Physics 480-1 Problem Set 6

February 22, 2024

Due: Friday, February 16 at 4 pm

1. Matter Wave Interferometer Phase Shift Calculation. In this problem, we will calculate how the phase shift of a matter wave interferometer depends on various external influences. We will consider the interferometer configuration discussed in class, where a $\pi/2$ pulse applied at time $t = 0$ acts as an initial beam splitter (transferring momentum $n\hbar k$), a π pulse applied at time $t = T$ acts as a mirror, and a $\pi/2$ pulse applied at time $t = 2T$ acts as a final beam splitter. In between these pulses, the matter waves evolve under the following Lagrangian:

$$L = \frac{1}{2}m\left[\dot{\boldsymbol{r}} + \boldsymbol{\Omega} \times (\boldsymbol{r} + \boldsymbol{R}_e)\right]^2 - mgz - \frac{1}{2}mT_{xx}x^2 - \frac{1}{2}mT_{yy}y^2 - \frac{1}{2}mT_{zz}z^2 \quad (1)$$

This Lagrangian accounts for the fact that the Earth is rotating with angular velocity $\boldsymbol{\Omega} = (0, \Omega_y, \Omega_z)$, which leads to Coriolis and centrifugal forces. The z axis is defined to be normal to the surface of the Earth. The vector $\boldsymbol{r}$ denotes $\boldsymbol{r} = (x, y, z)$. Coordinates are chosen so that the point $(x, y, z) = (0, 0, 0)$ is located on the surface of the Earth at position $\boldsymbol{R}_e = (0, 0, R_e)$ relative to the center of the Earth, where R_e denotes the Earth's radius. At point $(x, y, z) = (0, 0, 0)$, the Earth's gravitational field is taken to have magnitude g along the z axis. The Earth possesses gravity gradients, which means that the gravitational acceleration varies depending on possession. The Lagrangian accounts for gradients T_{xx}, T_{yy}, and T_{zz} along the x, y, and z axes, respectively. There will also be higher order terms (in position) in Earth's gravitational potential, but for simplicity we neglect these here.

Throughout this problem, it is suggested that you use Mathematica (or something equivalent) for the calculations, which would be time-consuming to do by hand.

(a) Based on the above Lagrangian, find the Euler-Lagrange equations of motion.

(b) We will calculate the interferometer phase shift to fourth order in the time T. Therefore, it will be useful to express the trajectories $x(t)$, $y(t)$, and $z(t)$ as power series in time and keep terms up to fourth order in time. To fourth order in time, find the trajectories for the upper and lower arms of the interferometer throughout the interferometer duration (note that the expressions for these trajectories will include many terms). Using these trajectories, calculate the propagation, laser, and separation phases to fourth order in T (hint: remember that for the separation phase, you should use the canonical momentum). Keep the initial positions and velocities in the x, y, and z dimensions as free variables.

(c) Sum together the propagation, laser, and separation phases to find the total interferometer phase shift. Use the Expand function in Mathematica to write the total phase shift as a sum of multiple terms.

(d) Numerically evaluate each of the terms from your result in (c) for the following example parameters: $k = \frac{2\pi}{780\text{ nm}}$, $n = 2$, $g = 9.8$ m/s^2, $R_e = 6.4 \times 10^6$ m, $T_{zz} = -\frac{2g}{R_e}$, $T_{xx} = T_{yy} = \frac{g}{R_e}$, $\Omega_y = 7 \times 10^{-5} \cos\theta_{lat}$ rad/s and $\Omega_z = 7 \times 10^{-5} \sin\theta_{lat}$ rad/s for a latitude angle θ_{lat} of 40 degrees, initial positions in all dimensions of 1 mm, initial velocities in the x and y dimensions of 1 mm/s and in the z direction of 13.7 m/s (corresponding to particles that are launched upward), $T = 1.4$ s, and $m = 1.4 \times 10^{-25}$ kg (corresponding to the mass of a Rb-87 atom). Note which terms make the largest contributions to the phase shift for these example parameters.