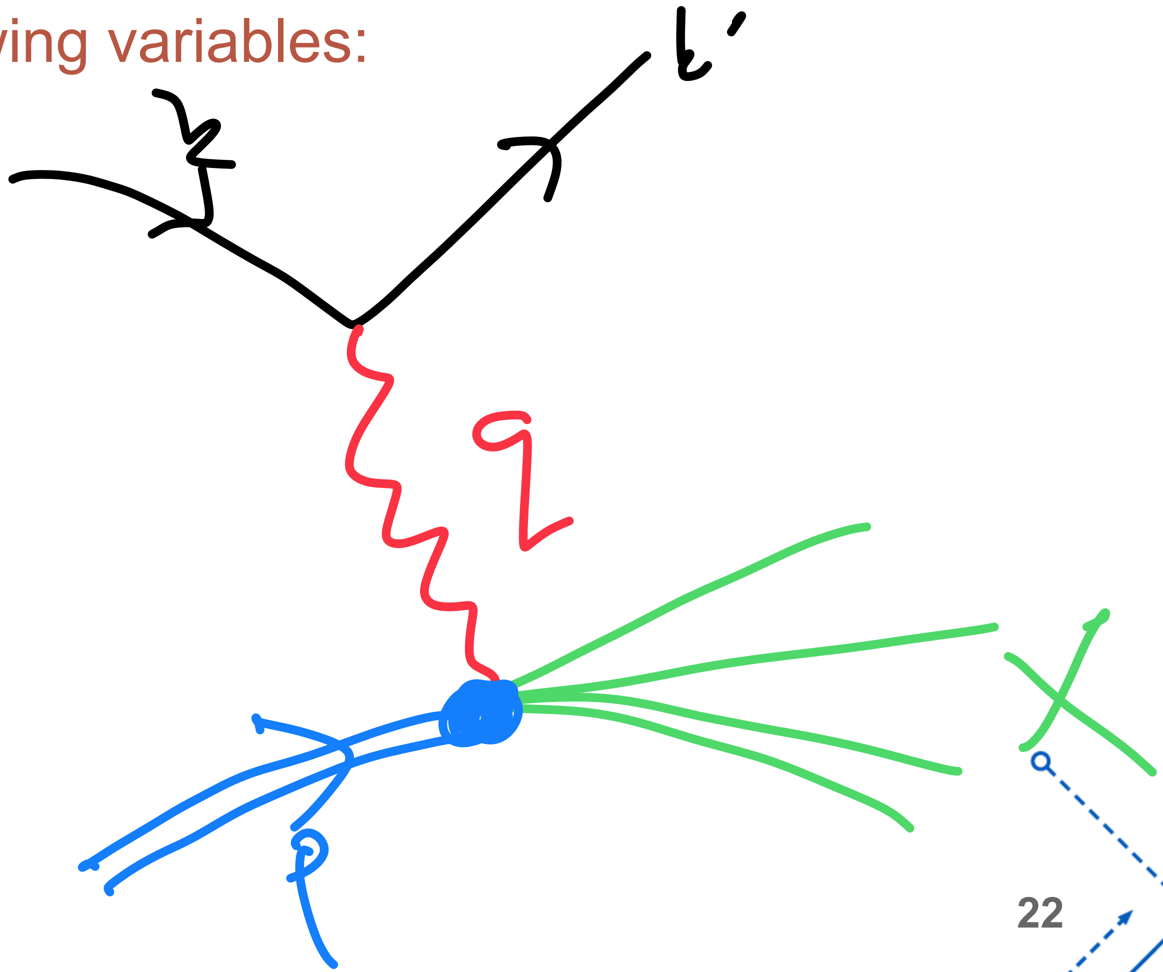


Deep Inelastic Scattering

Deep inelastic scattering (DIS) occurs when a lepton interacts with a hadron, traditionally we think about electron-proton scattering, but modern experiments also consider different possibilities.

DIS can be expressed in terms of the following variables:

$$Q^2 = -q^2$$
$$M^2 = p^2$$
$$\nu = p \cdot q$$
$$x = \frac{Q^2}{2\nu}$$
$$y = \frac{q \cdot p}{k \cdot p}$$



Deep Inelastic Scattering

Deep inelastic scattering (DIS) occurs when a lepton interacts with a hadron, traditionally we think about electron-proton scattering, but modern experiments also consider different possibilities.

DIS can be expressed in terms of the following variables:

