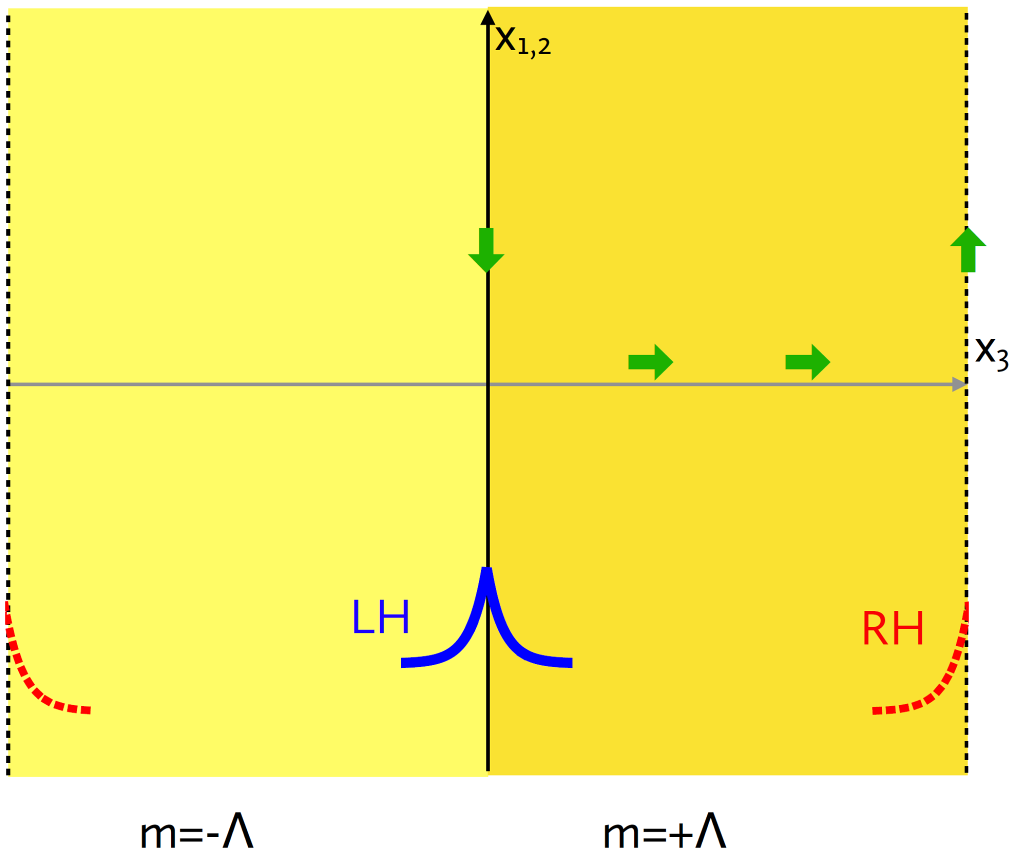
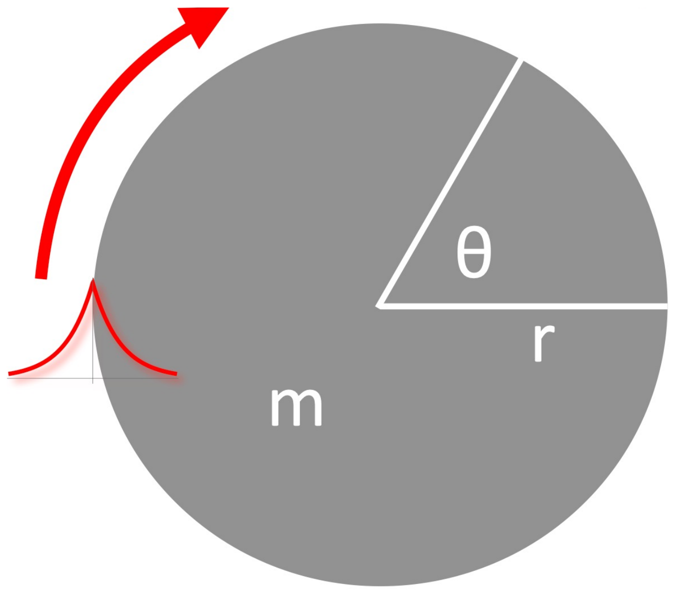
Weyl fermion on a lattice: regulating lattice chiral gauge theory at $\theta = 0$





Srimoyee Sen,

Iowa State University

Seminar on 12/05/2024, Fermi Lab

Based on *Phys.Rev.Lett.* 132 (2024) 14, 141604
and on arXiv 2412.02024

with David Kaplan, UW

Takeaway message before I begin

Regulating the standard model (or chiral gauge theories) on the lattice has remained one of the nagging unsolved problems over the last forty years.

