Computation of Eigen-Emittances
(and Optics Functions!)
from Tracking Data

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Issues

- How to find normal mode emittances (eigen-emittances) when optics functions are not known?

- Eigen-emittances as well as optics functions can be determined from covariance matrix.

- How to suppress halo contribution to covariance matrix in a self-consistent way to obtain emittances of the beam core?

- Iterative procedure for nonlinear fit of the particle distribution in the phase space with a Gaussian or other smooth function.

- Bonus point: how big the error can be when using mechanical momenta instead of canonical ones?

Definitions

Phase space vector:

$$\underline{z} = \{x, P_x, y, P_y, s - c\beta_0 t, \delta\}$$

Canonical momenta in units of the reference value $p_0 = mc\beta_0\gamma_0$:

$$P_x = (p_x + \frac{e}{c} A_x) / p_0$$

Energy deviation (disguised as momentum)

$$\delta = (\gamma - \gamma_0) / \beta_0^2 \gamma_0$$

Assume (for now) there is no tails and compute covariance matrix (Σ- matrix)

$$\Sigma_{i,j} = \frac{1}{N} \sum_{k=1}^{N} \zeta_i^{(k)} \zeta_j^{(k)}, \quad \zeta_i^{(k)} = z_i^{(k)} - \bar{z}_i, \quad \bar{z}_i = \frac{1}{N} \sum_{k=1}^{N} z_i^{(k)}, \quad i = 1, \ldots, 6$$

Basic assumption: particle distribution is a function of quadratic form

$$\Phi(\underline{\zeta}) = (\underline{\zeta}, \Sigma^{-1} \underline{\zeta}) \equiv \sum_{i=1}^{6} \zeta_i (\Sigma^{-1} \underline{\zeta})_i = \sum_{i,j=1}^{6} \Sigma_{ij}^{-1} \zeta_i \zeta_j$$

Eigen-Emittances from Σ- matrix

With Σ- matrix known, how to find the normal mode emittances?

- Σ- matrix has positive eigenvalues but they are useless unless the matrix of transformation to diagonal form is symplectic (generally not the case)
- solution suggested by theory developed by V.Lebedev & A.Bogacz :

Consider a product $\Omega = \mathrm{S}\Sigma^{-1}$ of inverse Σ- matrix and symplectic unity matrix

$$\mathrm{S} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

Matrix Ω has purely imaginary eigenvalues which are inverse eigen-emittances :

$$\lambda_{2m-1} = -\frac{i}{\varepsilon_m}, \quad \lambda_{2m} = \frac{i}{\varepsilon_m}, \quad m = 1, 2, 3$$

(All mathematics will be presented in a MAP note)

Eigen-Vectors of Matrix Ω

Using real and imaginary parts of eigen-vectors $\underline{v}_i^{/} \equiv \mathrm{Re}\,\underline{v}_i, \quad \underline{v}_i^{//} \equiv \mathrm{Im}\,\underline{v}_i$ as columns we can build a matrix:

$$\mathrm{V} = \{\underline{v}_1^{/}, -\underline{v}_1^{//}, \underline{v}_3^{/}, -\underline{v}_3^{//}, \underline{v}_5^{/}, -\underline{v}_5^{//}\}$$

which is symplectic, $V^t S V = S$, and brings Ω to diagonal form:

$$\mathrm{V}^{-1}\Omega\,\mathrm{V} = \mathrm{S}\,\Xi, \qquad \Xi = \mathrm{diag}\left(\frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_1}, \frac{1}{\varepsilon_2}, \frac{1}{\varepsilon_2}, \frac{1}{\varepsilon_3}, \frac{1}{\varepsilon_3}\right).$$

The quadratic form Φ takes the form:

$$\Phi = (\underline{\zeta}, \Sigma^{-1}\underline{\zeta}) \rightarrow (\underline{\xi}, \Xi\underline{\xi}) = \sum_{m=1}^{3} \frac{\xi_{2m-1}^2 + \xi_{2m}^2}{\varepsilon_m} = 2\sum_{m=1}^{3} \frac{J_m}{\varepsilon_m}, \qquad \underline{\xi} = \mathrm{V}^{-1}\underline{\zeta}$$