$$Q^2 = -q^2$$

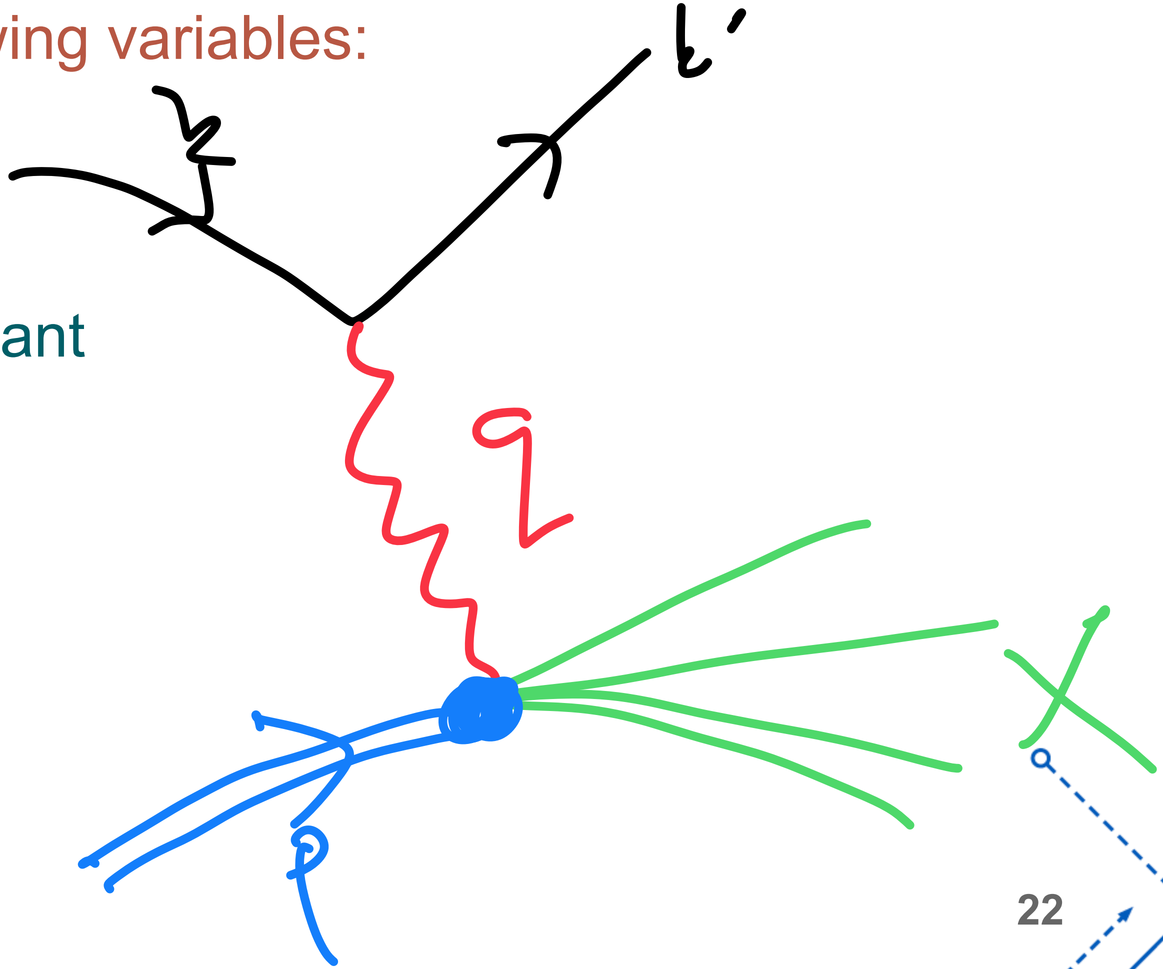
$$M^2 = p^2$$

$$\nu = p \cdot q$$

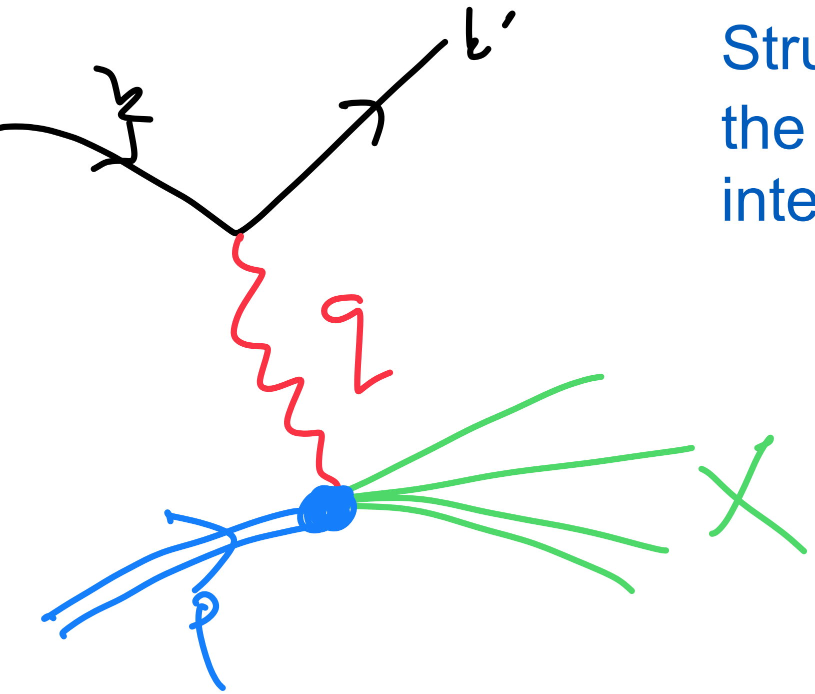
$$x = \frac{Q^2}{2\nu}$$

$$y = \frac{q \cdot p}{k \cdot p}$$

Most important variables

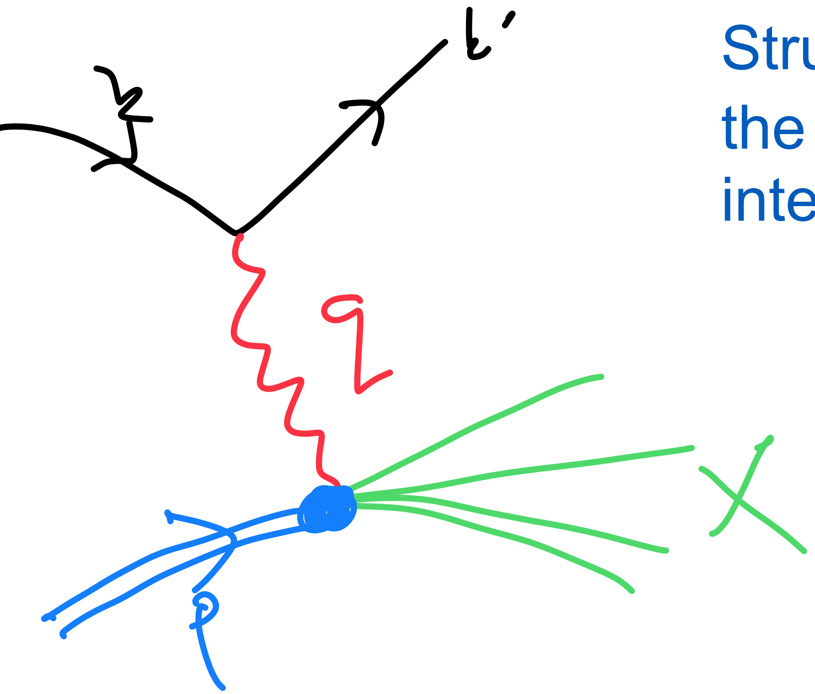


Structure functions



Structure Functions $F_i(x, Q^2)$ parameterize our ignorance about the detailed form of the proton, and instead formulate a general interaction which respects current conservation.

Structure functions



Structure Functions $F_i(x, Q^2)$ parameterize our ignorance about the detailed form of the proton, and instead formulate a general interaction which respects current conservation.

For instance, for charged lepton scattering $lp \to lX$ the cross section can be written as follows (this if for a EM current with $Q^2 < M_Z^2$)

$$\frac{d^2\sigma}{dxdy} = \frac{8\pi\alpha^2 ME}{Q^4}\left[\left(\frac{1+(1-y)^2}{2}\right)2xF_1^{em} + (1-y)(F_2^{em} - 2xF_1^{em}) - \frac{M}{2E}xyF_2^{em}\right]$$

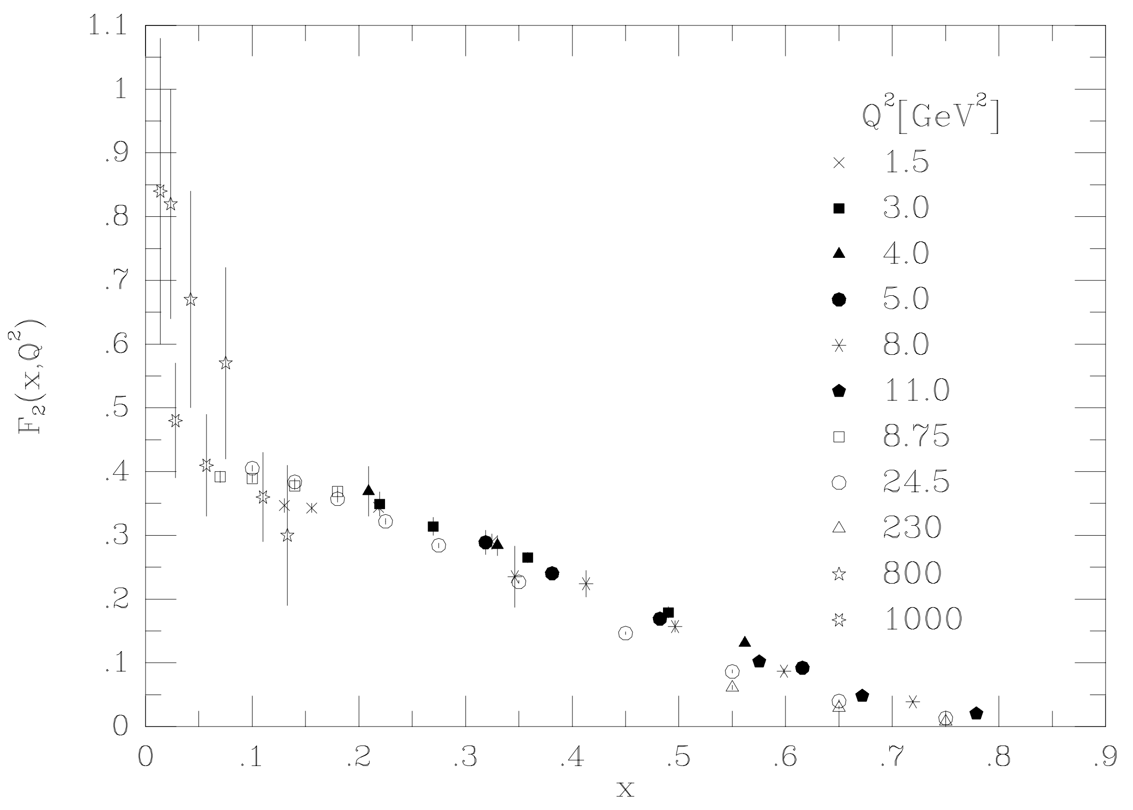
Bjorken Limit

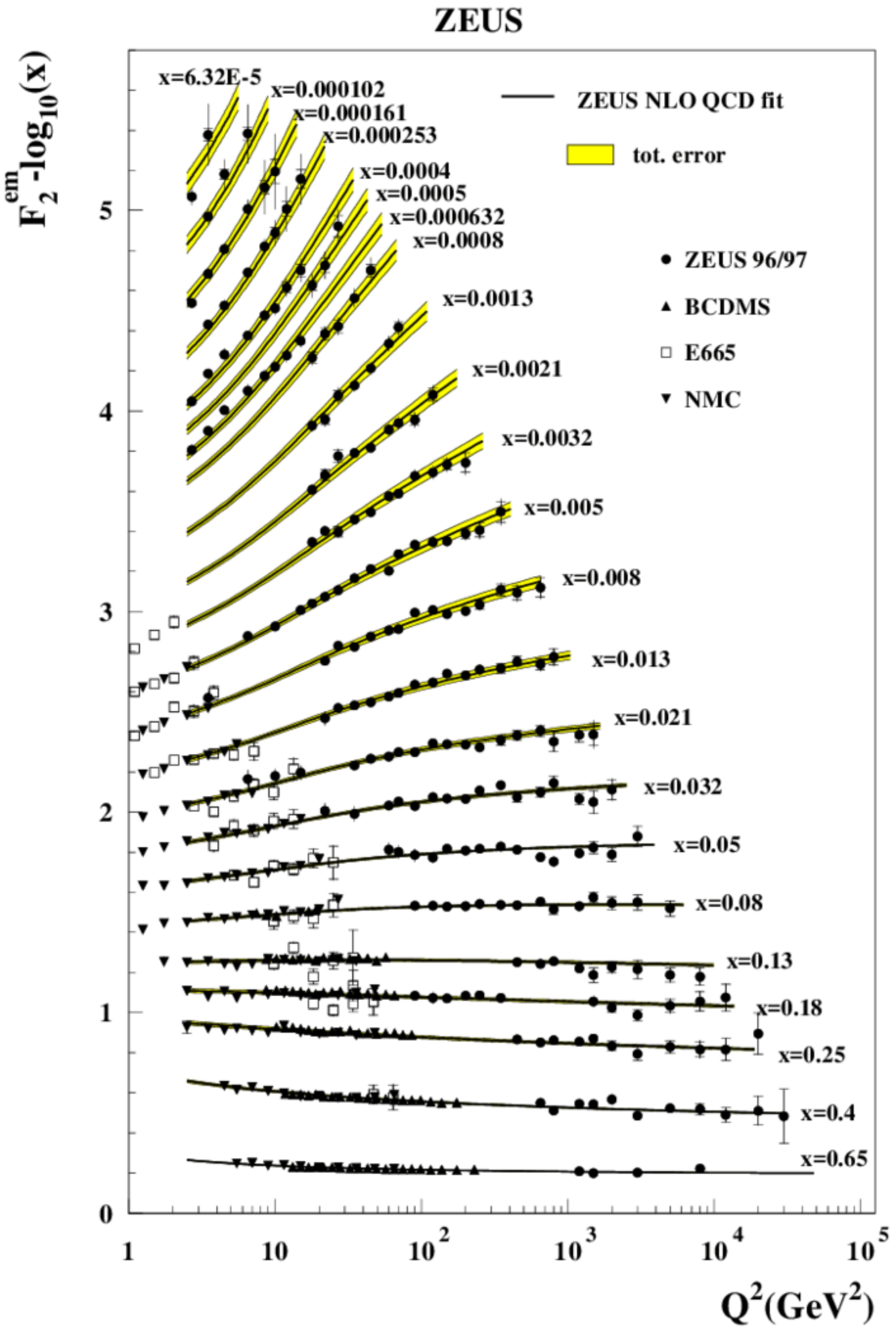
An interesting limit occurs when we take $Q^2, \nu \to \infty$ with x fixed. This is called the Bjorken limit. In this limit the structure functions are found to have an approximate scaling law, they depend on only the dimensionless variable x

$$F_i(x, Q^2) \to F_i(x)$$

This scaling is evidence for point like particles within the proton (i.e. there is no further scale Q_0 inducing a dependence on the ratio Q/Q_0).

