Anomalous $\mathbb{Z}_N$ Gauge Theory and Oblique Topological Insulators

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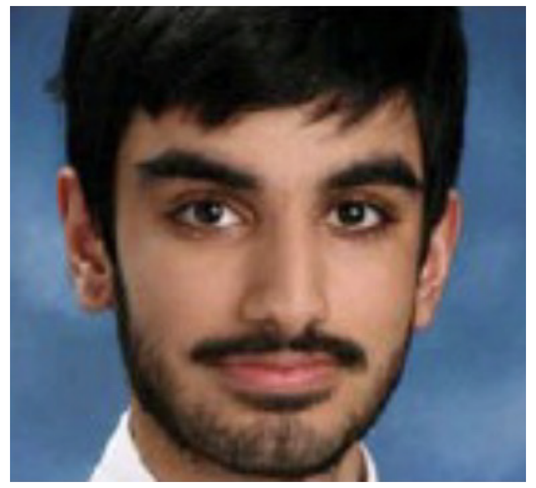

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B. Moy, H. Goldman, R. Sohal and E. Fradkin, SciPost 14, 023 (2023)

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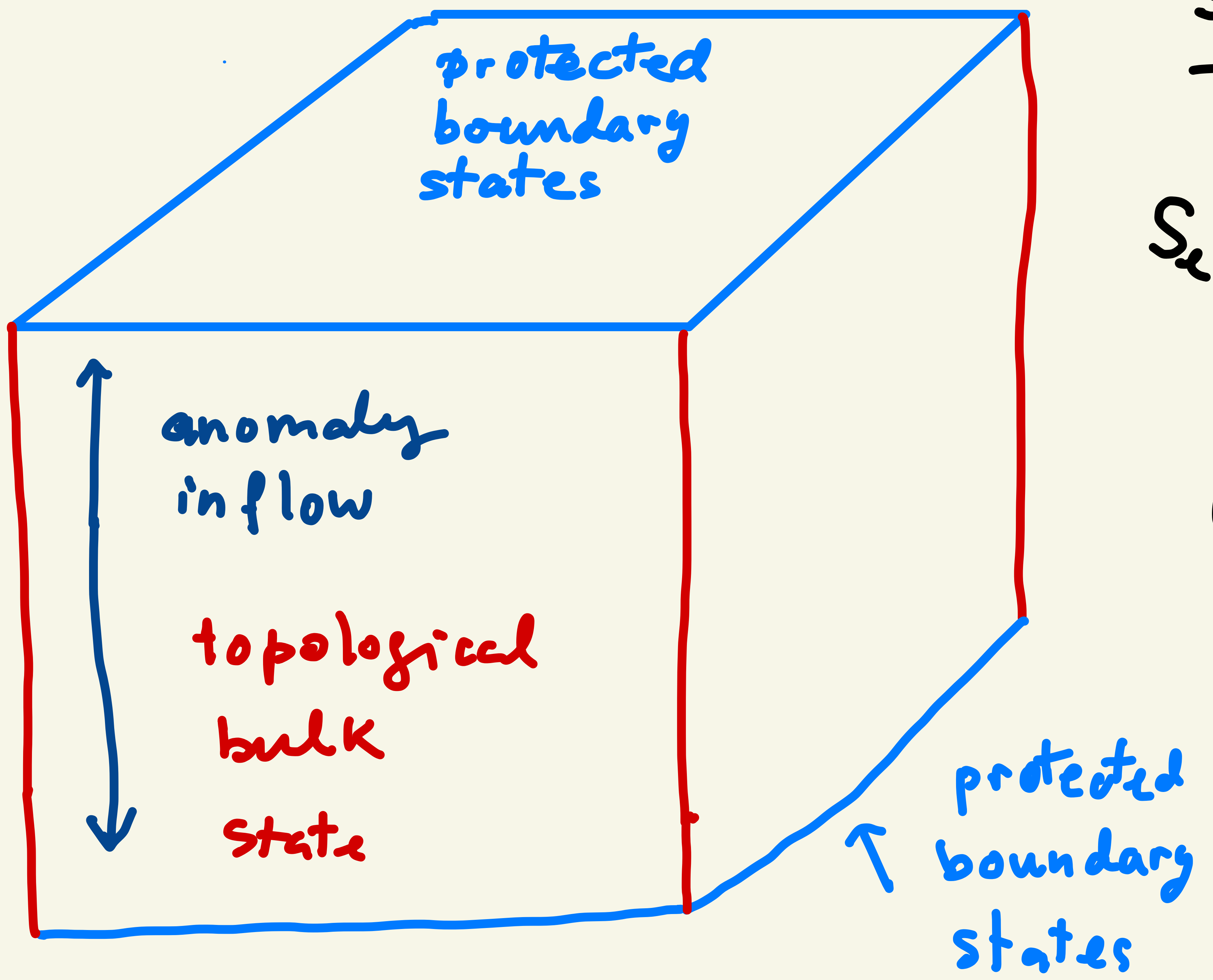
Topological Phases of Matter