Eigen-vectors provide information on β- and dispersion functions :

$$\beta_{xm} = \left|(\underline{v}_{2m})_1\right|^2, \quad \beta_{ym} = \left|(\underline{v}_{2m})_3\right|^2, \quad \beta_{sm} = \left|(\underline{v}_{2m})_5\right|^2, \quad m = 1, 2, 3$$

$$D_x = \frac{x}{\delta} = \frac{V_{16}V_{55} - V_{15}V_{56}}{V_{66}V_{55} - V_{65}V_{56}}, \quad D_y = \frac{y}{\delta} = \frac{V_{36}V_{55} - V_{35}V_{56}}{V_{66}V_{55} - V_{65}V_{56}}.$$

Canonical vs Mechanical Momenta

Suppose that (in canonical variables) the distribution is such that:

$$<x^2>=<y^2>=\sigma^2, \quad <P_x^2>=<P_y^2>=\sigma_p^2, \quad \text{all correlations}=0$$

Now if we use mechanical momenta in solenoidal field ($K=B_z/2B\rho$):

$$<p_x y>=-<p_y x>=K\sigma^2, \quad <p_x^2>=<P_x^2>+K^2<y^2>=\sigma_p^2+K^2\sigma^2=<p_y^2>$$

and for eigen-emittances we obtain wrong values:

$$\varepsilon_{1,2}^2=\varepsilon_0^2[1+2K^2\beta_\perp^2 \pm 2|K|\beta_\perp\sqrt{1+K^2\beta_\perp^2}], \quad \varepsilon_0=\sigma_p\sigma, \quad \beta_\perp=\sigma/\sigma_p$$

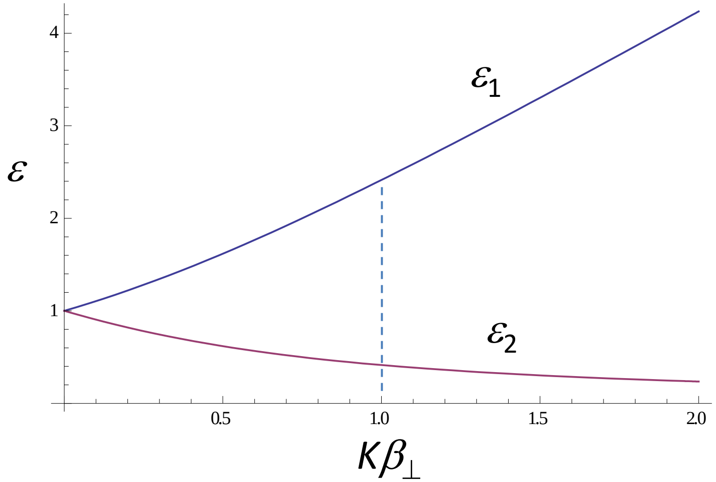
However, the 4D emittance remains correct:

$$\varepsilon_1\varepsilon_2=\varepsilon_0^2$$

Matched $\beta_\perp$ in a solenoid:

$$\beta_\perp=\frac{2B\rho}{B_z} \rightarrow K\beta_\perp=1$$

Use canonical momenta!



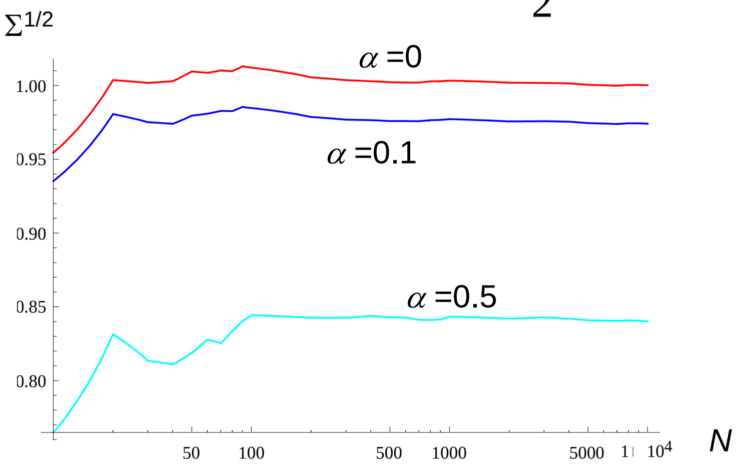
How to Suppress Halo Contribution?