Evidence for Bjorken Scaling





The Parton Model

Motivated by Bjorken scaling we can introduce a simple model of the proton which will form the foundation for our study with QCD

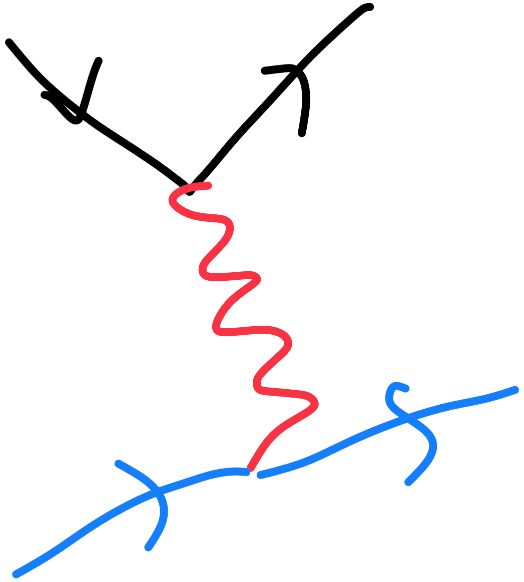
We begin taking an "infinite momentum frame" in which the proton is moving very fast $p^\mu \approx (P,0,0,P)$, with $P >> M$

The photon scatters off a pointlike quark which is moving parallel to the proton and carries a momentum fraction $p_q = \xi p$.

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 + (1-y)^2\right) F_1 + \frac{(1-y)}{x}(F_2 - 2xF_1)\right]$$

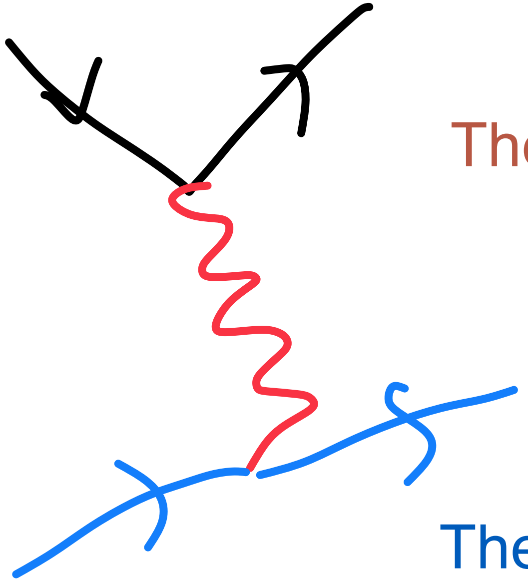
The Parton Model

Now we can compute this process at lowest order since its just an electron scattering off a quark $e^-(k) + q(p_q) \to e^-(k') + q(p_q')$.



The Parton Model

Now we can compute this process at lowest order since its just an electron scattering off a quark $e^-(k) + q(p_q) \to e^-(k') + q(p'_q)$.



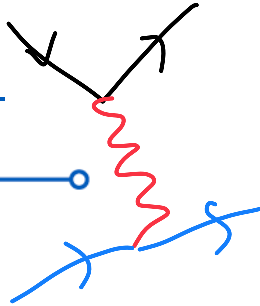
The spin averaged ME is of course a staple of intro QED lectures

$$\sum |\overline{\mathscr{M}}|^2 = 2e_q^2 \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$$

The hats remind us that we are talking about quark (not proton) momentum, i.e. $\hat{s} = (k + p_q)^2 = (k + \xi p)^2$.

The Parton model

$$\sum |\overline{\mathscr{M}}|^2 = 2e_q^2 \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$$



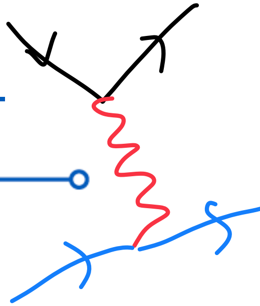
The differential cross section for 2→2 scattering is

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{1}{16\pi\hat{s}} \sum |\overline{\mathscr{M}}|^2$$

So our only job is to re-write the Mandelstam invariants in terms of the DIS variables

The Parton model

$$\sum |\overline{\mathscr{M}}|^2 = 2e_q^2 \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$$



The differential cross section for 2→2 scattering is

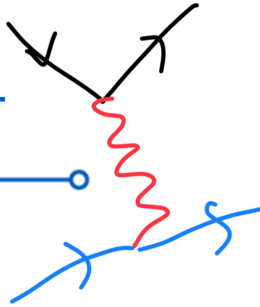
$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{1}{16\pi\hat{s}} \sum |\overline{\mathscr{M}}|^2$$

So our only job is to re-write the Mandelstam invariants in terms of the DIS variables

Namely $\hat{t} = -Q^2$, $\hat{u} = \hat{s}(y-1)$ and $\hat{s} = \xi Q^2/(xy)$.

The Parton model

$$\sum |\overline{\mathcal{M}}|^2 = 2e_q^2 \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}$$



The differential cross section for 2→2 scattering is

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{1}{16\pi\hat{s}} \sum |\overline{\mathcal{M}}|^2$$

So our only job is to re-write the Mandelstam invariants in terms of the DIS variables

Namely $\hat{t} = -Q^2$, $\hat{u} = \hat{s}(y-1)$ and $\hat{s} = \xi Q^2/(xy)$.

So that $\dfrac{d\hat{\sigma}}{dQ^2} = \dfrac{2\pi\alpha e_q^2}{Q^2}\left(1 + (1-y)^2\right)$

Doubly differential cross sections

$$\frac{d\hat{\sigma}}{dQ^2} = \frac{2\pi\alpha e_q^2}{Q^2}\left(1 + (1-y)^2\right)$$

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 + (1-y)^2\right)F_1 + \frac{(1-y)}{x}(F_2 - 2xF_1)\right]$$

Comparing our two formula we see that we are nearly there, but our expression in terms of form factors is doubly differential. We can write ours in this form using kinematics

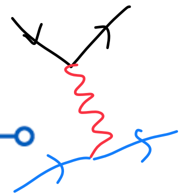
$$(p_q')^2 = (p_q + q)^2 = q^2 + 2p_q \cdot q = 0$$

So

$$-Q^2 + \xi 2\nu = 0 \implies \left(1 - \frac{\xi}{x}\right) = 0$$

And, $\xi = x$.

Doubly differential cross sections



Writing $\int_0^1 dx\delta(x-\xi) = 1$

$$\frac{d^2\sigma}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1+(1-y)^2\right)F_1 + \frac{(1-y)}{x}(F_2 - 2xF_1)\right]$$

We obtain $$\frac{d\hat{\sigma}}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^2}\left(1+(1-y)^2\right)\frac{e_q^2}{2}\delta(\xi - x)$$

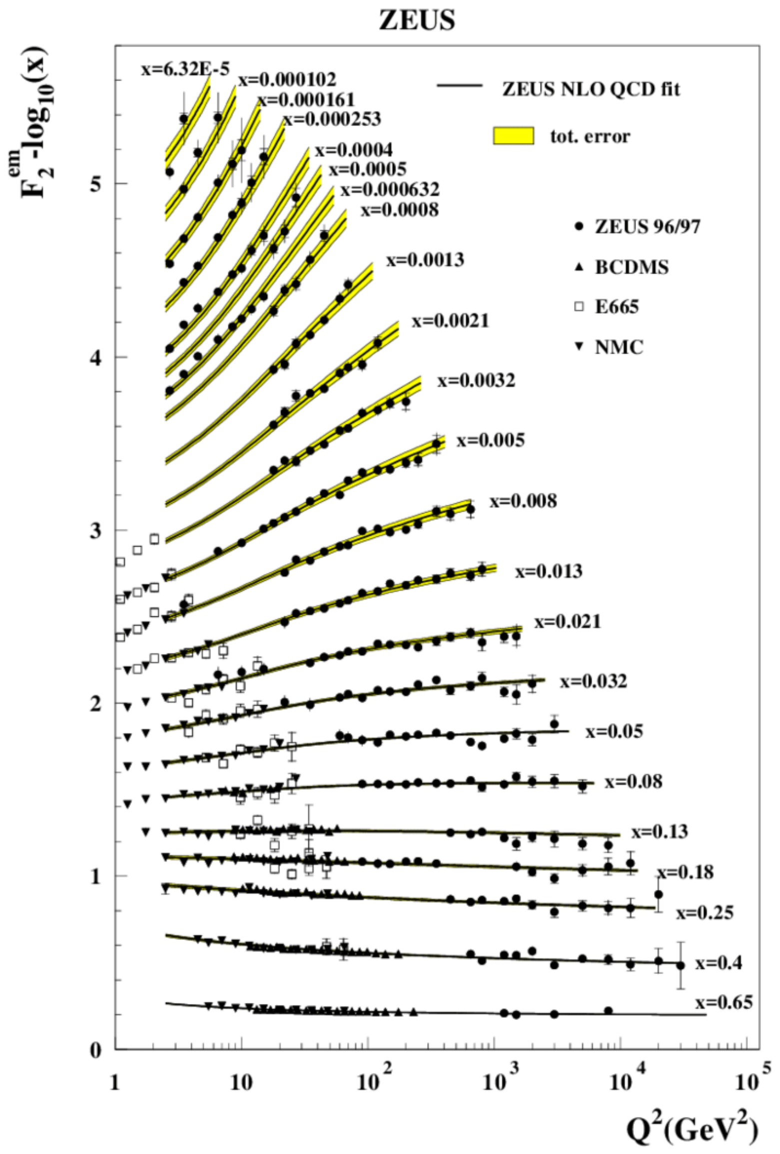
From which we can read off our structure functions

$$\hat{F}_2 = xe_q^2\delta(x-\xi) = 2x\hat{F}_1$$

This suggests that F_2 probes a quark constituent with momentum fraction $\xi = x$

$\hat{F}_2$

Recalling our plot of the structure function we saw that it was a distribution in x rather than a delta function. So quarks carry a range of momentum inside the proton.