- Gapped phases of matter with non trivial topology, typically are insulating
- Two types: *symmetry protected* (SPT) and *symmetry enhanced* (SET) topological phases
- SPTs: conventional bulk states with an *anomaly* with a *topologically invariant response* to a U(1) gauge field; example 3+1 D TIs, Bi_2Se_3
- SPTs have *boundary states* (protected by a global symmetry) that cannot exist on the heir own and are required to cancel an anomaly
- SETs: topological phases of matter with non-trivial topological order and *fractionalized excitations*
- Broken Time-Reversal SETs have massless boundary states (CFTs); example: fractional quantum Hall states of 2DEGs in large magnetic fields

Schematic Picture of a 3D Symmetry Protected Topological Insulator



<u>3+1 D TI</u> $\mathbb{Z}_2$

$$S_{eff} = \int_{M} d^4x \, \frac{\theta}{8\pi^2} F_{\mu\nu} F^{\mu\nu *}$$

$$\theta = \begin{cases} \pi & \text{topological} \\ 0 & \text{trivial} \end{cases}$$

$Bi_2 Se_3$

Free fermions : boundary states are Dirac (Weyl) fermions

This Talk

- Much is known in 2+1 dimensions (FQH states and their relatives)
- 3+1 D Topological Insulators: free fermion models, e.g. the theory of time-reversal invariant $\mathbb{Z}_2$ topological insulators, *without* bulk topological order
- Extensions to systems with bulk topological order are usually based on very special lattice models (e.g. Walker-Wang models) or using effective field theories derived from parton constructions.
- The bulk-edge correspondence of topological orders (and types) is actually poorly understood
- The universal responses in most cases are not known
- I will discuss a theory of fractionalized 3D TI's based on a $\mathbb{Z}_N$ lattice gauge theory with a rational θ angle which exhibits the phenomenon known as *oblique confinement*
- This theory generically has phases with both bulk and surface topological orders
- Understanding these questions turns out to be closely related to the problem of the role of higher-form global symmetries
- We constructed effective hydrodynamic TQFT's of these phases and of their gapped boundaries.

Schematic Picture

UV — Lattice Gauge Theory

Emergent one $-$ form
global symmetry G

Symmetry "breaking"
$G \mapsto G_{\text{oblique}}$

Topological Order :
$G_{\text{gauge}} = G/G_{\text{oblique}}$ — IR

$\mathbb{Z}_N$ gauge theories with a rational θ=2πp/q term

J. Cardy and E. Rabinovici, 1982

$$Z = \mathrm{Tr}\exp\Big[-\frac{1}{4g^2}\sum_{x,\mu\nu}F_{\mu\nu}^2 - i\frac{N\theta}{32\pi^2}\sum_{x,\tilde{x}}f(x-\tilde{x})\epsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} + iN\sum_{x,\mu}n_\mu a_\mu\Big]$$

$$F_{\mu\nu} = \Delta_\mu a_\nu - \Delta_\nu a_\mu + 2\pi s_{\mu\nu}, \qquad s_{\mu\nu}\in\mathbb{Z}$$

n_μ : current of bosons of a charge *N* field that breaks U(1) $\longrightarrow \mathbb{Z}_N$

$m_\mu = 1/2\, \varepsilon_{\mu\nu\lambda\rho}\, \Delta_\nu s_{\lambda\rho}$: current (worldlines) of magnetic monopoles

Periodicity $\quad a_\mu \to a_\mu + 2\pi p_\mu, \qquad s_{\mu\nu} \to s_{\mu\nu} + \Delta_\mu p_\nu - \Delta_\nu p_\mu$