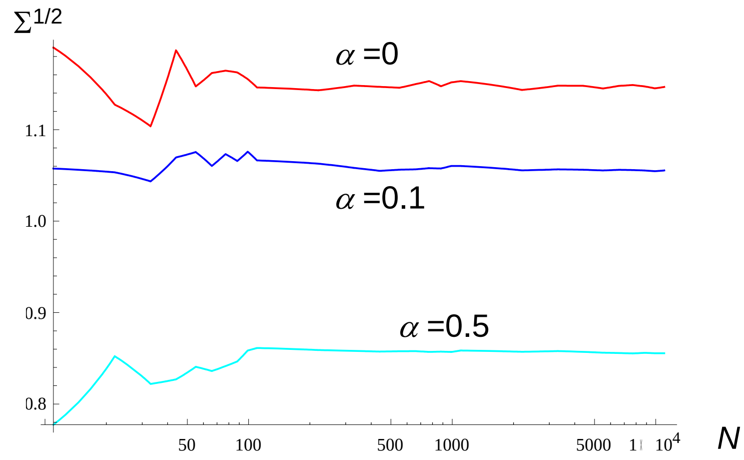
And to do this in a self-consistent way?

- a simple heuristic method is to introduce weights proportional to some degree of the distribution function. This leads to an iterative procedure

$$\bar{z}_i = \sum_{k=1}^{N} w_k z_i^{(k)} / \sum_{k=1}^{N} w_k, \quad \zeta_i^{(k)} = z_i^{(k)} - \bar{z}_i, \quad \Sigma_{i,j} = \sum_{k=1}^{N} w_k \zeta_i^{(k)} \zeta_j^{(k)} / \sum_{k=1}^{N} w_k, \qquad (1)$$

For Gaussian $w_k = \exp[-\frac{\alpha}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1}\underline{\zeta}^{(k)})]$, α being a fitting parameter (0<α<1)





Square root of Σ from eq.(1) averaged over 25 realizations of 1D Gaussian distribution with σ =1as function of the number of particles *N*.

Square root of Σ from eq.(1) averaged over 25 realizations of superposition of 1D Gaussian distributions with σ =1(90%) and σ =3(10%)

This method is imprecise and ambiguous ⇒ something based on a more solid foundation is needed.

Nonlinear Fit of the Klimontovich Distribution

$$G(\underline{z}) = \frac{1}{N}\sum_{k=1}^{N}\delta_{6D}(\underline{z}-\underline{z}^{(k)}) \equiv \frac{1}{N}\sum_{k=1}^{N}\prod_{i=1}^{6}\delta(z_i - z_i^{(k)})$$

We want to approximate it with a smooth function, e.g. Gaussian

$$F(\underline{\zeta}) = \frac{\eta}{(2\pi)^{n/2}\sqrt{\det\Sigma}}\exp[-\frac{1}{2}(\underline{\zeta},\Sigma^{-1}\underline{\zeta})]$$

where η is the fraction of particles in the beam core, via the minimization problem

$$\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}|F-G|^2\, dz_1\dots dz_n = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}(F^2-2FG)dz_1\dots dz_n + \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}G^2 dz_1\dots dz_n \to \min$$

or the maximization problem for the 1st term in the r.h.s. taken with the opposite sign

$$M(\bar{z},\Sigma,\eta) = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}(2FG-F^2)dz_1\dots dz_n =$$

$$\frac{\eta}{(2\pi)^{n/2}\sqrt{\det\Sigma}}\left\{\frac{2}{N}\sum_{k=1}^{N}\exp[-\frac{1}{2}(\underline{\zeta}^{(k)},\Sigma^{-1}\underline{\zeta}^{(k)})]-\frac{\eta}{2^{n/2}}\right\}\to \max$$

For n=6 there is n(n+3)/2+1=28 fitting parameters – convergence too slow

Rigorous Iterative Procedure

By differentiating $M(\bar{z},\Sigma,\eta)$ w.r.t. fitting parameters we recover equations which can be solved iteratively.

