Machine-assisted discovery of integrable systems

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Machine-assisted discovery of integrable symplectic mappings

T. Zolkin, Y. Kharkov, and S. Nagaitsev
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Integrable symplectic maps with a polygon tessellation

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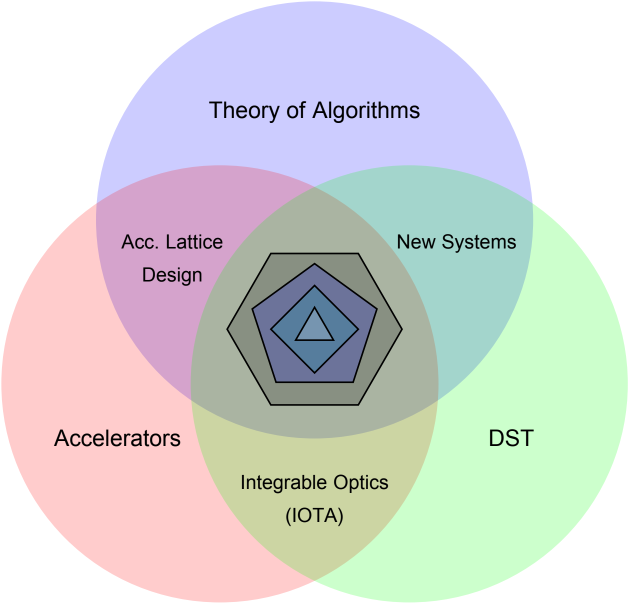
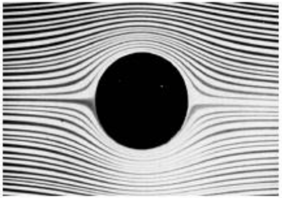


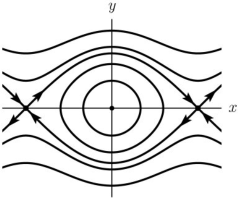
Theory of Algorithms
Acc. Lattice
Design
New Systems
Accelerators
DST
Integrable Optics
(IOTA)

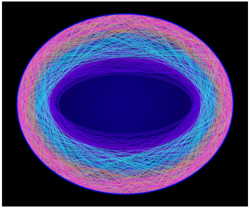
1. Integrability

A system is considered *integrable* if it possesses as many conserved quantities (*integrals of motion*) as degrees of freedom. This is often a consequence of underlying symmetries, as explained by Noether's theorem. These symmetries lead to regular, predictable motion and preclude chaotic behavior in phase space.

- Exact solutions (often approximate more complex behaviors)
- Predictability and reduced complexity
- Symmetries and conservation laws







Phase space: periodic, quasi-periodic and chaotic orbits

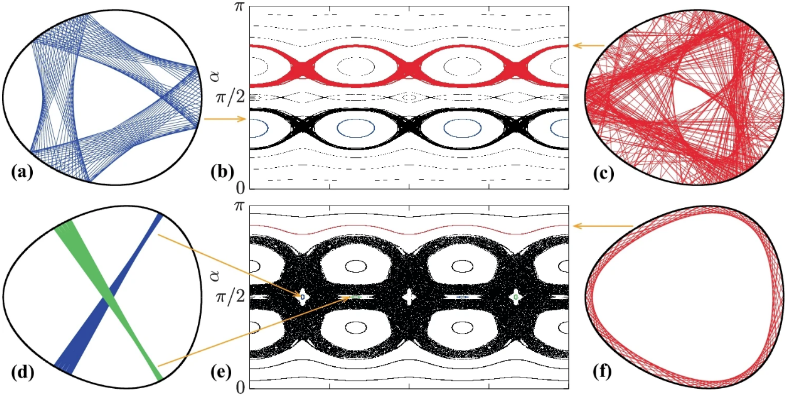


Figure: D.R. da Costa et al. *Braz J Phys* **52**, 75 (2022)

RF Cavities and Financial Markets

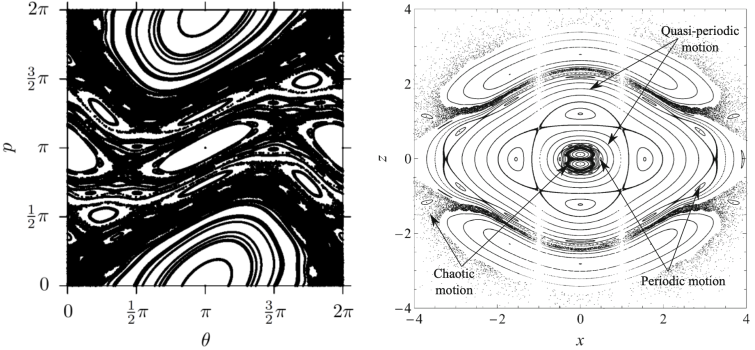
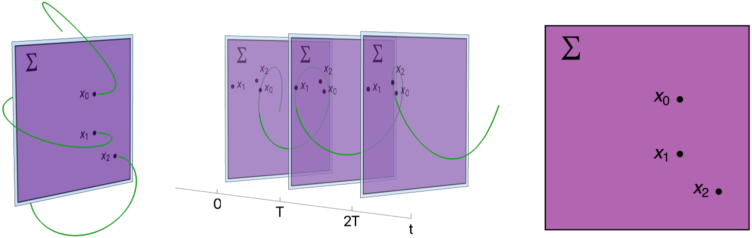