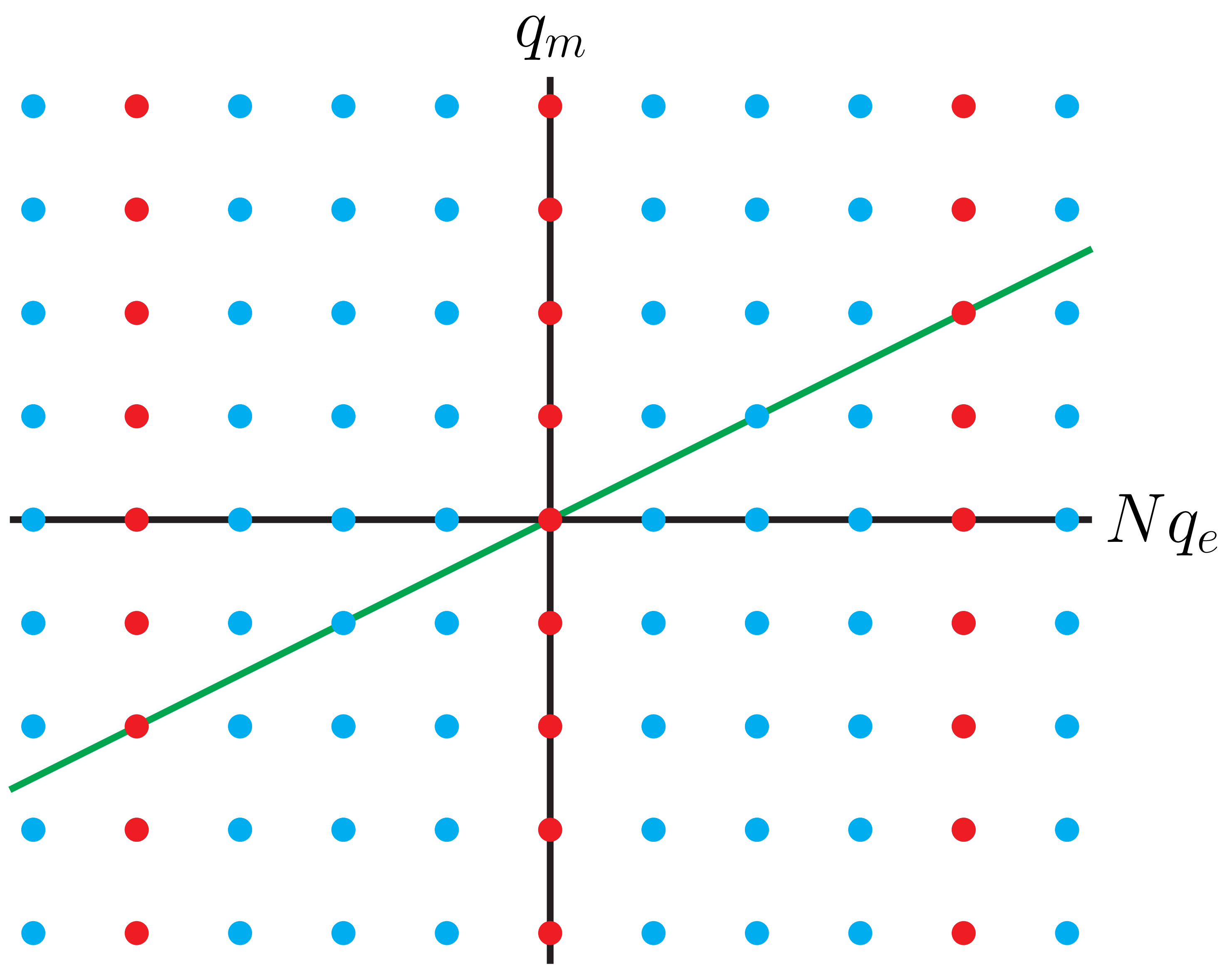
Global 1-form "electric" symmetry $\quad a_\mu \to a_\mu + \frac{2\pi}{N}\eta_\mu, \quad \eta_\mu\in\mathbb{Z}, \quad \Delta_\mu\eta_\nu - \Delta_\nu\eta_\mu = 0$

Charge *k* Wilson loops transform as $\quad W_\gamma = \exp(ik\oint_\gamma a), \quad W_\gamma \to \exp\left(i\frac{2\pi}{N}k\right)W_\gamma$

Dyons

- Charge-monopole composites (q_e, q_m), electric charge $N(q_e+q_m\ \theta/2\pi)$ and magnetic charge q_m
- Witten effect
- The statistics of an (n, m) bosonic dyon is $(-1)^{Nnm}$
- *Bosonic* dyons can condense (n, m) condenses $\Longleftrightarrow$ Nnm is *even.*
- *Fermionic* dyons can only condense after *pairing*

Dyon Condensation and Phase Diagram (CR 1982)