Naive Parton Model

Putting this altogether we can define the naive Parton model

1) $q(\xi)d\xi$ represents the probability that a quark q carries momentum fraction ξ between ξ and $\xi + d\xi$
2) The virtual photon scatters incoherently off the quark constituents

The proton structure functions are then obtained by weighting the sum over all individual quark distributions

$$F_2(x) = 2xF_1(x) = \sum_{q,\bar{q}} \int_0^1 d\xi q(\xi) x e_q^2 \delta(x - \xi) = \sum_{q,\bar{q}} e_q^2 x q(x)$$

Naive Parton model

Cross sections for photonic, charged current and neutral current DIS are now written in terms of the unknown $q(x)$ functions, and given enough measurements these equations can be inverted to fit the probability distribution functions.

Naive Parton model

Cross sections for photonic, charged current and neutral current DIS are now written in terms of the unknown $q(x)$ functions, and given enough measurements these equations can be inverted to fit the probability distribution functions.

The simplest setup would be to use the quark model and to say that a proton consists of two up quarks and one down and have two distributions to fit $u(x)$ and $d(x)$.

Naive Parton model

Cross sections for photonic, charged current and neutral current DIS are now written in terms of the unknown $q(x)$ functions, and given enough measurements these equations can be inverted to fit the probability distribution functions.

The simplest setup would be to use the quark model and to say that a proton consists of two up quarks and one down and have two distributions to fit $u(x)$ and $d(x)$.

However, this is too simplistic to do a good job of describing real data, since we know about pair creation in field theories, a more realistic setup would be to define the proton as having three **valence** quarks and an infinite **sea** of light $q\bar{q}$ pairs.

Naive Parton model

Cross sections for photonic, charged current and neutral current DIS are now written in terms of the unknown $q(x)$ functions, and given enough measurements these equations can be inverted to fit the probability distribution functions.

The simplest setup would be to use the quark model and to say that a proton consists of two up quarks and one down and have two distributions to fit $u(x)$ and $d(x)$.

However, this is too simplistic to do a good job of describing real data, since we know about pair creation in field theories, a more realistic setup would be to define the proton as having three **valence** quarks and an infinite **sea** of light $q\bar{q}$ pairs.

When probed at a scale Q the sea contains all quark flavors with $m_q << Q$.

Valence and sea quarks in NPM

We define (at a scale $O(1 \text{ GeV})$)

$$u(x) = u_V(x) + S(x)$$
$$d(x) = d_V(x) + S(x)$$
$$S(x) = \bar{u}(x) = \bar{d}(x) = s(x) = \bar{s}(x)$$

We also impose the following sum rules

$$\int_0^1 dx\, u_V(x) = 2 \quad \text{and} \quad \int_0^1 dx\, d_V(x) = 1$$

Total momentum

It’s interesting to consider the following total

$$P_{\sum q} = \sum_q \int_0^1 dx\ x(q(x) + \overline{q}(x)) = \int_0^1 dx\ x(u_V(x) + d_V(x) + 6S(x))$$

Total momentum

It's interesting to consider the following total

$$P_{\sum q} = \sum_q \int_0^1 dx\ x(q(x) + \overline{q}(x)) = \int_0^1 dx\ x(u_V(x) + d_V(x) + 6S(x))$$

Experimentally we find $P_{\sum q} \approx 0.5$. So where is the missing momentum of the proton??

Total momentum

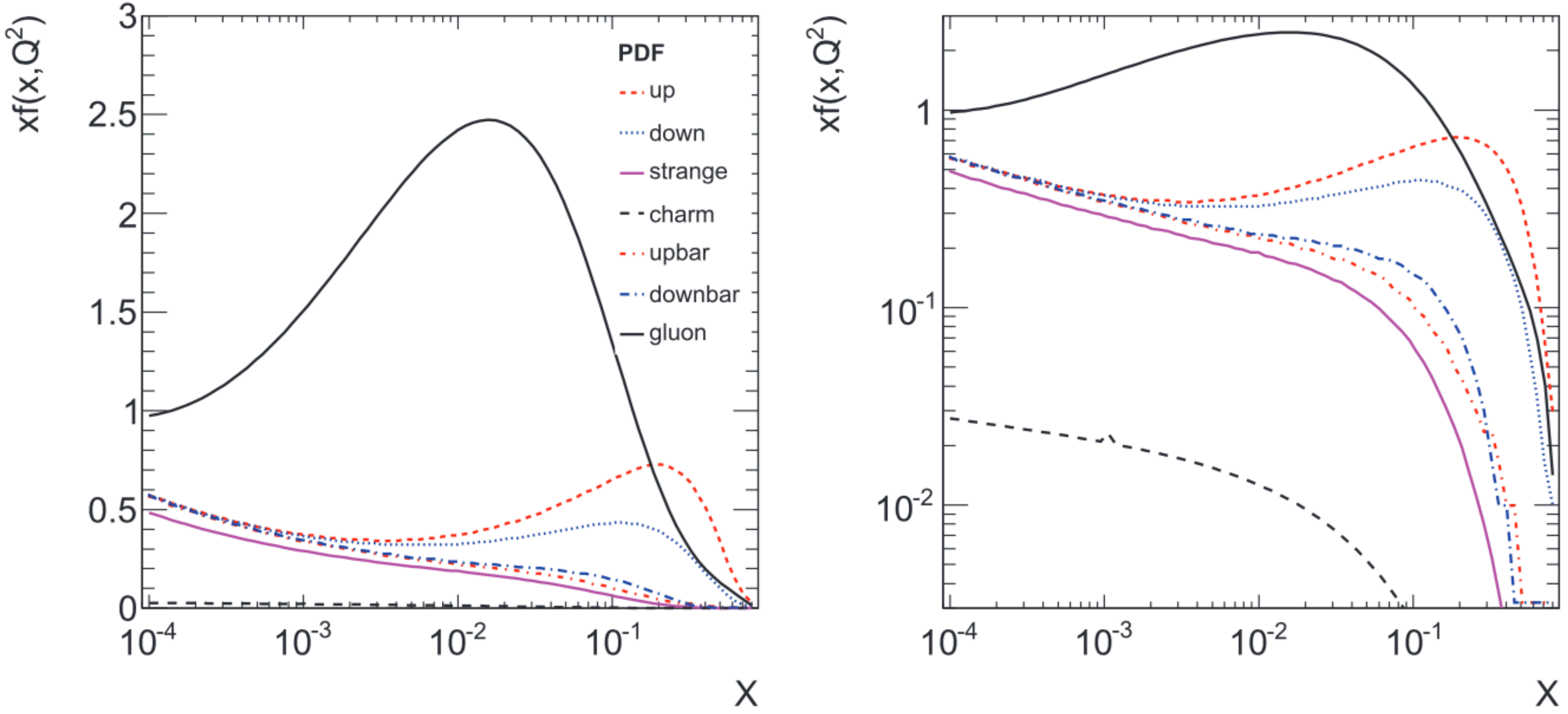
It's interesting to consider the following total

$$P_{\sum q} = \sum_q \int_0^1 dx\ x(q(x) + \overline{q}(x)) = \int_0^1 dx\ x(u_V(x) + d_V(x) + 6S(x))$$

Experimentally we find $P_{\sum q} \approx 0.5$. So where is the missing momentum of the proton??

The GLUON! And gluons carrying a whopping fraction of the total momentum of the proton.

A modern PDF set



Here's the CT14 PDF set (taken from the black book) , at $Q^2 = 2\text{GeV}^2$

University at Buffalo The State University of New York

UNDERSTANDING HADRON STRUCTURE

Parton Model from Field Theory

Hadronic tensor

Previously we calculated in the infinite momentum frame, however we can formulate the problem covariantly too. We introduce the hadronic tensor $W^{\mu\nu}$ which the matrix element squared is proportional to, i.e. $|\mathscr{M}|^2 \propto L_{\mu\nu}W^{\mu\nu}$

Here $L_{\mu\nu}$ is the leptonic tensor and can be calculated once and for all

$$L_{\mu\nu} = 4e^2(k_\mu k'_\nu + k_\mu k'_\nu - g_{\mu\nu}(k \cdot k'))$$

We can't write down such a nice form for the hadronic piece, but we can define a tensor which is consistent with EM current conservation $q \cdot W = 0$

$$W^{\mu\nu}(p,q) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) W_1(x, Q^2) + \left(p^\mu + \frac{1}{2x}q^\mu\right)\left(p^\nu + \frac{1}{2x}q^\nu\right) W_2(x, Q^2)$$

Hadronic tensor

$\nu = p \cdot q$

We can relate the hadronic tensor to our em structure functions defined in the last section

$$F_1(x, Q^2) = W_1(x, Q^2)$$
$$F_2(x, Q^2) = \nu W_2(x, Q^2)$$

Next we introduce the following vectors $p, n,$ and a two-dimensional transverse vector k_T defined such that

$$p^2 = n^2 = n \cdot k_T = p \cdot k_T = 0$$

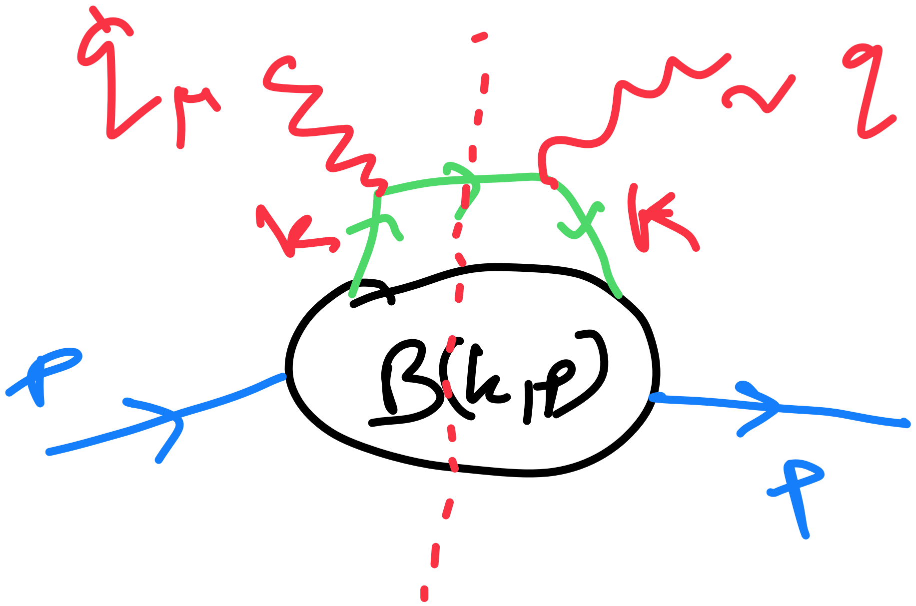
If we ignore the mass of the proton we can identify p as the incoming momentum of the target $p^\mu = (P,0,0,P)$