Recent developments provide major step forward that has brought us very close to solving this problem.

Seems to suggest, the regulator only possible when QCD θ parameter, $\theta = 0$

That's the whole talk!

Plan of the talk

- What are chiral gauge theories?
- Why is it hard to formulate them on the lattice?
- Why do we want to formulate them on the lattice?
- A few of the past attempts, that are yet to work or don't work.
- What is new and why it could be the best bet.
- What are the remaining challenges? Hints about how the world works?

Chiral gauge theories

Even dimensional world with massless fermions and gauge field.

Chiral symmetry is gauged.

Fermion mass terms transform under gauge transformation.

So, a simple mass term is disallowed.

Example: The standard model.

Good chiral symmetry is essential

Chiral gauge theories on the lattice

A non-perturbative formulation of such theories is desirable.

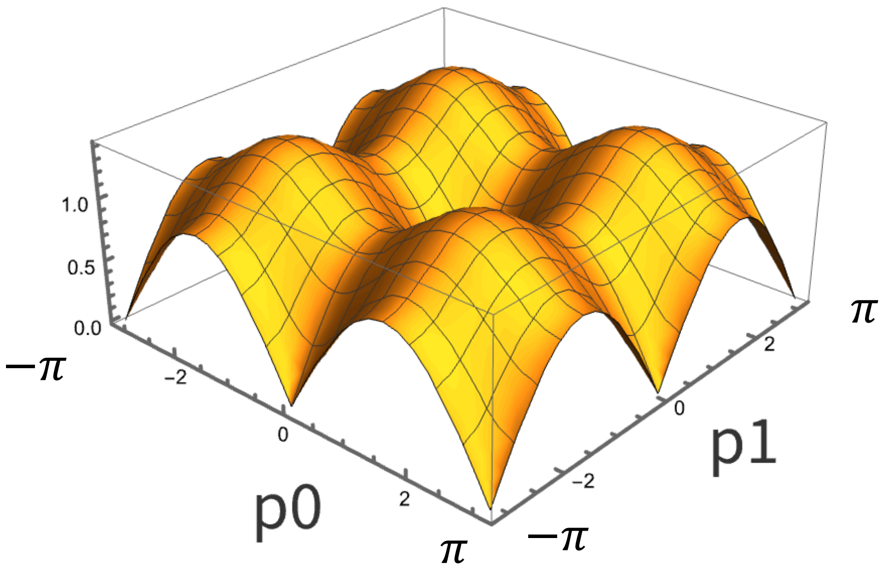
Why? It's a question of principle. Can it be done or not?

Can allow us to simulate SM and also BSM models with strong chiral gauge sector.

Phenomenological questions?

May end up teaching us something about how the world works!

Backing up: Even global chiral symmetry is hard (but solved)

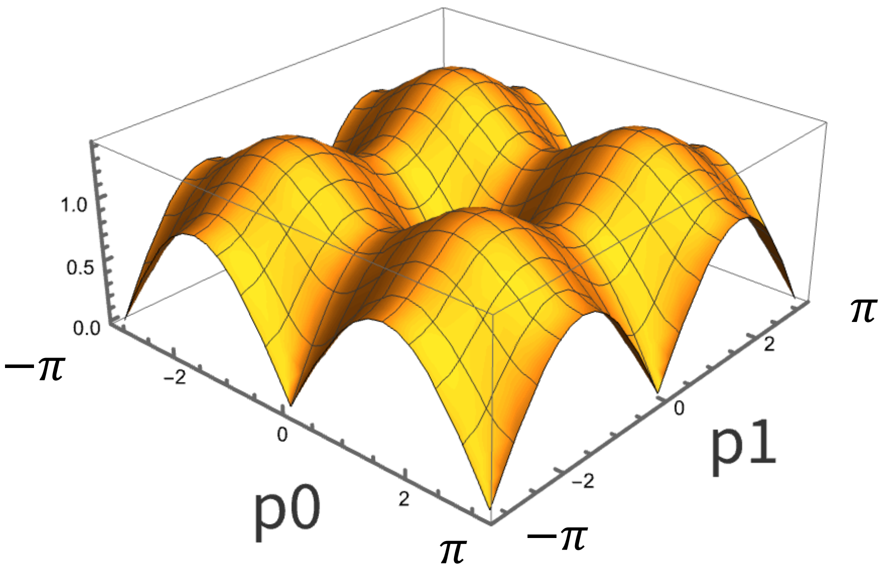


Lattice Brillouin zones

Simulating massless Dirac fermions on the lattice, was a challenge due to fermion doubling.