- Oblique Confinement ('t Hooft 1981): if an (n,m) dyon condenses Wilson loops with test charges (q_e, q_m) are *confined unless* $q_m n - q_e m = 0$
- $\mathbb{Z}_N$ theory with $\theta \neq 0$: Oblique confined phases have $L = \gcd(Nn, m)$ distinct deconfined Wilson loops
- Emergent, broken, $\mathbb{Z}_L$ global one-form symmetry
- Total one-form symmetry: $\mathbb{Z}_{Nn}$ (electric) $\times$ $\mathbb{Z}_m$ (magnetic) $\longrightarrow$ $G_{oblique} = \mathbb{Z}_{Nn/m}$

"Coulomb" Gas representation in 3+1 D

$$Z = \sum_{\{n_\mu, m_\mu\}} \exp\Big[-\frac{2\pi^2}{g^2} \sum_{R,R'} m_\mu(R)\, G(R-R')\, m_\mu(R')$$

$$-\frac{1}{2}N^2 g^2 \sum_{r,r'} \left(n_\mu(r) + \frac{\theta}{2\pi} m_\mu(r) \right) G(r-r') \left(n_\mu(r') + \frac{\theta}{2\pi} m_\mu(r') \right)$$

$$+ iN \sum_{R,r} m_\mu(R)\, \Theta_{\mu\nu}(R-r)\, n_\nu(r) \Big]$$

n_μ: particle world lines; m_μ: monopole worldlines

- Cardy & Rabonovici: dyon loops of length P with charge (n, m) proliferates if the energy cost of nucleating it is smaller than the entropy gain

$$\frac{2\pi}{Ng^2} m^2 + \frac{Ng^2}{2\pi} \left(n + \frac{\theta}{2\pi} m \right)^2 < \frac{C}{N}$$

Lattice ("UV") Duality and Modular Symmetry

Modular parameter $\tau = \frac{\theta}{2\pi} + i\frac{2\pi}{Ng^2}$

$$\mathcal{S}\ \text{duality}: \quad (n,m) \mapsto (-m,n) \ \Rightarrow\ \tau \mapsto -\frac{1}{\tau} \quad \text{duality}$$

$$\mathcal{T}\ \text{duality}: \quad (n,m) \mapsto (n-m,n) \ \Rightarrow\ \tau \mapsto \tau + 1 \quad \text{periodicity}$$

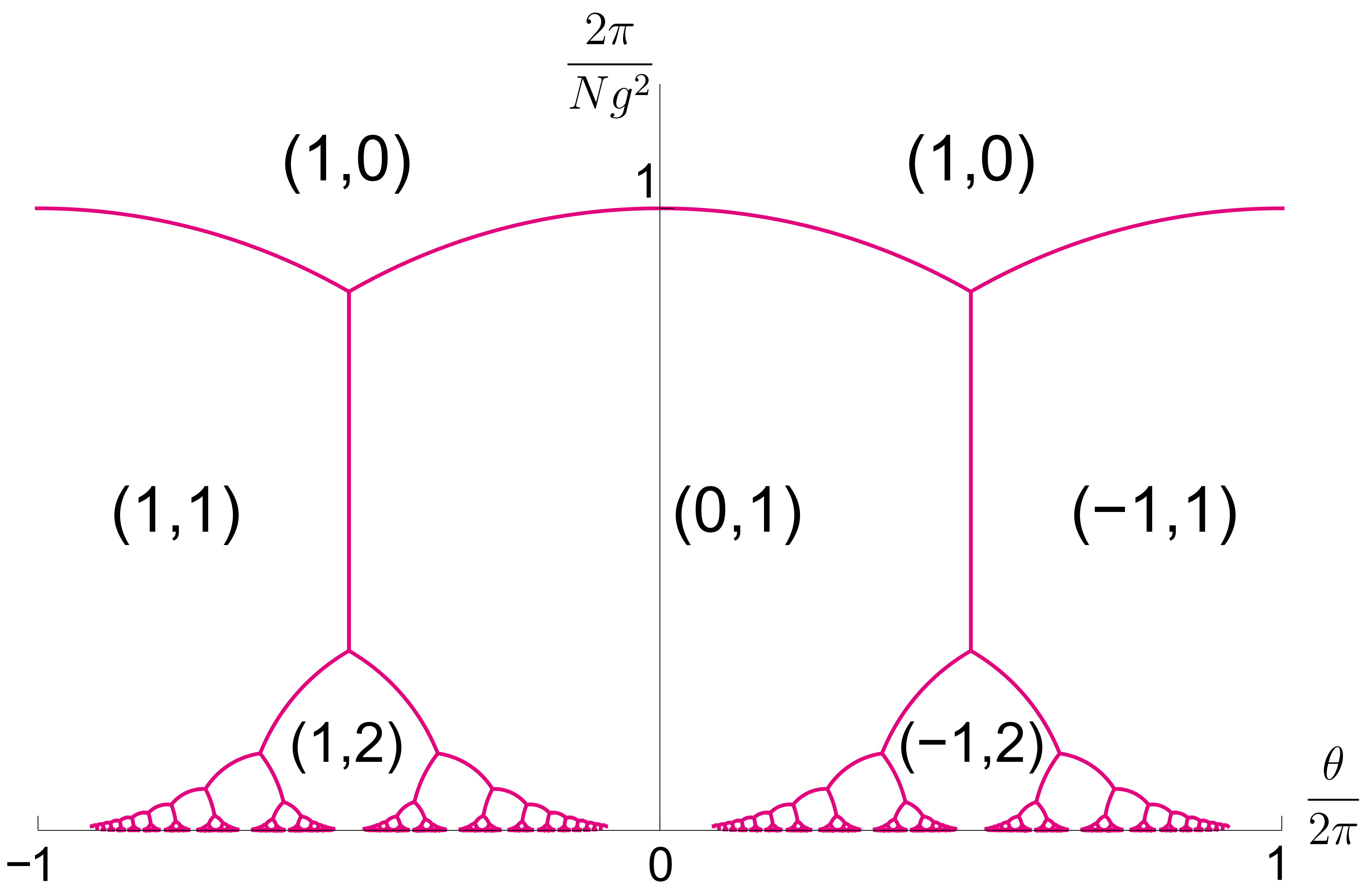
$\mathcal{S}$ and $\mathcal{T}$ generators of $PSL(2,\mathbb{Z})$