Figure: Chirikov standard map (left) and Huang-Li finance model (right), W. Szumiǹski *Nonlinear Dyn* **94**, 443-459 (2018)

2. Mappings of the plane. Dot ($\dot{}$) vs. Prime ($'$)

Mappings arise naturally in many different situations:

- Reduction of phase space in ODE via Poincaré section.
- Stroboscopic Poincaré map for periodic systems.
- Numerical integration (i.e. symplectic integrators)



$$\dot{p} = F(p, q, r) \qquad \dot{p} = F(p, q, t) \qquad p' = f(p, q)$$

$$\dot{q} = G(p, q, r) \qquad \dot{q} = G(p, q, t) \qquad q' = g(p, q)$$

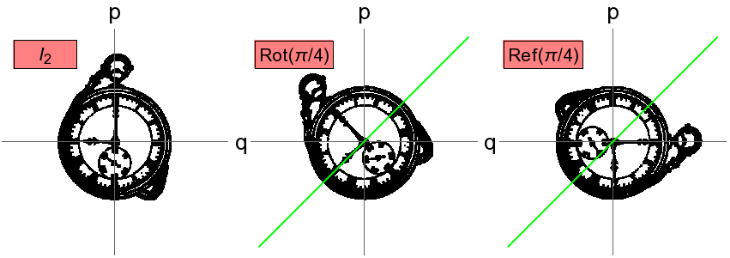
$$\dot{r} = H(p, q, r)$$

“Connect the dots”

C1. Symplectic map of the plane

We will consider area-preserving mappings of the plane

$$
\begin{aligned}
q' &= q'(q,p),\\
p' &= p'(q,p),
\end{aligned}
\qquad
\det \begin{bmatrix} \partial q'/\partial q & \partial q'/\partial p \\ \partial p'/\partial q & \partial p'/\partial p \end{bmatrix} = 1.
$$



Identity, Id	Rotation, Rot	Reflection*,**, Ref
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

C2. Standard form of the map

McMillan considered a special form of the map

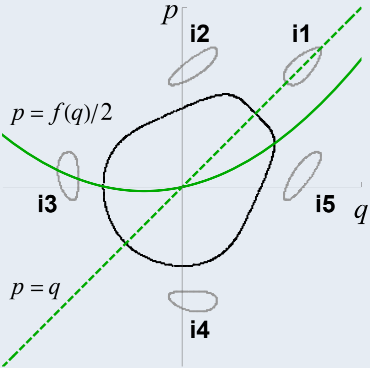
$$\mathrm{M}: \quad \begin{aligned} q' &= p, \\ p' &= -q + f(p), \end{aligned}$$

where $f(p)$ is called *force function* (or simply *force*).

- Invertible
- Symplectic for any $f(p)$
- Two symmetry lines:

$$l_1 : \; p = q$$

$$l_2 : \; p = f(q)/2$$



1D accelerator lattice with thin nonlinear lens, $\mathrm{T} = \mathrm{F} \circ \mathrm{M}$

$$\mathrm{M}: \quad \begin{bmatrix} y \\ \dot{y} \end{bmatrix}' = \begin{bmatrix} \cos\Phi + \alpha \sin\Phi & \beta \sin\Phi \\ -\gamma \sin\Phi & \cos\Phi - \alpha \sin\Phi \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix},$$

$$\mathrm{F}: \quad \begin{bmatrix} y \\ \dot{y} \end{bmatrix}' = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ F(y) \end{bmatrix},$$

where α, β and γ are Courant-Snyder parameters at the thin lens location, and, Φ is the betatron phase advance of one period.