Nielsen Ninomiya (1981): Cannot formulate Dirac fermion with exact chiral symmetry without an unwanted doubling of all fermion species.

What does this mean ? (still talking about global chiral symmetry)



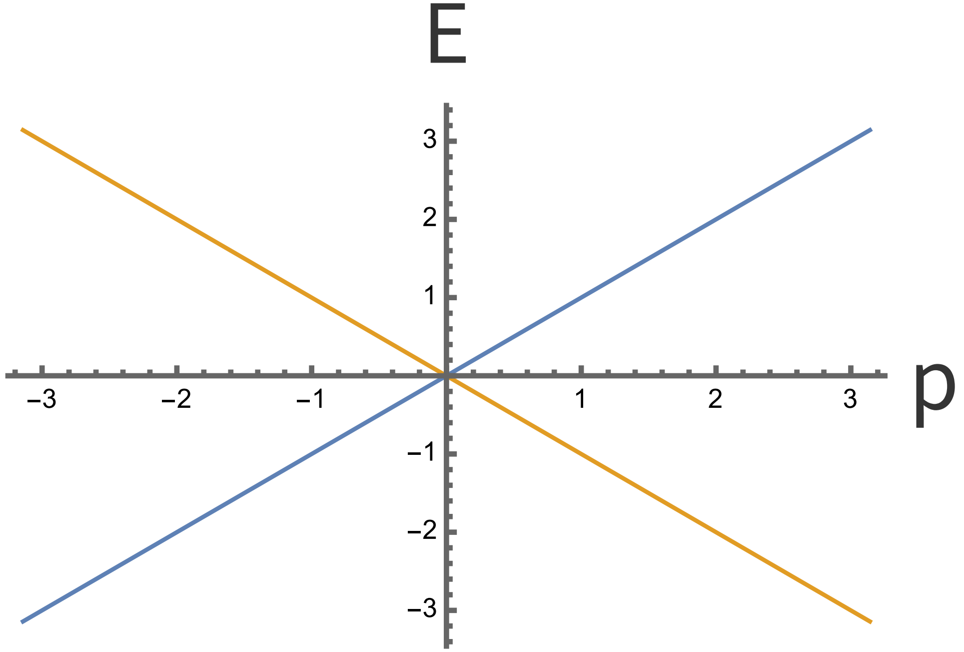
Lattice Brillouin zones

We want to formulate a Dirac fermion theory on the lattice.

We want to keep the fermions light, e.g. light quarks.

This combination i.e. "discrete space-time + massless fermion": extremely difficult to engineer.

Why chiral symmetry is hard: Dispersion (1 spatial dimension)



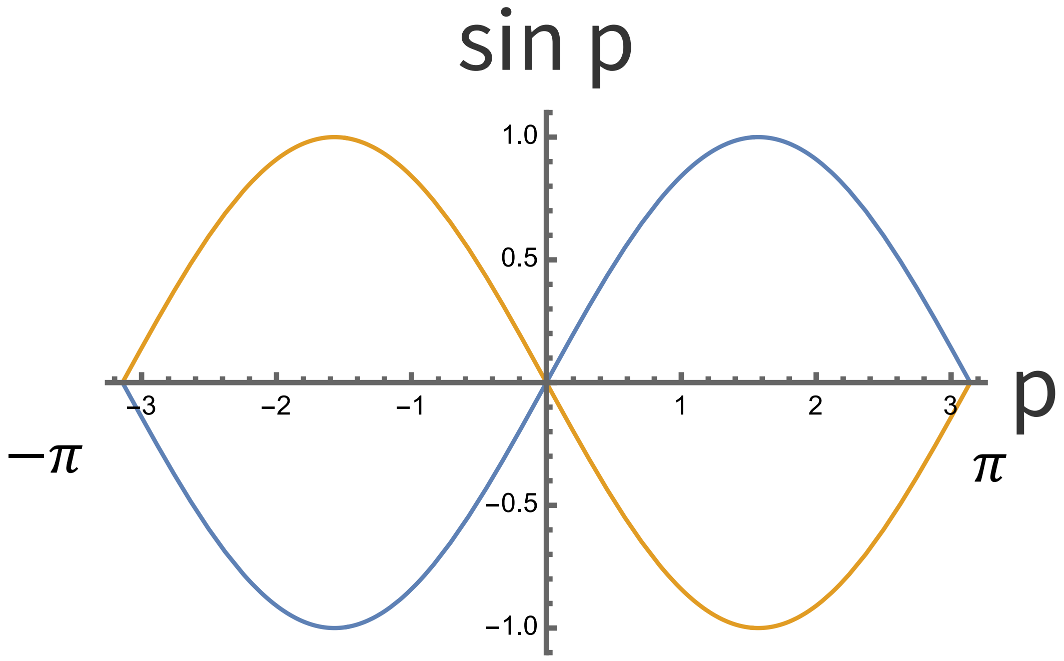
Continuum dispersion for a Dirac Hamiltonian

The no-go is better visualized using dispersion relation in Minkowski space-time (time continuous).