$$\tau \mapsto \tau' = \frac{a\tau + b}{c\tau + d}, \quad a,b,c,d \in \mathbb{Z},\ ad - bc \neq 0$$

$$Z[\tau] = Z\left[-1/\tau\right]\ ,\ Z[\tau] = Z[\tau+1]$$

- Generalization of Kramers-Wannier duality
- This symmetry generates the phase diagram
- (n, m) dyon condenses for θ=-2π n/m

Phase Diagram and Modular Invariance



- (1,0): Superfluid
- (0,1): Confinement
- (-1,1): fermions (trivial); bosons: SPT
- (±1,2): Oblique SET
- The meaning of the oblique phases depends on having fermions or bosons

Oblique Topological Order

Loop Operators:

$$\text{Wilson loops}: \quad W_\Gamma = \prod_{\ell\in\Gamma} \exp(i a_\mu(\ell))$$

$$\text{'t Hooft loops}: \quad T_{\tilde{\Gamma}} = \prod_{\tilde{\ell}\in\tilde{\Gamma}} \exp(i \tilde{a}_\mu(\tilde{\ell}))$$

$$\text{Dyon loops}: \quad D_\Gamma(q_e, q_m) = (W_\Gamma)^{N q_e} (T_{\tilde{\Gamma}})^{q_m}$$

$$\langle \mathcal{D}(q_e, q_m)\rangle = \begin{cases} \exp(-\mathrm{Area}), & (\mathrm{q_e, q_m})\ \text{braids with}\ (\mathrm{n, m}) \\ \exp(-\mathrm{Perimeter}), & (\mathrm{q_e, q_m})\ \text{braids trivially with}\ (\mathrm{n, m}) \end{cases}$$

- Dyon condensation iff q_m n - q_e m=0
- N $q_e \in \mathbb{Z} \Rightarrow$ L= gcd (Nn,m) distinct deconfined dyon loop operators

  with q_e=nk/L, q_m =mk/L, k=0, 1,…,L-1
- Non-trivial topological order if L >1

Oblique Topological Order

In 3+1 dimensions topological order is defined by braiding of loop and surface operators

Magnetic Surface Operators

$$U_\Sigma(\Phi_e) = \prod_{\square \in \Sigma} \exp(i\Phi_e f(\square))$$

$$\tilde{U}_\Sigma(\Phi_m) = \prod_{\tilde{\square} \in \Sigma} \exp(i\Phi_m \tilde{f}(\tilde{\square}))$$

Braiding a (Φ_e, Φ_m) surface operator with a (q_e, q_m) loop operator leads to a phase

$$\Theta_{\text{braid}} = 2\pi(Nq_e\Phi_m - \Phi_e q_m)\varphi(\Sigma, \Gamma) \qquad \text{linking number: } \varphi(\Sigma, \Gamma)$$

Physical Operators: Nn Φ_m - m $\Phi_e \in \mathbb{Z} \Rightarrow$ L= gcd(Nn, m) inequivalent flux tubes