Kicked rotator

$$\mathcal{H}[p, q, t] = \frac{p^2}{2\,m} + k\,\frac{q^2}{2} + F(q) \sum_{n=-\infty}^{\infty} \delta\left(\frac{t}{T} - n\right)$$

3. Integrable maps

A map T in the plane is called **integrable**, if there exists a non-constant real valued continuous functions $\mathcal{K}(q,p)$, called **integral**, which is invariant under T:

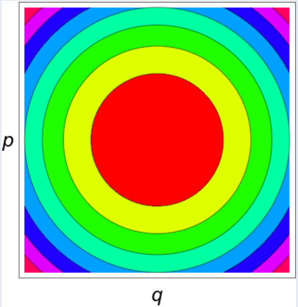
$$\forall\,(q,p): \qquad \mathcal{K}(q,p) = \mathcal{K}(q',p')$$

where primes denote the application of the map, $(q',p') = \mathrm{T}(q,p)$.

Example: Rotation transformation

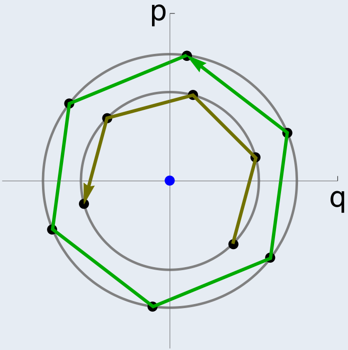
$$\mathrm{Rot}(\theta): \quad \begin{aligned} q' &= q\,\cos\theta - p\,\sin\theta \\ p' &= q\,\sin\theta + p\,\cos\theta \end{aligned}$$

has the integral $\mathcal{K}(q,p) = q^2 + p^2$.

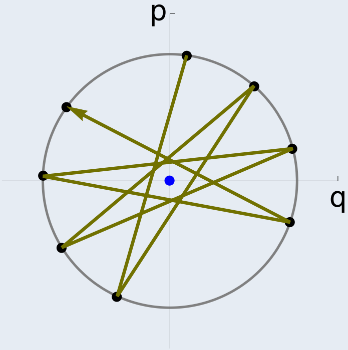


Dynamics on invariant curve/Rotation number

If θ is incommensurate with π, the iterations will result in an invariant curve being traced out. However, if θ and π are commensurate, the iterations will instead produce a discrete set of points.



$$\nu = \frac{\theta}{2\pi} = \frac{m}{n} \in \mathbb{Q}$$



$$\nu = \frac{\theta}{2\pi} \in \mathbb{R} \setminus \mathbb{Q}$$

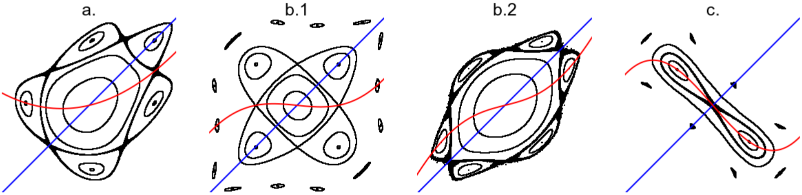
Symmetry lines and Invariant of motion

First symmetry line, $p = q$

$$\mathcal{K}(p, q) = \mathcal{K}(q, p)$$

Second symmetry line, $p = f(q)/2$

$$\mathcal{K}(p, q) = \mathcal{K}(-p + f(q), q)$$



Difficulties in the Search for Integrable Systems

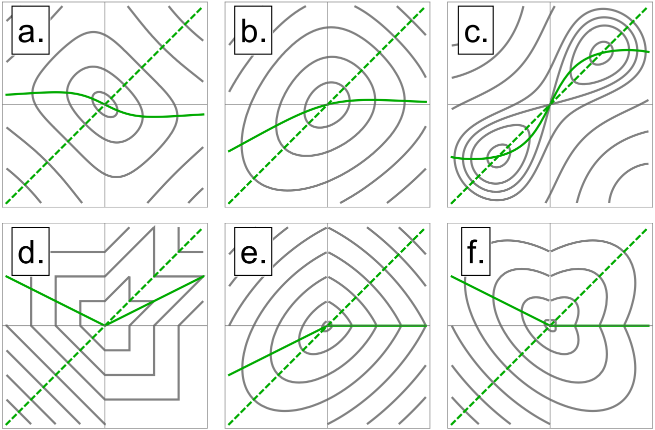
a. – c. McMillan-Suris mappings

For analytic $\mathcal{K}[p,q]$, the invariant of the integrable mapping can take only one of 3 forms: (I) biquadratic function of p and q [McMillan], (II) biquadratic exponential or (III) trigonometric polynomial:

