Relativistic diffusion equation for light propagation in LArTPCs.

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- Their proposal is based on the explicit derivation of the equation for the 1d case and adjustments for the 3d case to have the correct limiting behavior.
- They got a kind of Telegrapher's equation for the photon density φ:

$$\Delta\varphi = \frac{\partial^2\varphi}{v^2\partial t^2} + \left(\frac{2}{\lambda_{abs}} + \frac{3}{\lambda^*_{rs}}\right)\frac{\partial\varphi}{v\partial t} + \frac{1}{\lambda_{abs}}\left(\frac{1}{\lambda_{abs}} + \frac{3}{\lambda^*_{rs}}\right)\varphi$$

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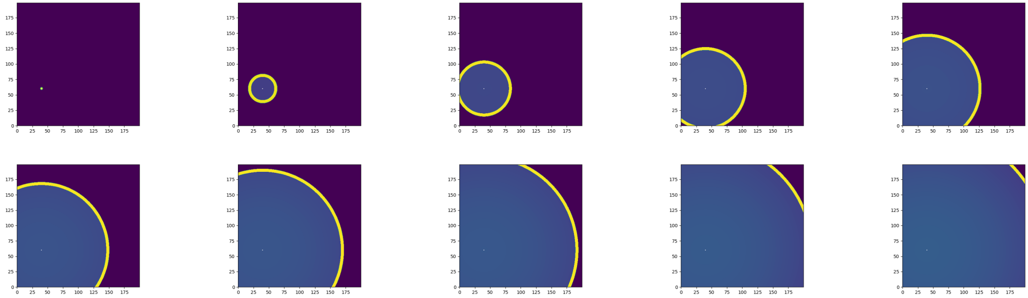
$$
\begin{aligned}
\varphi(t,\vec{x}) =& \frac{e^{\alpha t} e^{\gamma t}}{20\pi} \Bigg[(8 - 3e^{-\gamma t} + 2\gamma t + 4\gamma^2 t^2) \frac{\delta(vt - \|\vec{x} - \vec{x}_0\|)}{\|\vec{x} - \vec{x}_0\|^2} \\
&+ \frac{\gamma^2}{v} H(vt - \|\vec{x} - \vec{x}_0\|) \left(\frac{1}{v\sqrt{v^2t^2 - \|\vec{x} - \vec{x}_0\|^2}} I_1\left(\frac{\gamma\sqrt{v^2t^2 - \|\vec{x} - \vec{x}_0\|^2}}{\|\vec{x} - \vec{x}_0\|} \right) \right. \\
&\left. + \frac{4t}{v^2t^2 - \|\vec{x} - \vec{x}_0\|^2} I_2\left(\frac{\gamma\sqrt{v^2t^2 - \|\vec{x} - \vec{x}_0\|^2}}{\|\vec{x} - \vec{x}_0\|} \right) \right) \Bigg] \\
=& \varphi_{wave}(t,\vec{x}) + \varphi_{dif}(t,\vec{x}),
\end{aligned}
$$

where I_1 and I_2 are modified Bessel functions of the first kind.

Propagation of radiation

Photon density distribution inside the detector for times 0.1, 1.1, 2.1, 3.1, 4.1, 5.1, 6.1, 7.1, 8.1, 9.1 ns



Flow of photons

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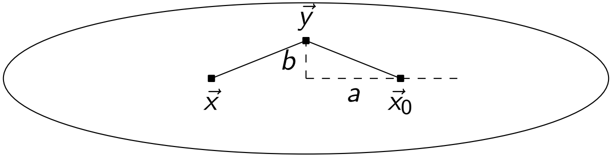
Let $t > t_0$ be some instant in time after the wave front passes $\vec{x}$. The photons reaching $\vec{x}$ at time t have all traveled the distance $vt > \|\vec{x} - \vec{x}_0\|$ and have been scattered at least once. These photons were emitted at $\vec{x}_0$, have traveled vt cm and reach $\vec{x}$ at time t. But the set of points $\vec{y}$ such that $\|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{x}_0\| < vt$ is an ellipsoid with foci $\vec{x}$ and $\vec{x}_0$. This means that all photons reaching $\vec{x}$ at time t suffered their **last scattering event (prior to reaching $\vec{x}$ at instant t)** inside this ellipsoid, which we call *causality ellipsoid*.



The causality ellipsoid defined by $\vec{x}$, $\vec{x}_0$ and $t > \|\vec{x} - \vec{x}_0\|/v$ has semi-major axis $a = vt/2$, semi-minor axis $b = \sqrt{v^2t^2 - \|\vec{x} - \vec{x}_0\|}/2$ and eccentricity $e = \|\vec{x} - \vec{x}_0\|/vt$.

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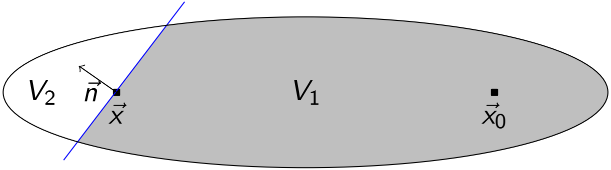
Suppose, now, that there is a plane wall W through $\vec{x}$. We consider the photon flux through the wall to be the **photon density times the average of the direction component pointing to $\vec{n}$ inside V_1.**



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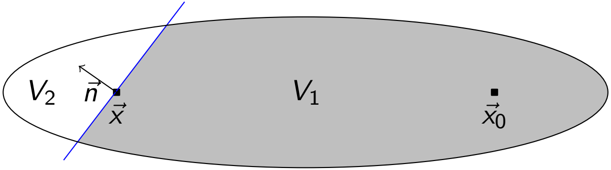
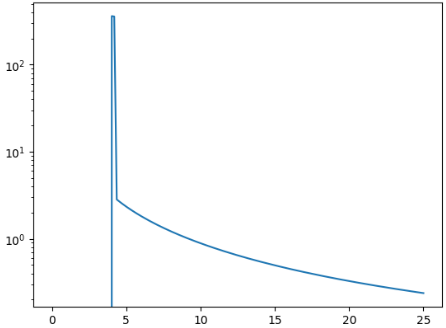
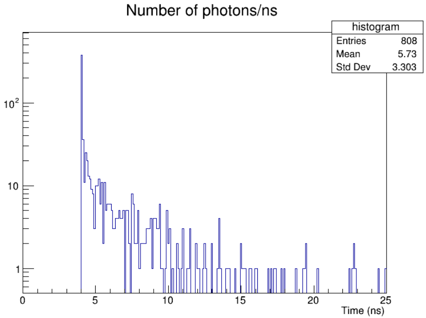


We then erase photons flowing from the outside to inside the detector by the analogous quantity on V_2.

Photon flux through a detector comparison

Telegrapher equation / Geant4 simulated data for photon flux through a SiPM at the wall of the detector for 100k photons pulse emitted at distance 86.8cm.





Total number of photons arriving at a detector

Telegrapher equation / Geant4 simulated data for total number of photons on photon counting square for sources in front of it at distances 10cm, ..., 600 cm