Emergent Gauss Law ∇. e=0 (mod Nn) ⇒ emergent electric $\mathbb{Z}_{Nn}$ one-form symmetry

Residual One-Form Symmetry

- Magnetic Monopoles ⇒ $\mathbb{Z}_m$ magnetic one-form symmetry
- Total emergent one-form symmetry is $\mathbb{Z}_{Nn} \times Z_m$
- $\mathbb{Z}_L$ topological order + mixed anomaly ⇒ $\mathbb{Z}_L$ broken to $\mathbb{Z}_{Nm/L}$ left over
- Higher Form SET ⟺ “Oblique TI”

Hydrodynamic TQFT of the (n, m) condensed phase

We used the lattice model to derive an effective hydrodynamic (IR) TQFT (formally valid in the limit $g^2 \mapsto \infty$)

$$Z_{\text{oblique}} = \int \mathcal{D}\tilde{a}_\mu \mathcal{D}\tilde{b}_{\mu\nu} \;\exp(-\tilde{S}[\tilde{a}_\mu, \tilde{b}_{\mu\nu}]$$

$$\tilde{S} = i\frac{m}{2\pi}\int \tilde{b}\wedge d\tilde{a} - i\frac{Nnm}{4\pi}\int \tilde{b}\wedge\tilde{b}$$

(A. Kapustin and N. Seiberg 2014)

invariant under the 2-form gauge transformation: $\tilde{b}\to\tilde{b}-d\lambda,\;\tilde{a}\to\tilde{a}+Nn\lambda$

Nnm even for bosons, *Nnm*+*Nn* even for fermions

Global 2-form symmetry $\mathbb{Z}_L$, $L=\gcd(Nn,m)$

IR duality: $S = i\frac{Nn}{2\pi}\int b\wedge da + \frac{Nnm}{4\pi}\int b\wedge b$

The TQFT has an IR electromagnetic duality *in each* (n, m) oblique phase

Bulk Topological Order of the (n, m) Oblique Confined Phases

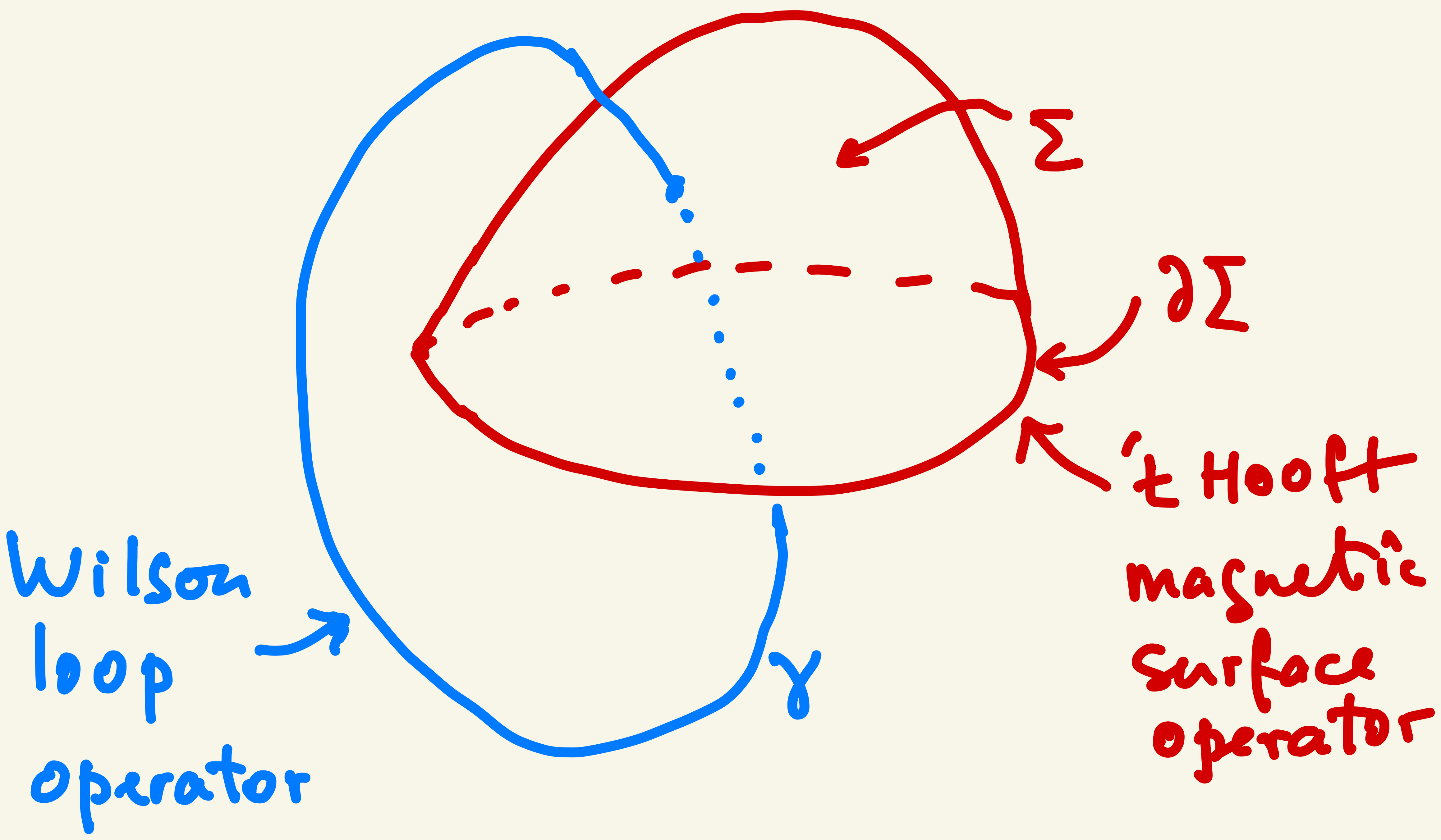
- Non-trivial linking number: Wilson operators on closed loops Γ attached to ’t Hooft magnetic operators on surfaces ∑ with boundary ∂∑=Γ
- Dyons of electric charge q attached to the end of a string of magnetic flux Φ=mq/(Nn)