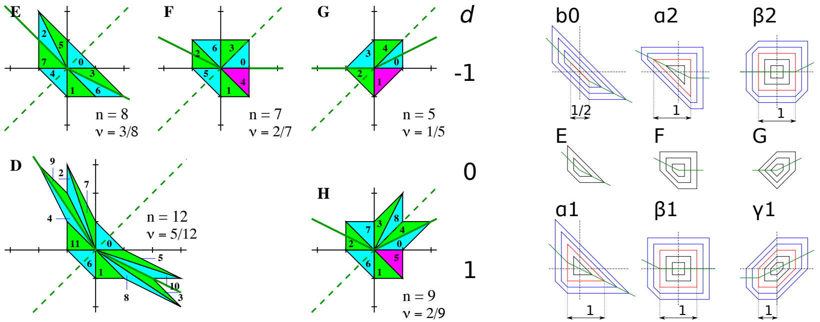
$$\begin{aligned}
\text{(I)}:\quad & \mathcal{K}[p,q] = \mathrm{A}\,p^2q^2 + \mathrm{B}\,(p^2q + p\,q^2) \\
& + \Gamma\,(p^2+q^2) + \Delta\,p\,q + \mathrm{E}\,(p+q) \\
\text{(II)}:\quad & \mathcal{K}[p,q] = \mathrm{A}\,e^{2p}e^{2q} + \mathrm{B}\,(e^{2p}e^{q} + e^{p}e^{2q}) \\
& + \Gamma\,(e^{2p}+e^{2q}) + \Delta\,e^{p}e^{q} + \mathrm{E}\,(e^{p}+e^{q}) \\
\text{(III)}:\quad & \mathcal{K}[p,q] = \mathrm{A}\,\cos[2(p+q)] + \mathrm{B}\,(\cos[2p+q] + \cos[p+2q]) \\
& + \Gamma\,(\cos[2p] + \cos[2q]) + \Delta\,\cos[p+q] + \mathrm{E}\,(\cos[p] + \cos[q])
\end{aligned}$$

d. – f. Brown-Knuth map and McMillan be-/two-headed ellipses

Linear systems (i.e. no dependence on amplitude) with piecewise linear force $f(q) = a\,q + b\,|q|$



4. Nonlinear mappings with polygon invariants



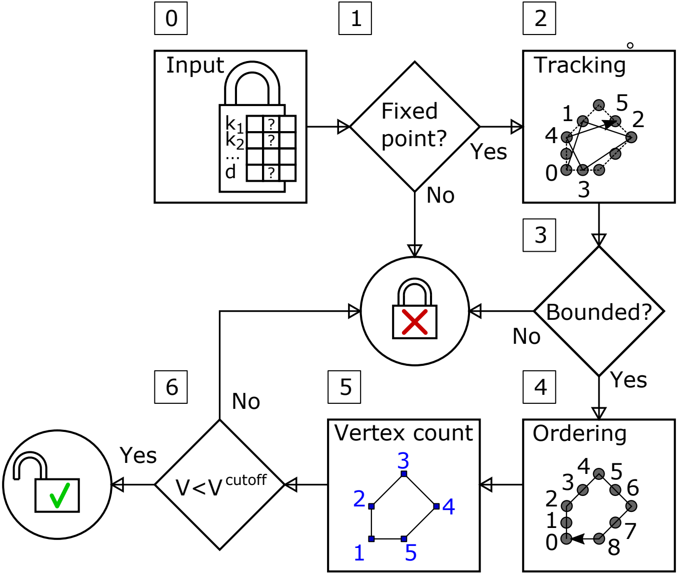
Heuristic Generalization of the CNR and Brown-Knuth Maps

By employing piecewise linear continuous functions with integer k_i

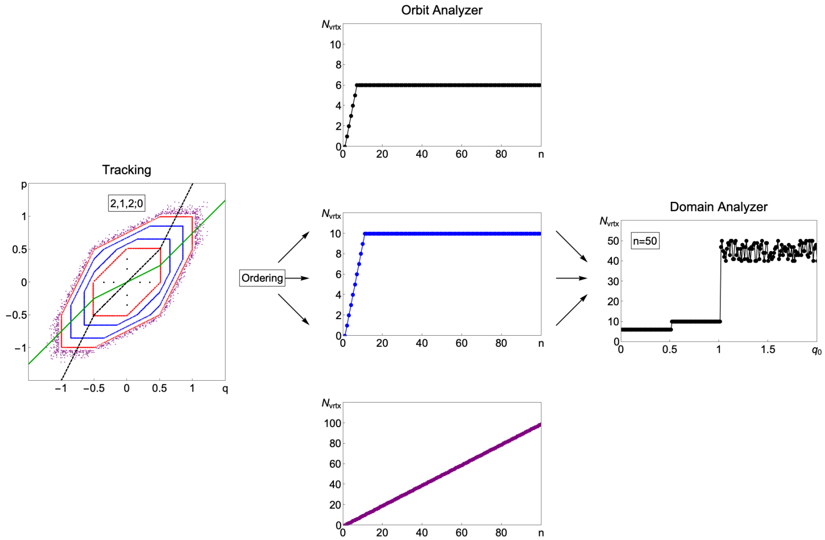
$$f(p) = k_i\, p + d, \qquad i = 1, \ldots, N,$$

we discovered new previously unknown integrable systems.