Then $n = 1/(2P)(1,0,0,-1)$

Hadronic tensor

Our vectors are useful since they project out structure functions,

$n_\mu n_\rho W^{\mu\rho} = \nu W_2 = F_2$ and $p_\mu p_\rho W^{\mu\rho} = \nu/(4x^2)(\nu W_2 - 2xW_1) = \nu/(4x^2)F_L$



$W^{\mu\nu}$ is then obtained from the above "handbag" diagram

$$W^{\mu\nu} = e_q^2 \int \frac{d^4k}{(2\pi)^4} \left[\gamma^\mu \gamma^\rho (k+q)_\rho \gamma^\nu\right]_{ij} \left[B(k,p)\right]_{ji} \delta((k+q)^2)$$

Hadronic tensor

$$W^{\mu\nu} = e_q^2 \int \frac{d^4k}{(2\pi)^4} \left[\gamma^\mu \gamma^\rho (k+q)_\rho \gamma^\nu\right]_{ij} \left[B(k,p)\right]_{ji} \delta((k+q)^2)$$

In terms of our basis vectors $$k^\mu = \xi p^\mu + \frac{k^2 + k_T^2}{2\xi} n^\mu + k_T^\mu$$

Such that the delta function becomes

$$\delta((k+q)^2) = \delta(k^2 + 2\xi\nu - 2q_T k_T + q^2)$$

Our Naive Parton model is recovered in the limit where the virtuality and transverse momentum tend to zero, i.e. $k^2 \to 0$ and $k_T \cdot q_T \to 0$ so that

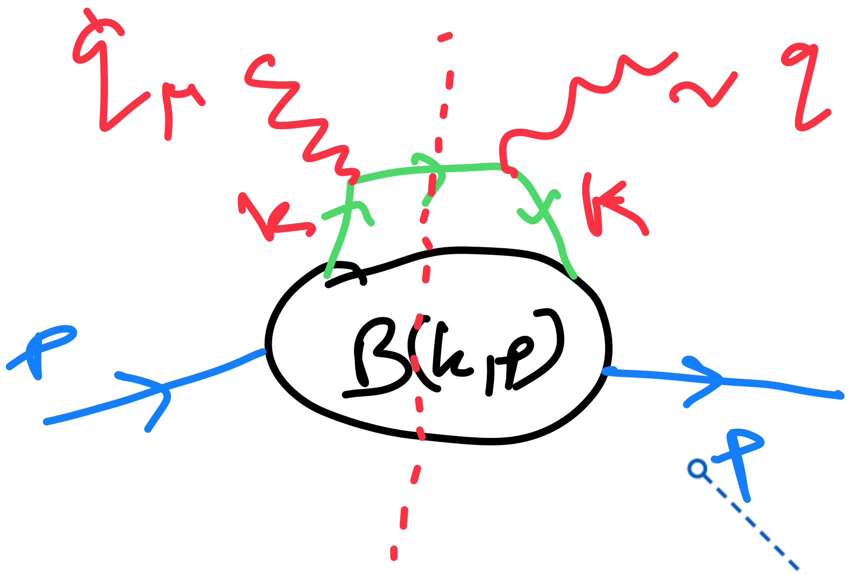
$$\delta((k+q)^2) \approx \delta(2\xi\nu - Q^2) = \frac{1}{2\nu}\delta(\xi - x)$$

Hadronic tensor in NPM limit

Our Naive Parton model is recovered in the limit where the virtuality and transverse momentum tend to zero, i.e. $k^2 \to 0$ and $k_T \cdot q_T \to 0$ so that

$$F_2 = \frac{e_q^2}{2} \int \frac{d^4k}{(2\pi)^4} n_\sigma n_\mu \left[\gamma^\mu \gamma^\rho (k+q)_\rho \gamma^\sigma\right]_{ij} \left[B(k,p)\right]_{ji} \delta(\xi - x)$$

$$F_2 = x e_q^2 \int \frac{d^4k}{(2\pi)^4} n_\sigma \left[\gamma^\sigma\right]_{ij} \left[B(k,p)\right]_{ji} \delta(\xi - x)$$
$$= e_q^2 x q(x)$$



Hadronic tensor in NPM limit

All we really have done so far is a more formal definition of our quark probability distribution function from a hadronic point of view.

$$q(x) = \int \frac{d^4k}{(2\pi)^4} n_\mu \mathrm{Tr}(\gamma^\mu B(k,p))\delta(\xi - x)$$

At this order we restore Bjorken scaling since the structure functions depend only a single variable x

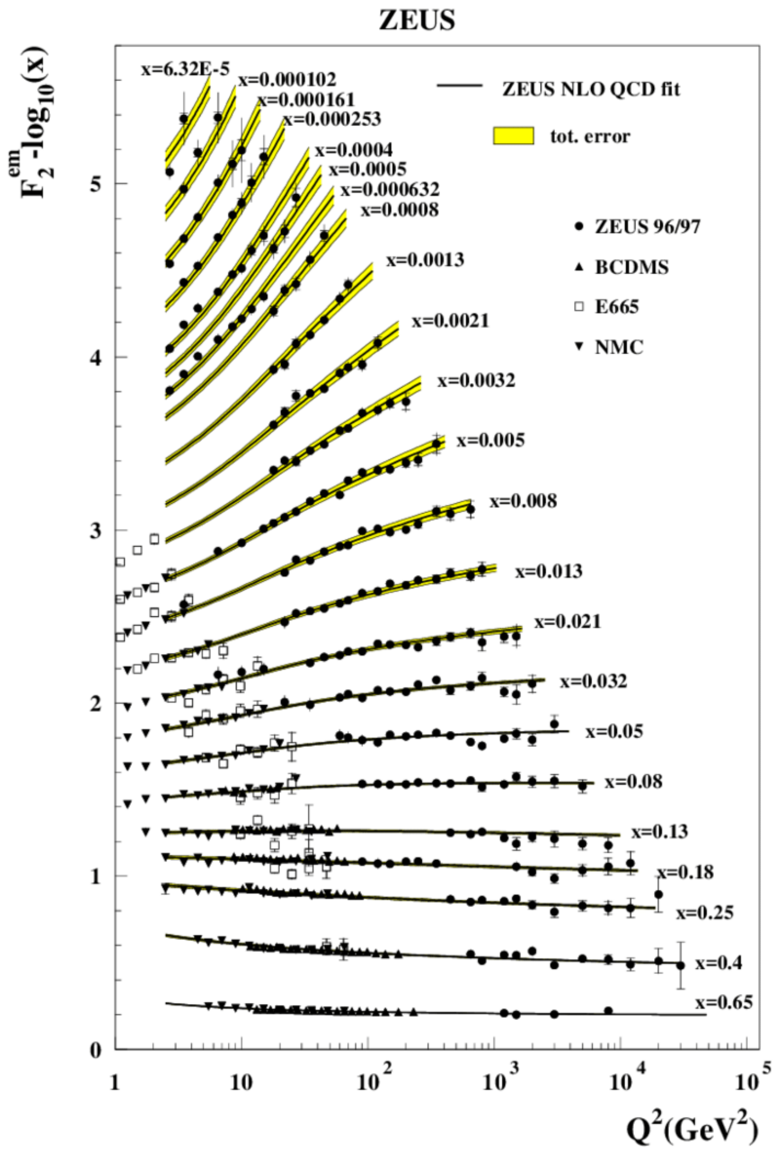
$$F_2(x, Q^2) \rightarrow F_2(x)$$

If we had calculated $F_L = F_2 - 2xF_1$ instead we would have found that $F_L \propto \nu$ which vanishes in the Bjorken limit as expected.

Departure from Bjorken scaling

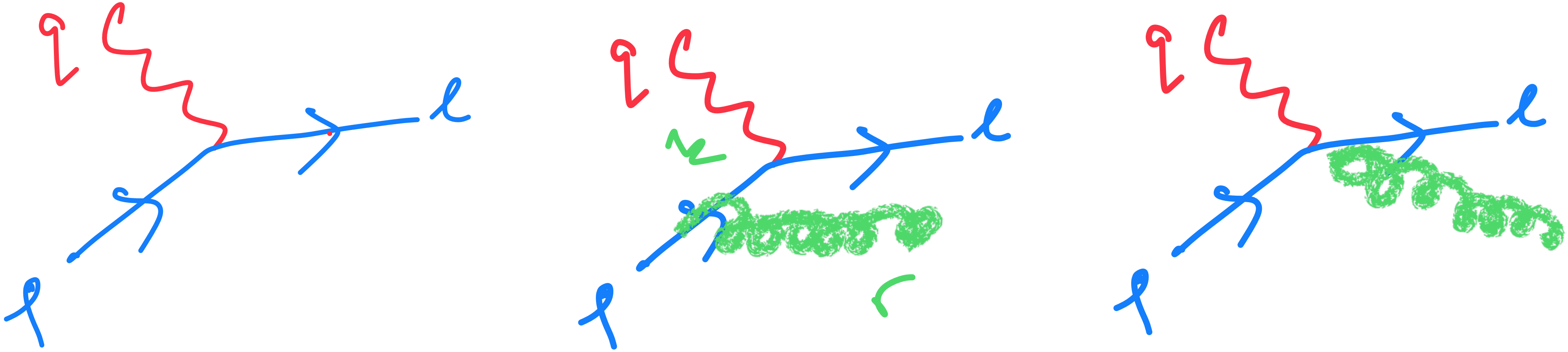
To say that Bjorken scaling holds generally would be a pretty weak statement!

While the larger-x behavior is flatter, as we go to smaller x its clear that there is a Q^2 dependence and scaling is violated.