$$W[\Sigma]^q = \exp\left(iq\oint_\Gamma a + im\int_\Sigma b\right)$$

- If q=Nn/L , with L=gcd(Nn, m), the surface operator is *invisible*
- Global electric symmetry $\mathbb{Z}_{Nn} \mapsto \mathbb{Z}_{Nn/L}$
- Global magnetic symmetry $\mathbb{Z}_{m} \mapsto \mathbb{Z}_{m/L}$
- Closed surface operators $U(\Sigma)^k$

$$U(\Sigma)^k = \exp\left(ik\oint_\Sigma b\right)$$

$$\left\langle W[\Gamma]^{pNn/L} U(\Sigma)^k \right\rangle = \exp\left(2\pi i \frac{pk}{L}\ell(\Gamma,\Sigma)\right) \Rightarrow \text{ bulk topological order} : \mathbb{Z}_L$$



non-trivial linking #

Bulk 2-form fractional magneto-electric effect

Probe the Cardy-Rabinovici theory with a flat fractional 2-form field

$$2\pi s_{\mu\nu} \to 2\pi s_{\mu\nu} + \frac{2\pi}{N} B_{\mu\nu}, \quad B_{\mu\nu} = \Delta_\mu l_\nu - \Delta_\nu l_\mu, \; l_\mu \in \mathbb{Z}$$

The effective action of the bulk TQFT has a universal fractional response

$$S_{\text{response}}[B] = -i \frac{Nn}{4\pi m} \int B \wedge B, \; B \text{ is a } \mathbb{Z}_N \text{ 2-form flat background field}$$

The prefactor is the *universal fractional magneto-electric response* of the oblique confinement phase