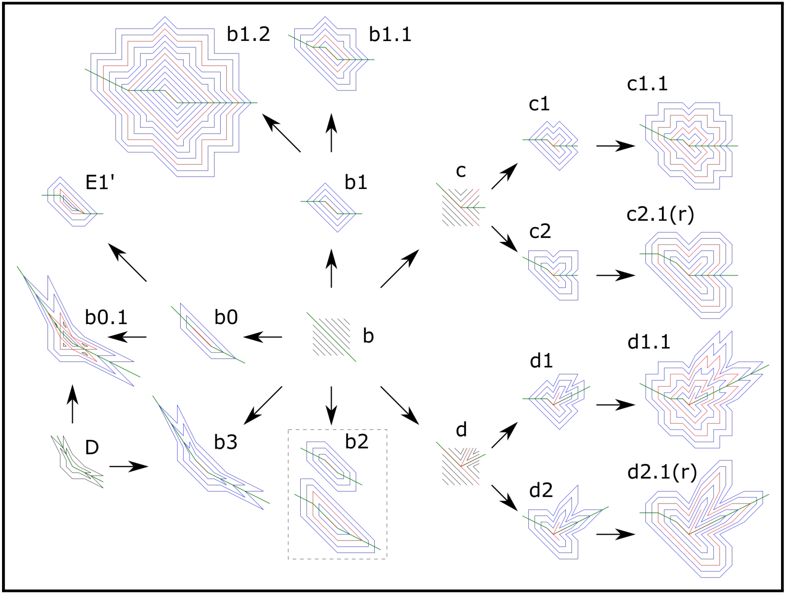
Algorithm 1. Phase Space Analysis/Polygon Identification



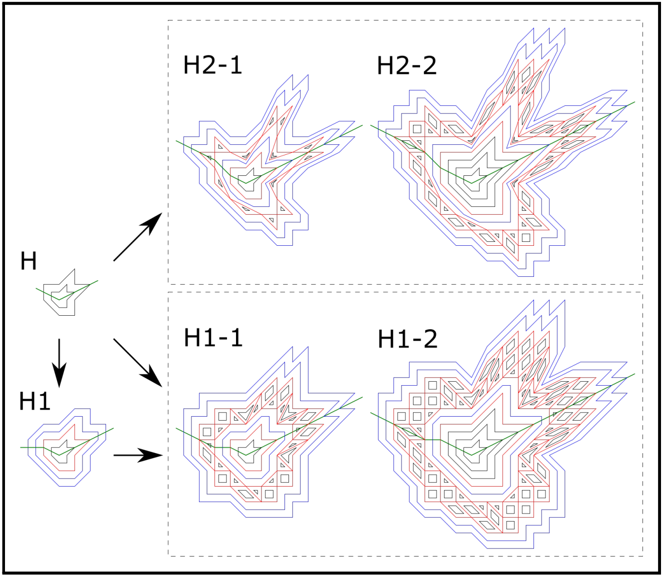
Example: $(k_1, k_2, k_3) = (2, 1, 2)$ with $d = 0$.



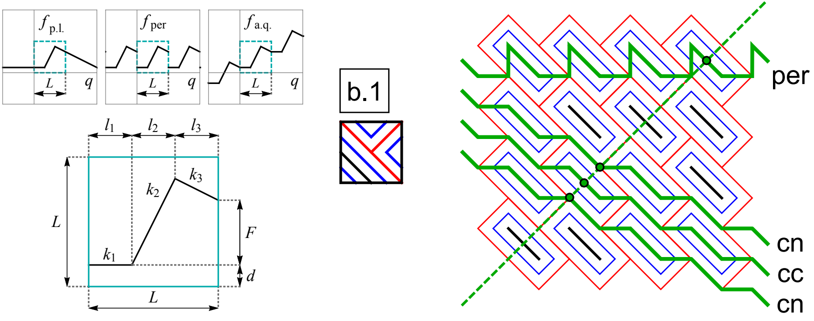
Results



Results (chains of linear islands)



5. Mappings on a torus, $\mathbb{T}^2$



From torus $\mathbb{T}^2$ to plane $\mathbb{R}^2$

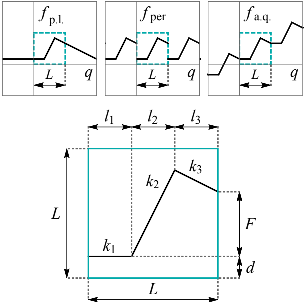
- Periodic "unwrapping"