For average values of coordinates the equations coincide with heuristic ones with α=1

$$\bar{z}_i = \sum_{k=1}^{N} z_i^{(k)} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)},\Sigma^{-1}\underline{\zeta}^{(k)})] / \sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)},\Sigma^{-1}\underline{\zeta}^{(k)})], \quad \zeta_i^{(k)} = z_i^{(k)} - \bar{z}_i$$

$$(\frac{1}{N}\sum_{k=1}^{N}\ldots \to \sum_{k=1}^{N} w_k \ldots / \sum_{k=1}^{N} w_k$$

for weighted particles)

We can keep η fixed (i.e. set the fraction of particles taken into account)

Then for Σ- matrix we get

$$\Sigma_{ij} = \frac{1}{N}\sum_{k=1}^{N} \zeta_i^{(k)}\zeta_j^{(k)} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)},\Sigma^{-1}\underline{\zeta}^{(k)})] / \left(\frac{1}{N}\sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)},\Sigma^{-1}\underline{\zeta}^{(k)})] - \frac{\eta}{2^{n/2+1}} \right)$$

For $\eta \to 1$ some damping is necessary in *n*=6 case to avoid oscillations:

$$\Sigma^{(i)} = (1-d)\Sigma^{(i-1)} + d\Sigma^{(formula)}, \quad d \approx 0.8$$

(Again, mathematics will be presented in a MAP note)

Rigorous Iterative Procedure (cont'd)

We can try to find the optimal fraction of particles η for the fit.

From equation $\frac{d}{d\eta} M(\bar{z}, \Sigma, \eta) = 0$ we get

$$\eta = \frac{2^{n/2}}{N} \sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1}\underline{\zeta}^{(k)})], \quad \zeta_i^{(k)} = z_i^{(k)} - \bar{z}_i$$

Equations for average values of coordinates remain the same,

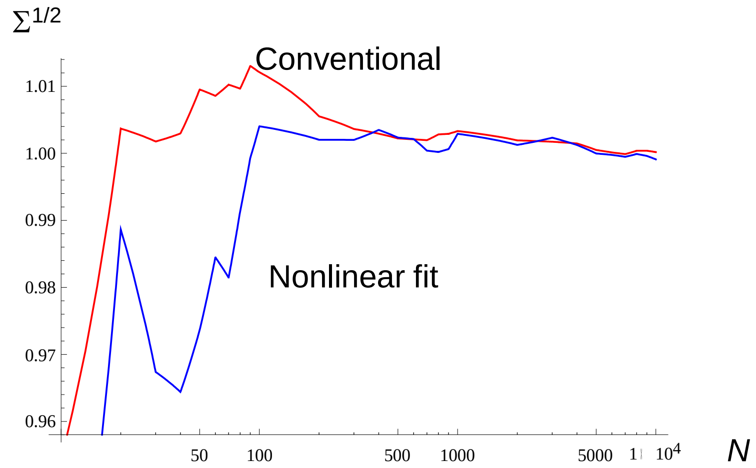
whereas for Σ- matrix we obtain expression with an extra factor of 2 (!) compared to the heuristic one