QCD and the Parton model

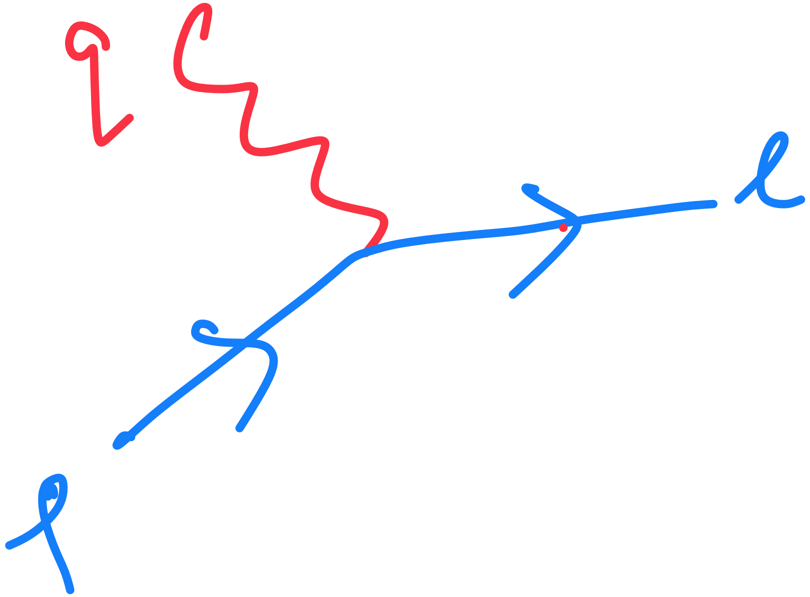
QCD breaks Bjorken scaling by allowing for emissions of gluons which **do not** have small transverse momentum k_T.



Let's think about the emission of a gluon in our diagrams

QCD and the Parton model

At LO this is the calculation we have already performed in our notation

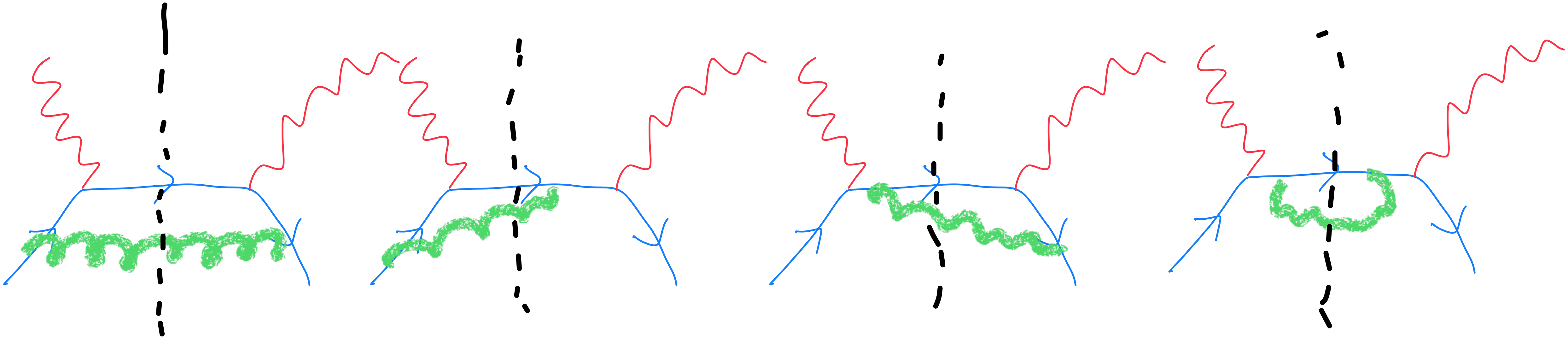


$$\hat{F}_2(x) = e_q^2 \delta(1 - x)$$

The hat reminds us that we are talking about the structure function for a quark rather than a proton and here $\xi = 1$.

The NLO handbag

There are 4 contributions when we square up the contributions with an emitted gluon.

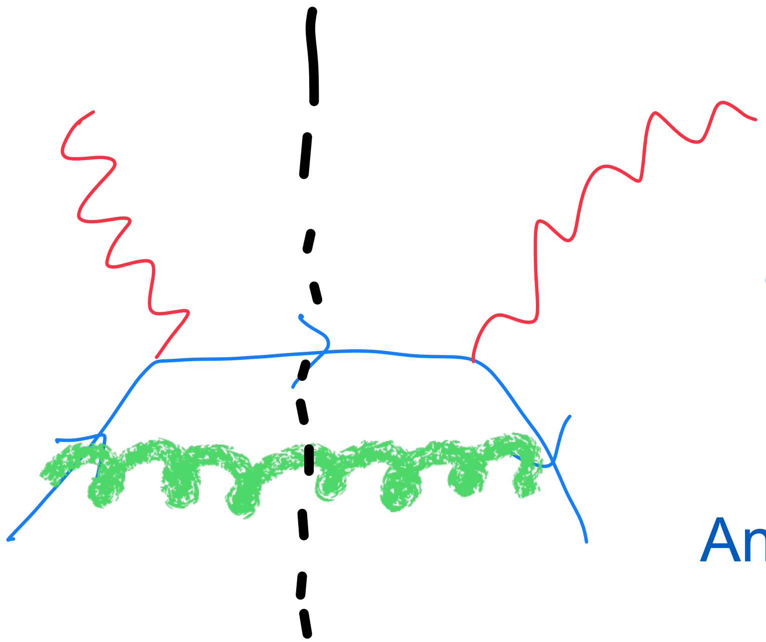


Focusing on the first diagram and writing $\gamma^*(q) + q(p) \to q(r) + g(l)$ the final state phase space is

$$d\Phi_2 = \int \frac{d^4 r}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \delta^+(r^2)\delta^+(l^2)(2\pi)^4 \delta^{(4)}(p + q - r - l)$$

The NLO handbag

$$d\Phi_2 = \int \frac{d^4 r}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \delta^+(r^2)\delta^+(l^2)(2\pi)^4 \delta^{(4)}(p+q-r-l)$$



Tidying up the phase space.

$$d\Phi_2 = \frac{1}{4\pi^2} \int d^4 k \delta^+((p-k)^2)\delta^+((k+q)^2)$$

And writing $k^\mu = \xi p^\mu + \frac{k_T^2 - |k^2|}{2\xi} n^\mu + k_T^\mu$

We can ultimately write

$$d\Phi_2 = \frac{1}{16\nu\pi^2} \int d\xi \int d(k^2) d(k_T^2) d\theta \delta(k_T^2 - (1-\xi)|k^2|)\delta\left(\xi - x - \frac{|k^2| + 2q_T \cdot k_T}{2\nu}\right)$$

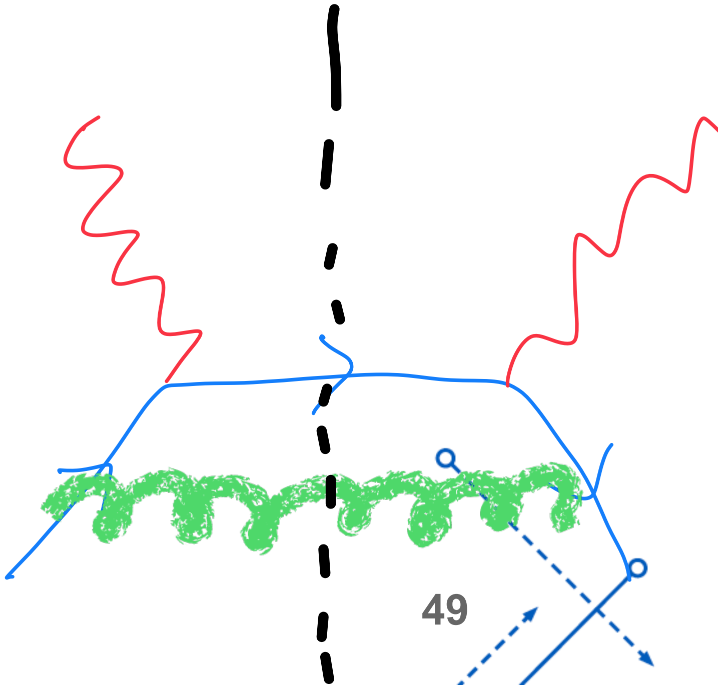
The NLO handbag

The matrix element can be calculated with the usual techniques. Projecting out F_2, and spin-averaging we have

$$\frac{1}{4\pi} n^\mu n^\rho \sum |\overline{\mathscr{M}}|^2_{\rho\sigma} = \frac{8 e_q^2 \alpha_S}{|k^2|} \xi P(\xi)$$

Where $P(\xi)$ is know as the **splitting function**

$$P(\xi) = C_F \frac{1+\xi^2}{1-\xi}$$



$$\frac{1}{4\pi} n^\mu n^\rho \sum |\overline{\mathcal{M}}|^2_{\rho\sigma} = \frac{8e_q^2\alpha_S}{|k^2|}\xi P(\xi)$$

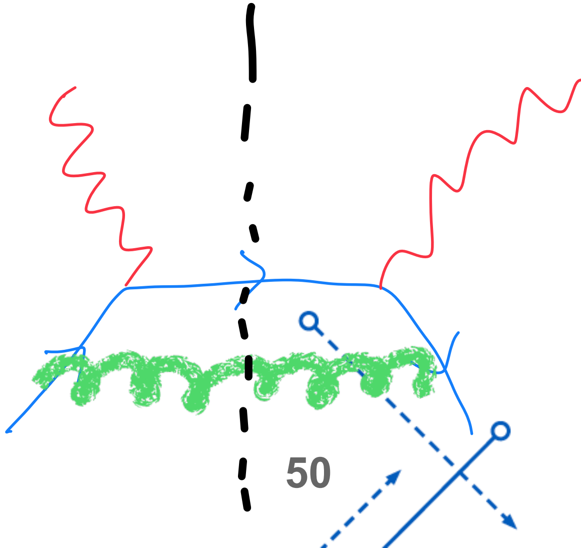
$$d\Phi_2 = \frac{1}{16\nu\pi^2}\int d\xi \int d(k^2)d(k_T^2)d\theta\delta(k_T^2 - (1-\xi)|k^2|)\delta\left(\xi - x - \frac{|k^2| + 2q_T\cdot k_T}{2\nu}\right)$$