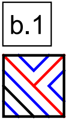
$$f_{\text{per}}(q) = f_{\text{p.l.}}(q \mod L) \mod L$$

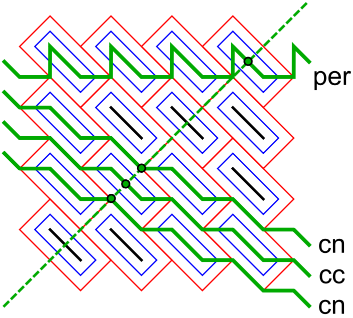
- *Arithmetical quasiperiodicity*

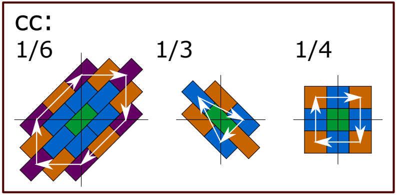
$$\forall\, q: \;\; f_{\text{a.q.}}(q+L) = f_{\text{a.q.}}(q) + F, \quad F = f_{\text{p.l.}}(L) - f_{\text{p.l.}}(0) = \text{const.}$$

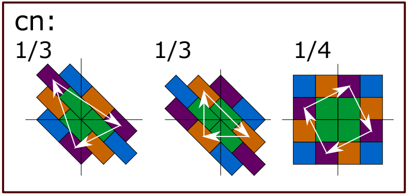
Global mode locking: central cell vs. central node