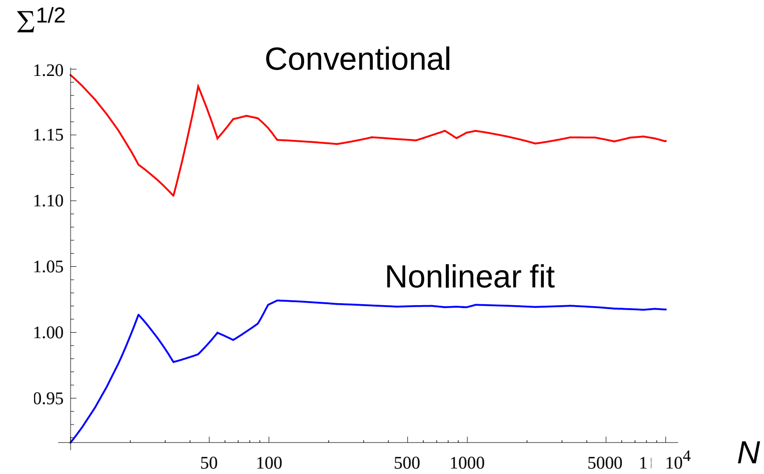
$$\Sigma_{ij} = 2\sum_{k=1}^{N} \zeta_i^{(k)} \zeta_j^{(k)} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1}\underline{\zeta}^{(k)})] / \sum_{k=1}^{N} \exp[-\frac{1}{2}(\underline{\zeta}^{(k)}, \Sigma^{-1}\underline{\zeta}^{(k)})]$$

Damping is not necessary in this case.

For n=6 in all cases just 20-30 iterations are required to achieve precision $\leq 10^{-6}$, it takes *Mathematica* ~13 seconds with N= 10^4 on my home PC. For a Fortran or C code it will be a fraction of a second.

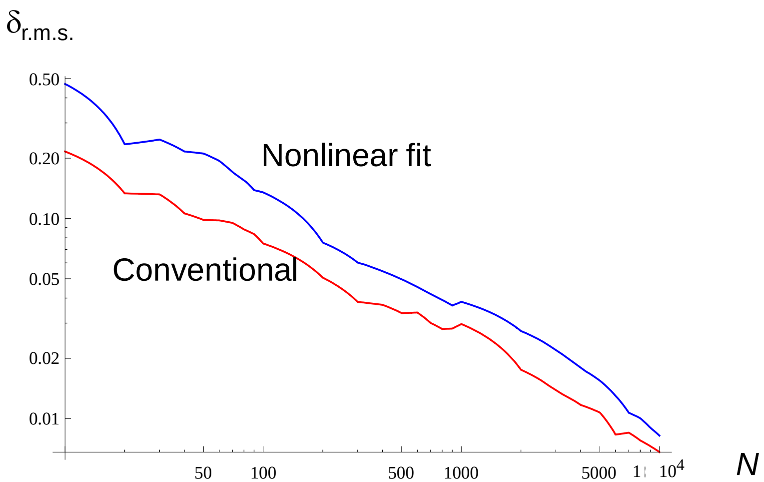
1D Precision Test

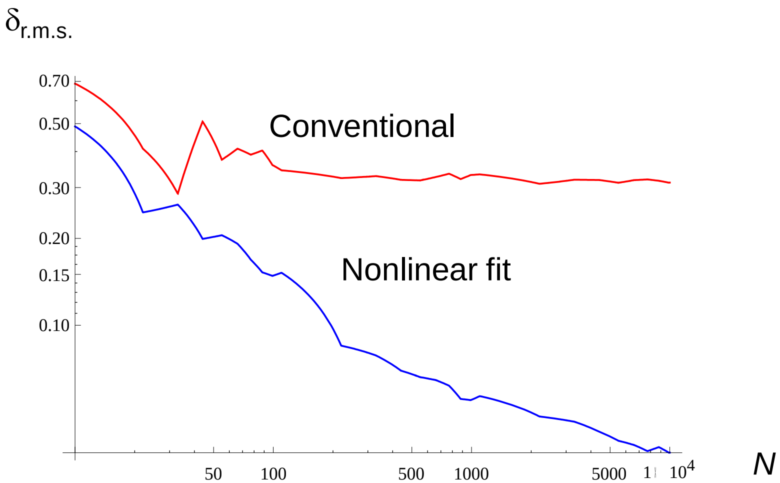




Square root of Σ averaged over 25 realizations of 1D Gaussian distribution with σ =1as function of the number of particles *N*.