Surface Topological order and Anomaly Inflow

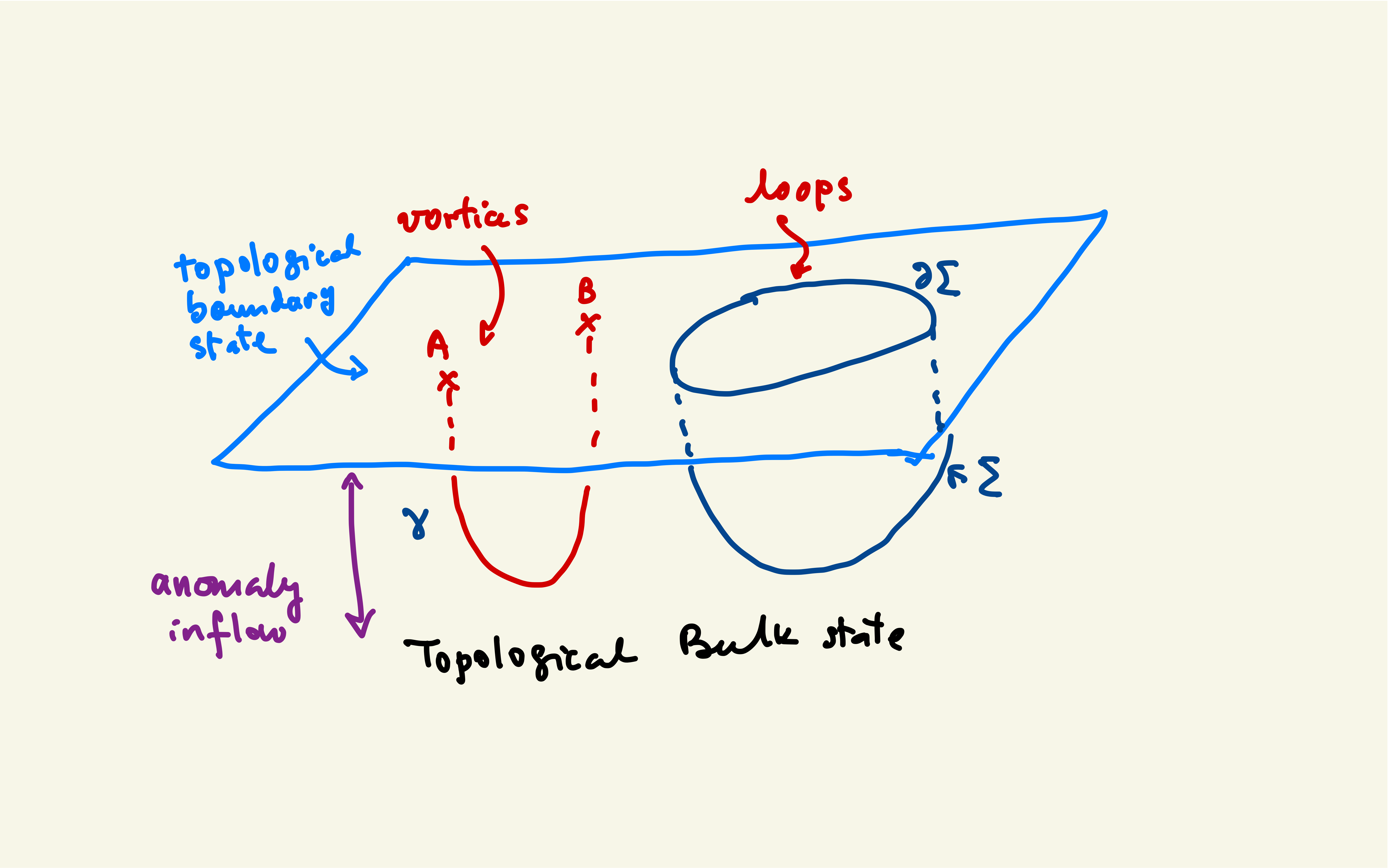
$$\tilde{S} = i\frac{m}{2\pi}\int \tilde{b}\wedge d\tilde{a} - i\frac{Nnm}{4\pi}\int \tilde{b}\wedge\tilde{b}$$

- This bulk has a gauge anomaly at the boundary ∂M
- The anomaly is canceled by a U(1) boundary 1-form gauge field c_μ

$$S_{\partial M} = i\frac{Nnm}{4\pi}\int_{\partial M} c\wedge dc + i\frac{m}{2\pi}\int_{\partial M} c\wedge d\tilde{a}$$

Two possible boundary theories depending on boundary conditions

Bulk-Boundary Correspondence



Surface Topological order and Anomaly Inflow

$$S_{\partial M} = i\frac{Nnm}{4\pi}\int_{\partial M} c \wedge dc + i\frac{m}{2\pi}\int_{\partial M} c \wedge d\tilde{a}$$

- Two possible boundary theories:
- A: $b_{\mu\nu} \mid_{\partial M}=0$
- B: $a_{\mu} \mid_{\partial M}=0$
- A has $\mathbb{Z}_{Nn}$ topological order
- B has $\mathbb{Z}_{m}$ topological order
- The bulk has $\mathbb{Z}_{L}$ topological order, L=gcd(Nn, m)
- Boundary Hall Conductivity: A: σ_{xy}=m/Nn, B: σ_{xy}=Nn/m

The bulk theory is *invariant* under the IR duality and the boundary states map into each other

Conclusions

- Oblique confined phases of $\mathbb{Z}_N$ gauge theories have non trivial $\mathbb{Z}_L$ bulk topological order L=gcd(Nn, m)
- The special case L=1 is an SPT
- Two possible boundary states with different topological order
- The boundary states are FQH states that cannot exist in 2+1 D
- Derived a hydrodynamic TQFT of the bulk and boundary states with matching anomalies
- There is an interesting versions of this theory in 1+1 D and 2+1 D
- Possible extensions to 3+1 D theories with non-abelian braiding?