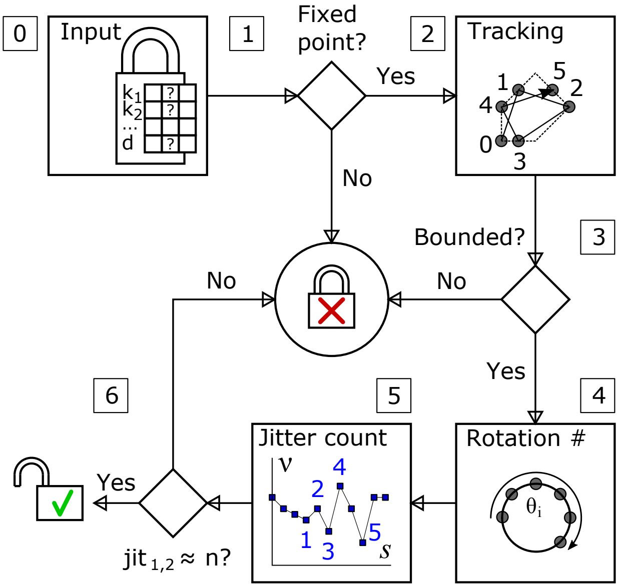




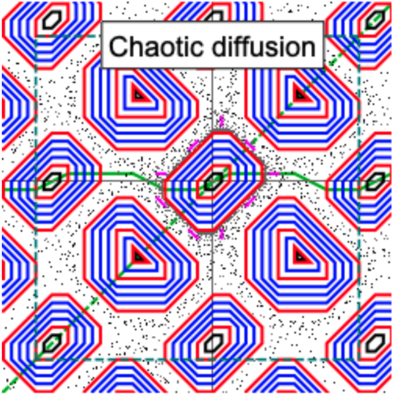


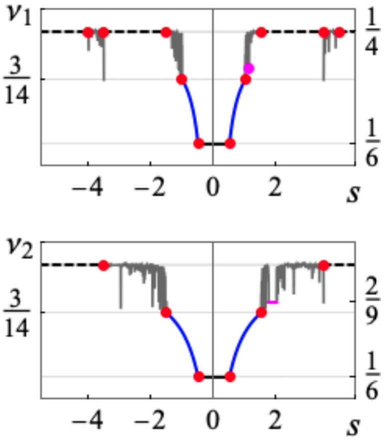


Algorithm 2. Piecewise Monotonic Rotation Number

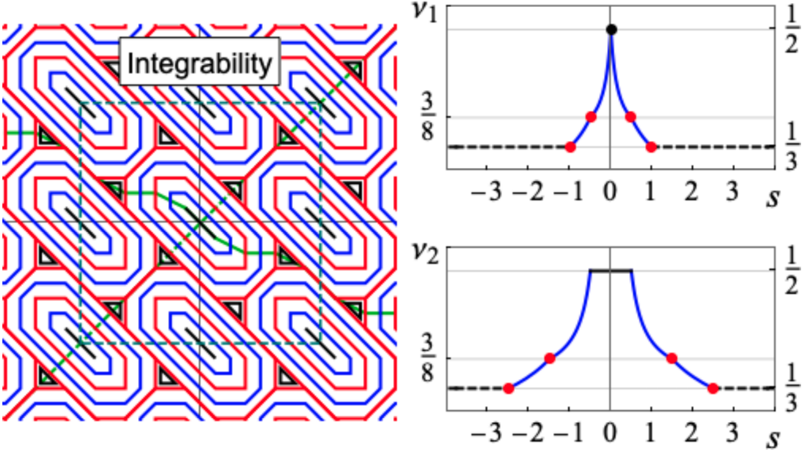


Example: Chaotic diffusion

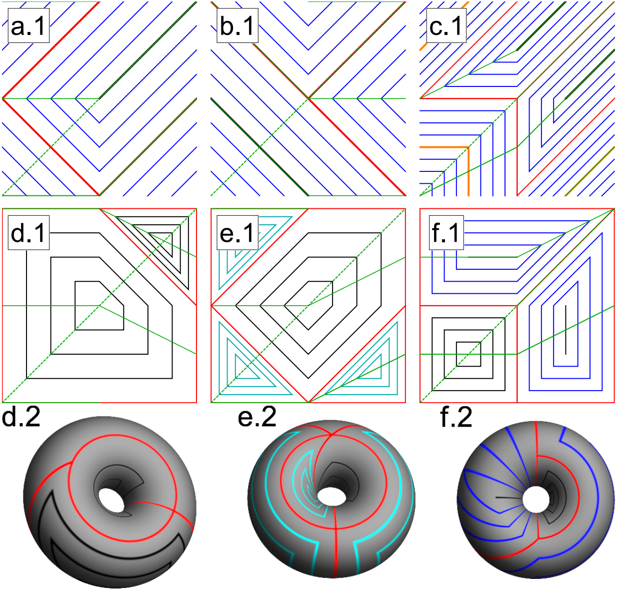




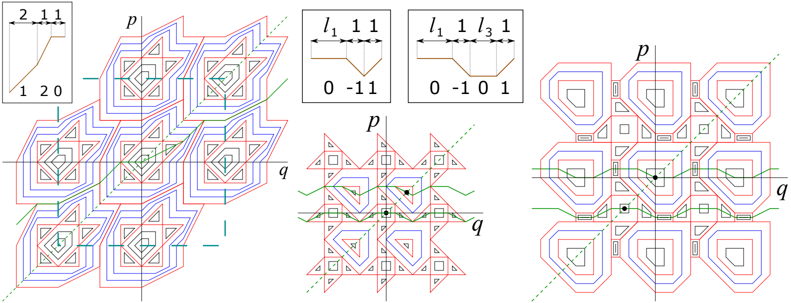
Example: Integrable map



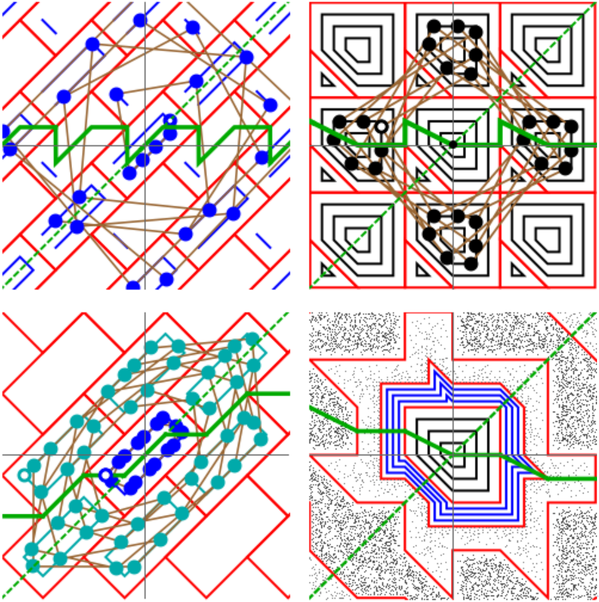
Results (Torus)



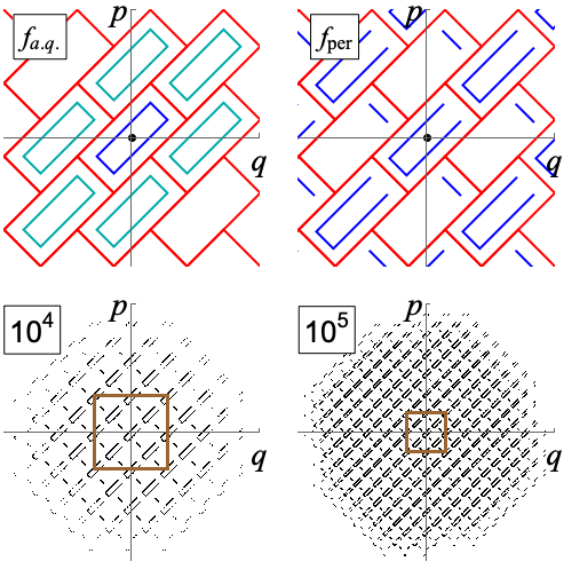
Results (Tessellation & Fibration by Polygons)