Square root of Σ averaged over 25 realizations of superposition of 1D Gaussian distributions with σ =1(90%) and σ =3(10%)

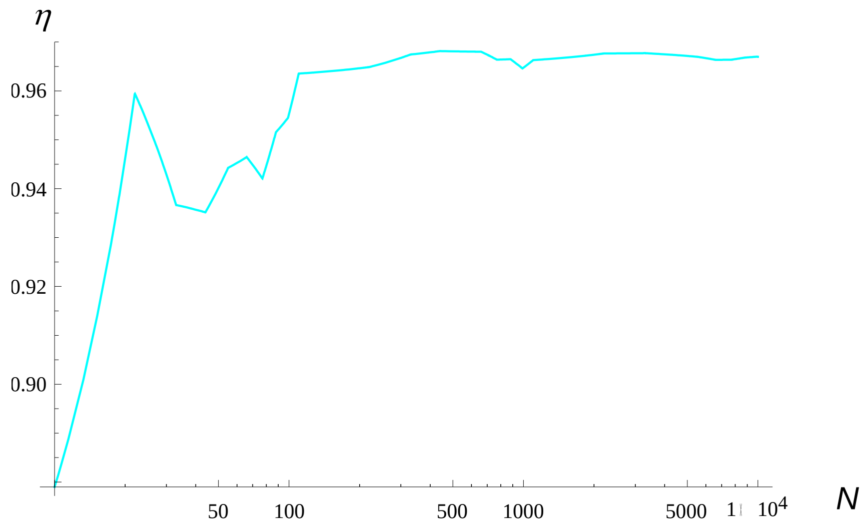




R.m.s. error in $\Sigma^{1/2}$ from above

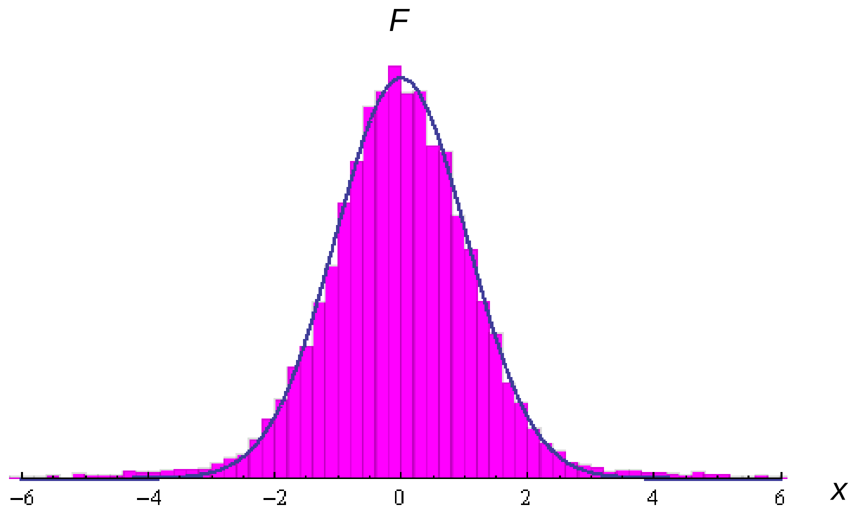
R.m.s. error in $\Sigma^{1/2}$ from above

1D Precision Test (cont'd)



Fraction of beam in the core averaged over 25 realizations of superposition of 1D Gaussian distributions with σ =1(90%) and σ =3(10%) .

With N =10^4 η =0.967: 2/3 of the σ =3 component were absorbed by the core and only 1/3 rejected.



Data histogram for one of realizations and fitted distribution function

The Algorithm

- Decide if the design trajectory (e.g. $\bar{z}_i = 0$) should be taken as the reference or the average coordinates should be computed along with covariance matrix.

- Compute average coordinates $\bar{z}_i$ (if needed), the covariance matrix Σ and (optionally) the optimal fraction of particles η in the same iterative process.