Using the phase space delta functions to eliminate the k_T and θ integrations and integrating the spin-averaged matrix element over the phase space we get the following definition for $\hat{F}_2$

$$\hat{F}_2 = e_q^2 \frac{\alpha_S}{2\pi^2}\int_0^{2\nu}\frac{d|k^2|}{|k^2|}\int_{\xi_-}^{\xi_+} d\xi \frac{\xi P(\xi)}{\sqrt{(\xi_+ - \xi)(\xi - \xi_-)}}$$

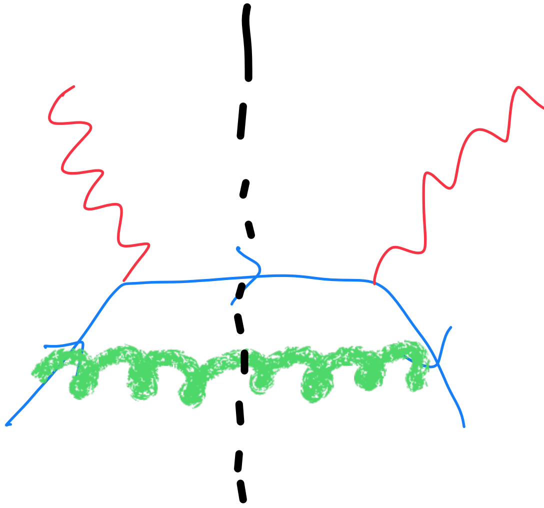
Where $\xi_\pm = x + z - 2x \pm \sqrt{4x(1-x)z(1-z)}$

And $z = |k^2|/(2\nu)$



Divergences in $\hat{F}_2$

$$\hat{F}_2 = e_q^2 \frac{\alpha_S}{2\pi^2} \int_0^{2\nu} \frac{d|k^2|}{|k^2|} \int_{\xi_-}^{\xi_+} d\xi \frac{\xi P(\xi)}{\sqrt{(\xi_+ - \xi)(\xi - \xi_-)}}$$



$\hat{F}_2$ is logarithmically divergent as $k^2 \to 0$, and needs regulation, for simplicity we'll use a cutoff κ^2.

So that the regulated integral is

$$\hat{F}_2 = e_q^2 \frac{\alpha_S}{2\pi^2} \int_{\kappa^2}^{2\nu} \frac{d|k^2|}{|k^2|} \int_{\xi_-}^{\xi_+} d\xi \frac{\xi P(\xi)}{\sqrt{(\xi_+ - \xi)(\xi - \xi_-)}}$$

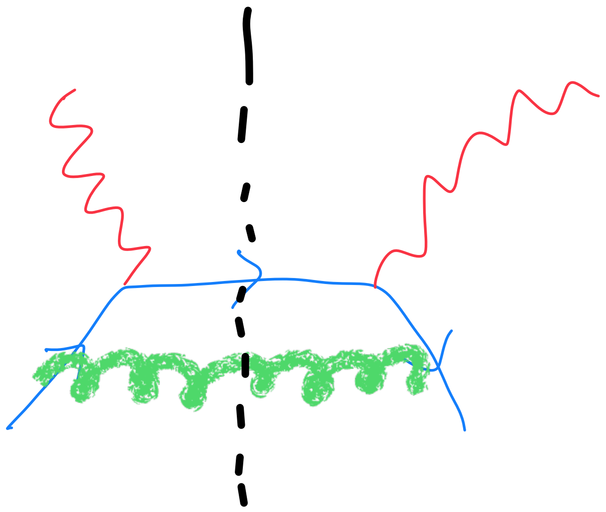
Divergences in $\hat{F}_2$

$$\hat{F}_2 = e_q^2 \frac{\alpha_S}{2\pi^2} \int_{\kappa^2}^{2\nu} \frac{d|k^2|}{|k^2|} \int_{\xi_-}^{\xi_+} d\xi \frac{\xi P(\xi)}{\sqrt{(\xi_+ - \xi)(\xi - \xi_-)}}$$

As $k^2 \to 0$ we have $z \to 0$ so $\xi_\pm = x + z - 2x \pm \sqrt{4x(1-x)z(1-z)} \;\; \to x$

Taylor expanding the numerator and using

$$\int_{\xi_-}^{\xi^+} \frac{d\xi}{\sqrt{(\xi_+ - \xi)(\xi_- - \xi)}} = \pi$$



We find the following logarithmically divergent pieces

$$\hat{F}_2|_{div} = e_q^2 \frac{\alpha_S}{2\pi} xP(x) \int_{\kappa^2}^{2\nu} \frac{d|k^2|}{|k^2|} = e_q^2 \frac{\alpha_S}{2\pi} xP(x) \; \ln\left(\frac{2\nu}{\kappa^2}\right)$$

(Almost) Total $\hat{F}_2$

So far we have looked at one contributing diagram out of four, but it turns out that only this diagram contributes a log in Q^2. We can therefore skip some details and write the total quark structure function as (noting $\ln 2\nu = \ln Q^2 - \ln x$)

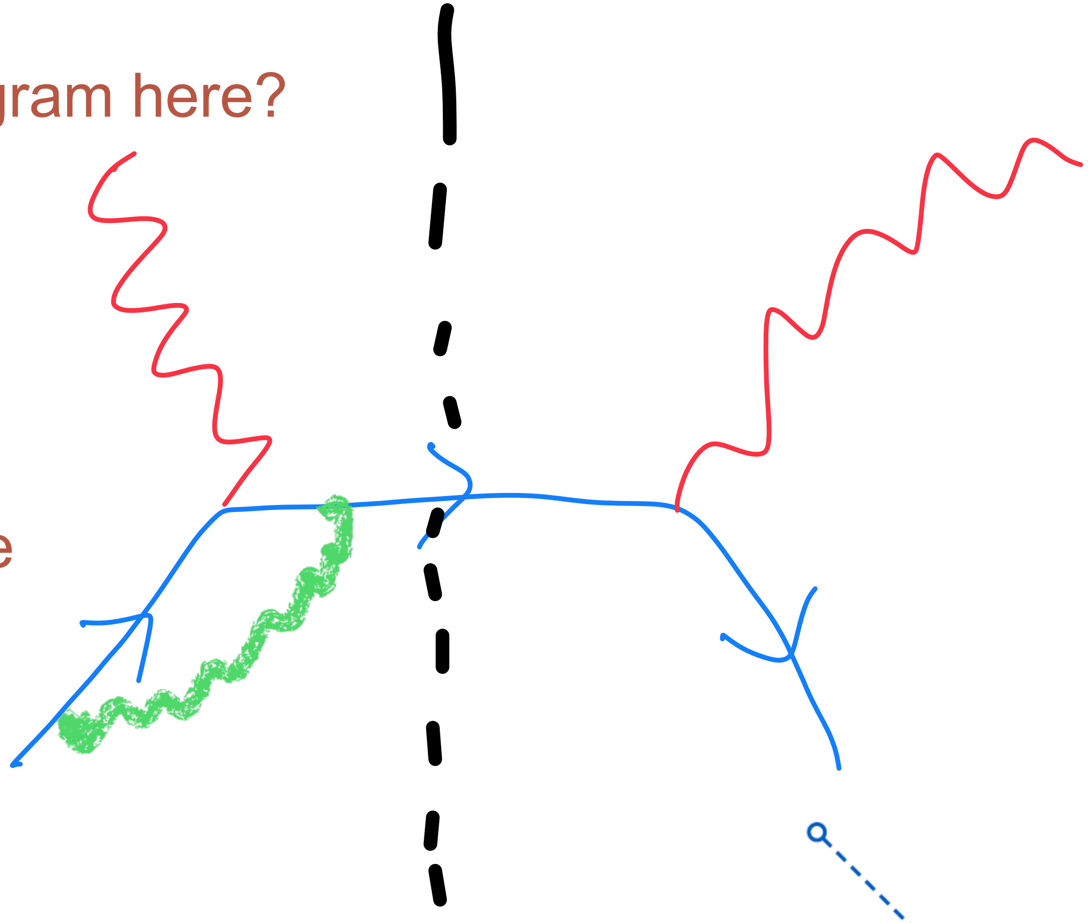
$$\hat{F}_2 = e_q^2 x \left[\delta(1-x) + \frac{\alpha_S}{2\pi} \left(P(x) \ln\left(\frac{Q^2}{\kappa^2}\right) + C(x) \right) \right]$$

Lots to unpack here, but the first and most important message is that **beyond leading order the structure function is Q^2 dependent**. And Bjorken scaling is broken by logarithms of Q^2

(Almost) Total $\hat{F}_2$

WAIT! Didn't we forget that there are virtual contributions too?

Yes ("we") did. What about corrections like the diagram here?