Periodic and arithmetically quasiperiodic unwrappings



Integrable diffusion



6. Smoothening procedure

$$q' = p$$
$$p' = -q + f(p)$$

$$f(q) = \sqrt{q^2 + \epsilon^2} \underset{|\epsilon|\to 0}{\longrightarrow} |q|$$

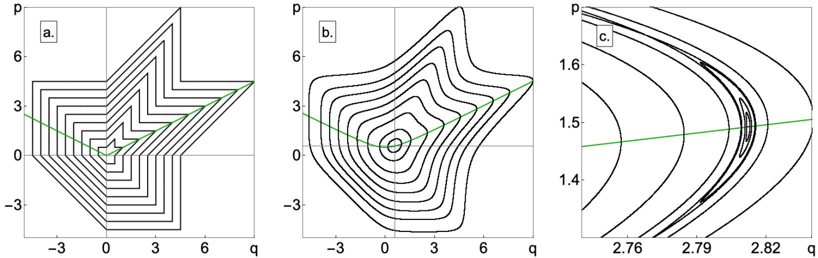


Figure: Left plot (a.) illustrates invariant level sets for Brown-Knuth map, force function $f(q) = |q|$. Middle plot (b.) displays invariant level sets for Cohen map, $f(q) = \sqrt{q^2 + 1}$. Right plot (c.) again provides invariant level sets for Cohen map, but on a different scale showing one of the island structures. Level sets for Cohen map are obtained by tracking. Green curve is the second symmetry line $p = f(q)/2$.

Application 1: Near-integrable systems via "smoothening"

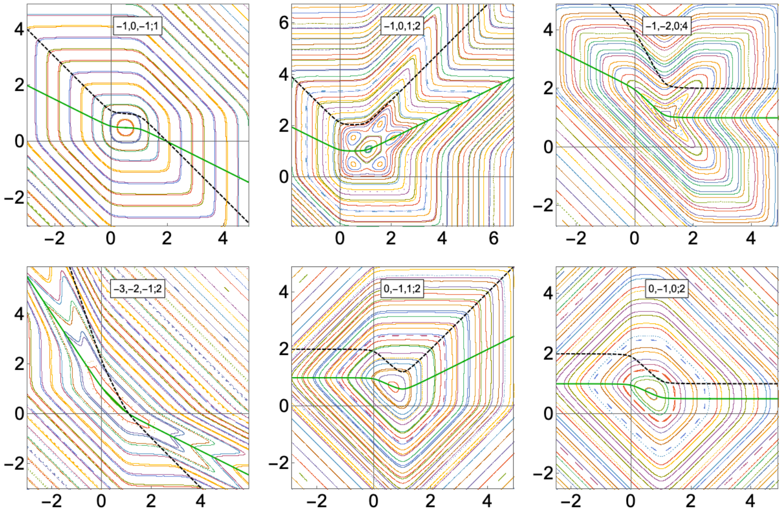
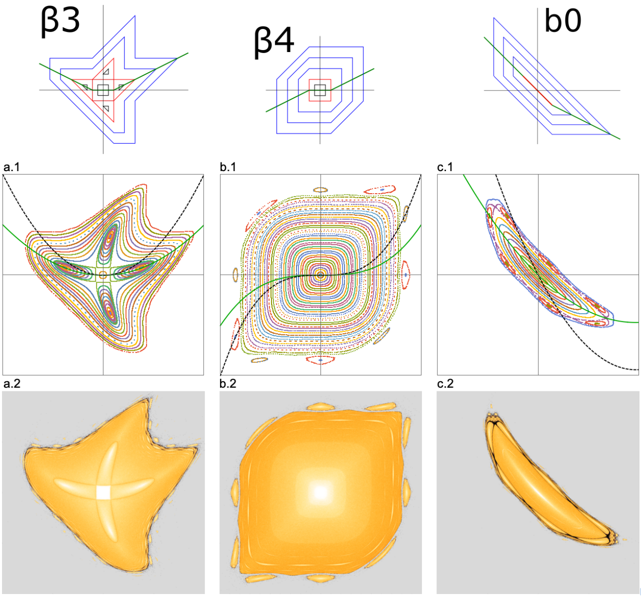


Figure: Examples of quasi-integrable systems produced by "smoothening" 3-piece integrable polygon maps using $\epsilon = 0.05$.

7. Application 2: Discrete perturbation theory



Thank you for your attention!

Questions?