I would suggest to perform calculations with η =1 and η = η_{optimal}

- Find eigen-emittances = imaginary parts of eigenvalues of matrix $\Omega^{-1} = -\Sigma \mathrm{S}$

- Normalize eigenvectors $(\underline{v}^*_{2m-1}, \mathrm{S}\underline{v}_{2m-1}) = -2i$, m =1,2,3 being the mode #

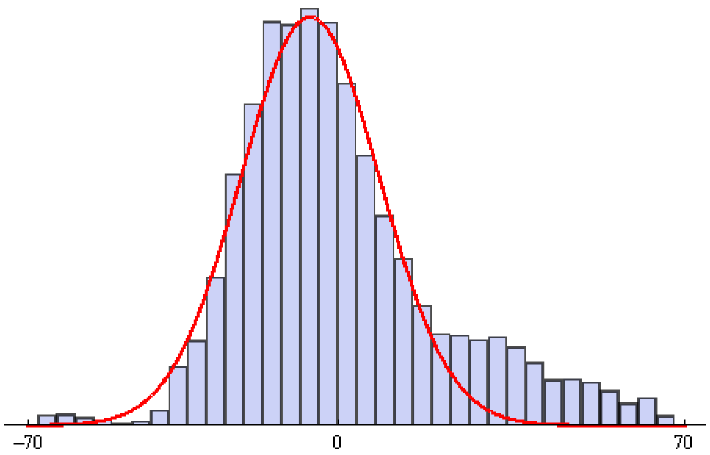
- To relate eigen-modes to the phase space planes compute and compare eigen-mode projections

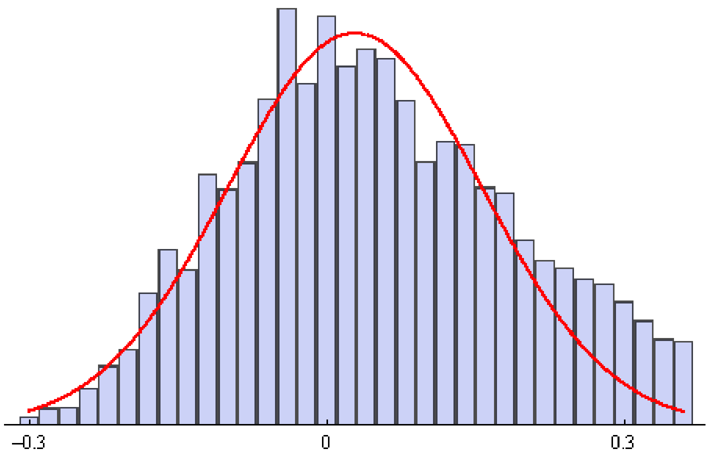
$$P(m \to p) = (\underline{v}'_{2m-1})_{2p-1}(\underline{v}''_{2m-1})_{2p} - (\underline{v}''_{2m-1})_{2p-1}(\underline{v}'_{2m-1})_{2p}$$

p =1,2,3 being the plane # (horz, vert, long)

Application to the Front End

μ+ longitudinal distributions right after the rotator (some old version by C.Y.):