However, since the virtual corrections share the LO topology (phase space) they will contribute a term proportional to $\delta(1-x)$

They can be accommodated by making the change

$$P(x) \to P(x) + K\delta(1-x)$$

Fixing K

If we had a quiet afternoon we could calculate K, however we can actually get an answer even quicker by noting (or being told that) the integral over the total (modified) $P(x)$ should vanish (this ensures baryon number conservation if you're interested)

As a result we write

$$P(x) = C_F\left(\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right)$$

Were the plus distribution is defined as follows.

$$\int_0^1 dx \frac{f(x)}{(1-x)_+} = \int_0^1 dx \frac{f(x)-f(1)}{1-x}$$ And . $$\frac{1}{(1-x)_+} = \frac{1}{1-x}, \quad 0 \leq x < 1$$

(Actual) Total $\hat{F}_2$ and F_2

Using our modified splitting function the result from before is now accurate to α_S

$$\hat{F}_2 = e_q^2 x \left[\delta(1-x) + \frac{\alpha_S}{2\pi}\left(P(x)\ln\left(\frac{Q^2}{\kappa^2}\right) + C(x)\right)\right]$$

To go from quark to proton we convolute our quark structure function with the distribution of quarks in the proton $q_0(\xi)$

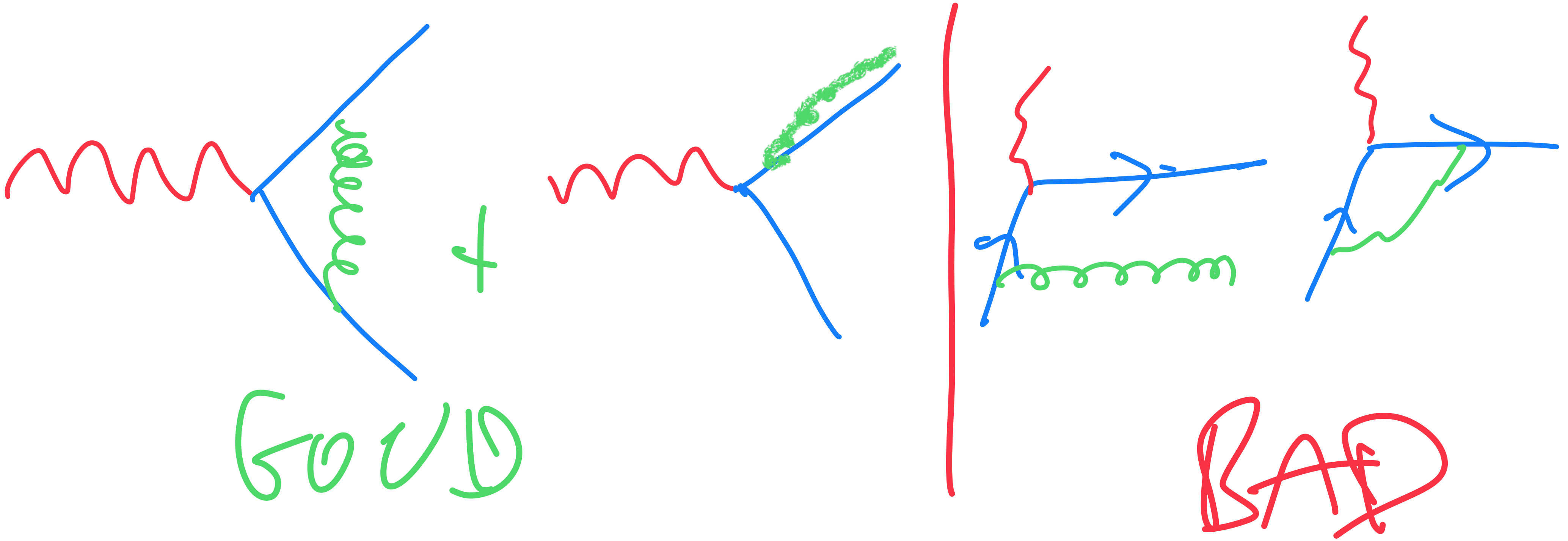
$$F_2(x, Q^2) = e_q^2 x \sum_{q,\bar{q}} \left[q_0(x) + \frac{\alpha_S}{2\pi}\int_x^1 \frac{d\xi}{\xi} q_0(\xi)\left(P\left(\frac{x}{\xi}\right)\ln\left(\frac{Q^2}{\kappa^2}\right) + C\left(\frac{x}{\xi}\right)\right)\right]$$

What about the whopping great singularity as we formally take $\kappa^2 \to 0$.
This singularity arises when the gluon is emitted collinearly to the quark.

Infrared safety

At this stage you might be fairly annoyed with me, or at least confused.

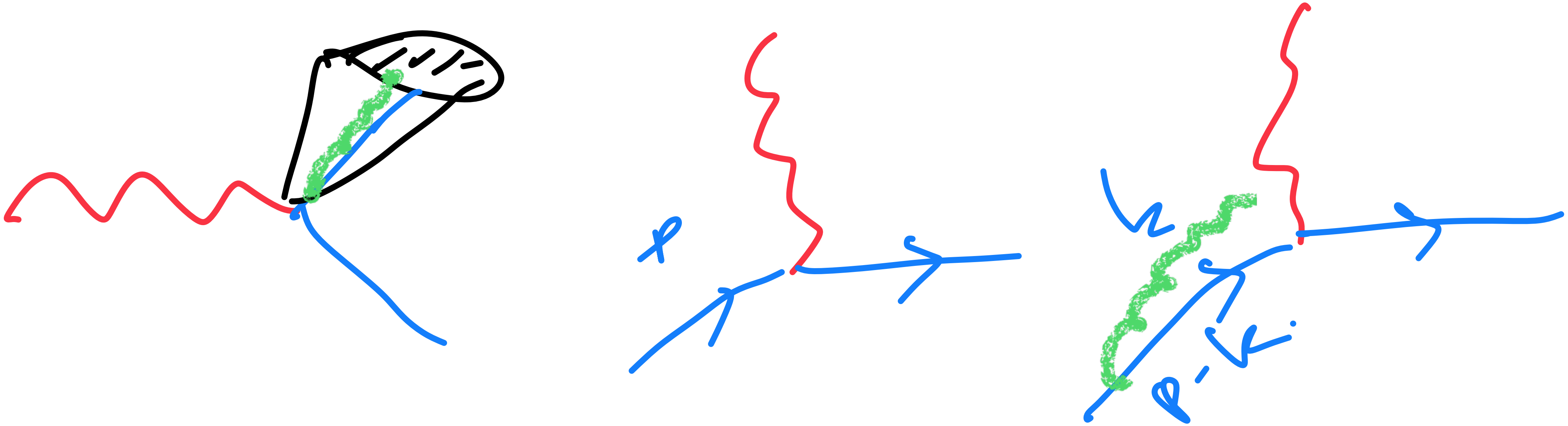
In the last section we saw how the process $\gamma^* \rightarrow 2$ jets, involved a cancellation between virtual and real contributions to render the whole thing finite. Why doesn't this happen here in this **very similar** process?



Infrared safety

The difference lies in the fact that we can cluster the final state in the former case into a jet. The jet has good properties and is safe in the IR limits. If we talked about final state observables associated with just the quark we would run back into IR issues.

In our calculation there is no clustering, the photon CAN distinguish the difference between a quark and a quark-gluon collinear pair with the same momentum.



Renormalization

$$F_2(x,Q^2) = e_q^2 x \sum_{q,\bar{q}} \left[q_0(x) + \frac{\alpha_S}{2\pi}\int_x^1 \frac{d\xi}{\xi} q_0(\xi) \left[P\left(\frac{x}{\xi}\right) \ln\left(\frac{Q^2}{\kappa^2}\right) + C\left(\frac{x}{\xi}\right) \right] \right]$$

We solve the issue by renormalization. In the same way as we think of the bare coupling as being an unmeasurable theoretical construct, we think about $q_0(x)$ as a bare distribution function. We define a renormalized distribution as follows,

$$q(x,\mu_F^2) = q_0(x) + \frac{\alpha_S}{2\pi}\int_x^1 \frac{d\xi}{\xi}\, q_0(\xi) \left[P\left(\frac{x}{\xi}\right) \ln\left(\frac{\mu_F^2}{\kappa^2}\right) + C\left(\frac{x}{\xi}\right) \right]$$

Then

$$F_2(x,Q^2) = e_q^2 x \sum_{q,\bar{q}} \left[q(x,\mu_F^2) + \frac{\alpha_S}{2\pi}\int_x^1 \frac{d\xi}{\xi} q(\xi,\mu_F^2)\, P\left(\frac{x}{\xi}\right) \ln\left(\frac{Q^2}{\mu_F^2}\right) + \mathcal{O}(\alpha_s^2) \right]$$

Factorization scale and scheme

The factorization scale μ_F plays a similar role to the renormalization scale μ_R, we can think of it as the scale at which the long-range IR physics is absorbed into the proton. Some things to note:

- The distribution $q(x, \mu_F)$ cannot be calculated from first principles in perturbation theory (although possibly with the lattice) since it contains non-perturbative long-range corrections.
- Effectively we have factorized the non-perturbative long-range physics and the short-range perturbative physics which depends on large momentum transfers. This type of factorization occurs beyond NLO and is vital for our ability to calculate cross sections for scattering processes.
- We chose to define our renormalized functions with the full finite piece $C(x/\xi)$ such that this was all absorbed into $q(x, \mu_F)$. This was a choice not a requirement. Like in UV physics this choice is referred to as a scheme.

Factorization scheme.

$$q(x,\mu_F^2) = q_0(x) + \frac{\alpha_S}{2\pi}\int_x^1 \frac{d\xi}{\xi}\, q_0(\xi)\left[P\left(\frac{x}{\xi}\right)\ln\left(\frac{\mu_F^2}{\kappa^2}\right) + C\left(\frac{x}{\xi}\right)\right]$$

The choice we made is referred to as the DIS scheme. A more common scheme for general collider physics is the $\overline{\mathrm{MS}}$ scheme in which only the divergent piece and a ubiquitous $\ln(4\pi) - \gamma_E$ is absorbed into $q_{\overline{MS}}(x,\mu_F)$

Once chosen, a scheme must be used in all cross sections, note that partonic cross sections will be different in different schemes.

Lecture 2 summary

We've covered a lot today.

- We saw how by defining a suitable jet algorithm we could meaningfully describe collider data with theoretical predictions. Jets must be defined in such a way as to preserve the delicate IR structure of a theory.
- Structure functions can be used to parametrize our ignorance of the composite nature of hadrons. They are constrained only by conservation laws of the theory and Lorentz structure.
- Bjorken scaling predicts that structure functions should be independent of Q^2 at fixed x (for large enough Q). We observe some regions in which scaling approximately holds, but in general scaling is violated.
- We understood scaling violation as arising from emissions of gluons in the full QCD theory.
- Initially we computed bare parton distribution functions, these require renormalization to render predictions finite. The scale at which this renormalization is performed is called the factorization scale, and sets the scale at which soft physics is absorbed into the proton.