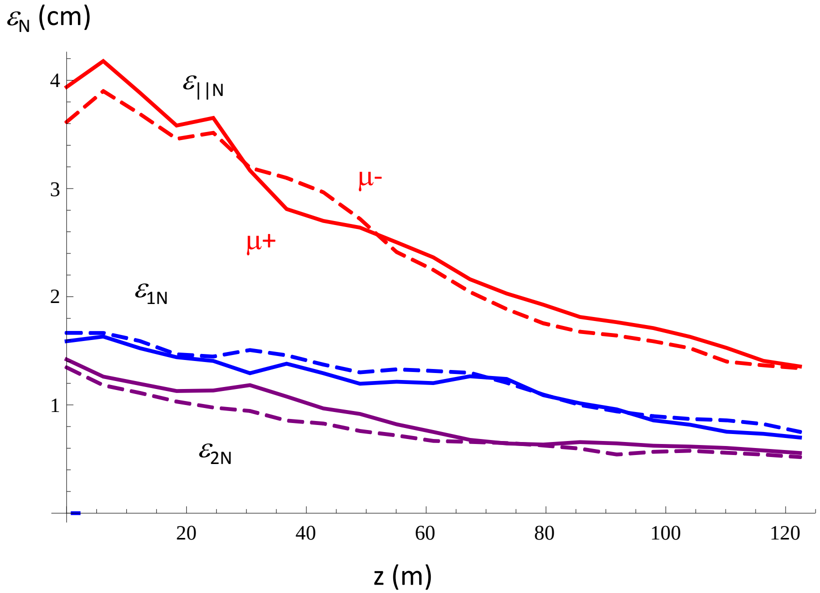




Red lines show projections of the fitted distribution for η =1.
– The long tails are obviously rejected even for η =1

η	$\varepsilon_{\|\|N}$ (cm)	ε_{1N} (cm)	ε_{2N} (cm)
1	3.94	1.59	1.42
η_{opt}=0.67	3.20	1.26	1.15

Application to HFOFO Snake

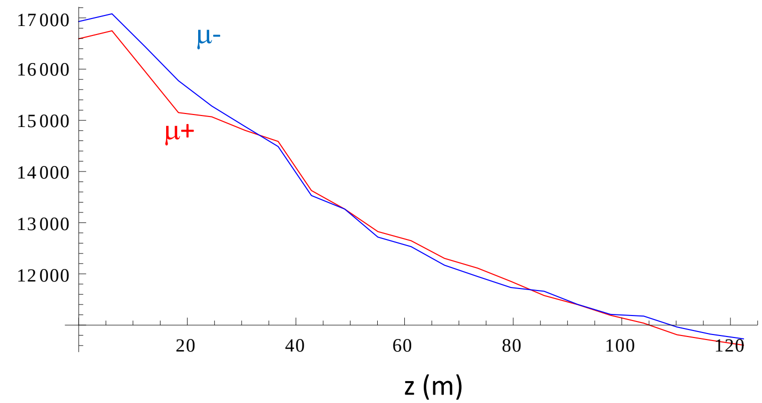


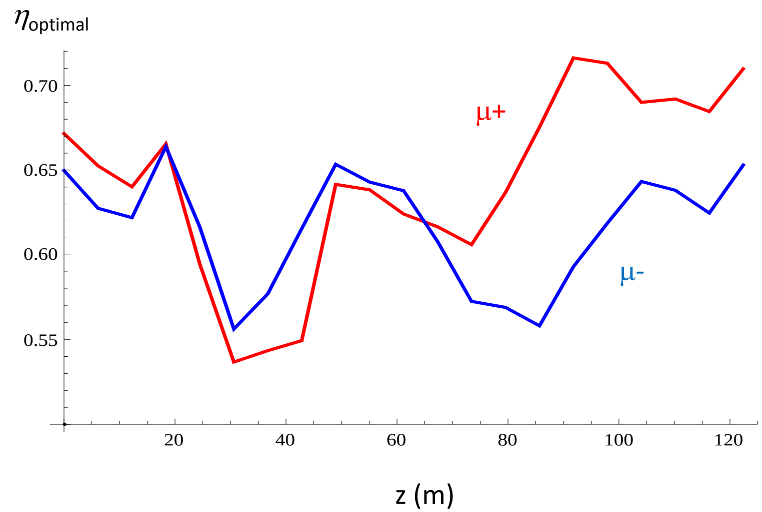
μ+ normalized emittances for η =1.

	$\varepsilon_{||N}$ (cm)	ε_{1N} (cm)	ε_{2N} (cm)
initial	3.94	1.59	1.42
final	1.36	0.70	0.56
ratio i/f	2.91	2.27	2.56

6D cooling factor = 16.88

N_μ in 150<p<300 MeV/c range





Summary

- The proposed algorithm is efficient and fast,
- I am ready to help with its implementation in G4BL and ICOOL.
- Performance of the HFOFO snake is really better than reported before (there was a mistake: *z* in m instead of cm)