Hamiltonian formulation.

$$E = \pm p$$

Brilliuoin zones (Dirac)



Two Dirac fermions

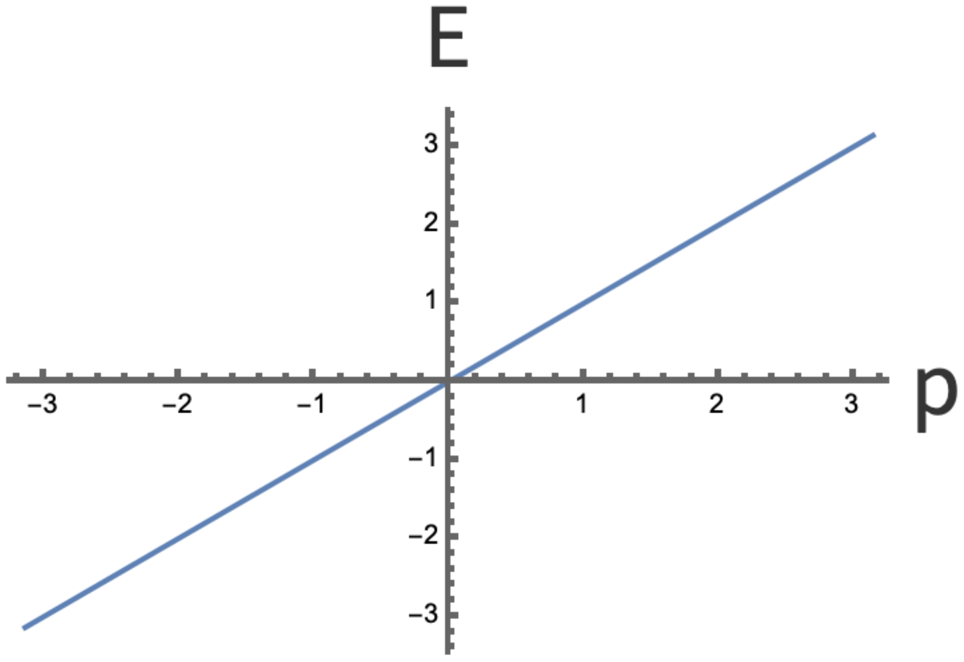
Lattice in space.

Time not discretized.

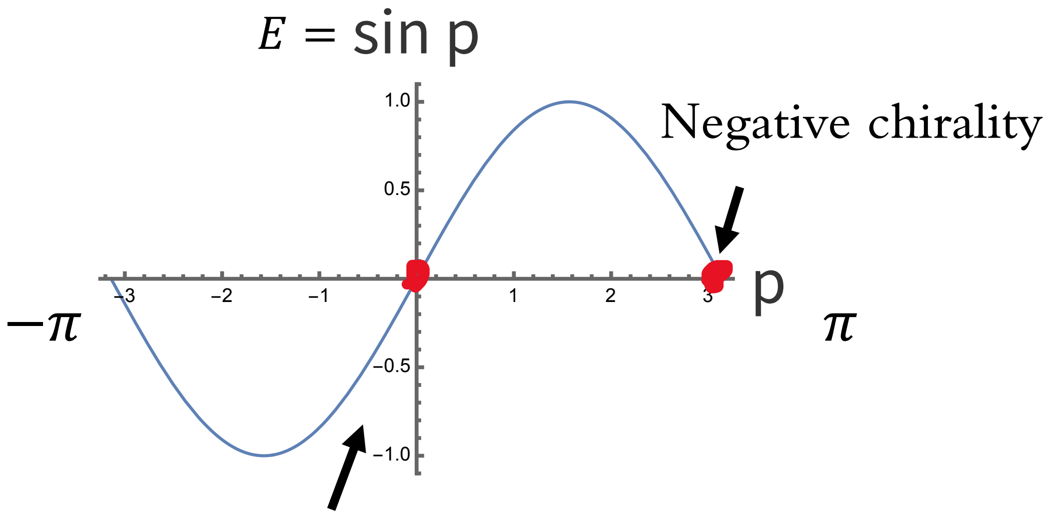
Solving the naively discretized Dirac Hamiltonian with eigenvalues $\pm \sin p$

$$E = \pm \sin p$$

Weyl fermion



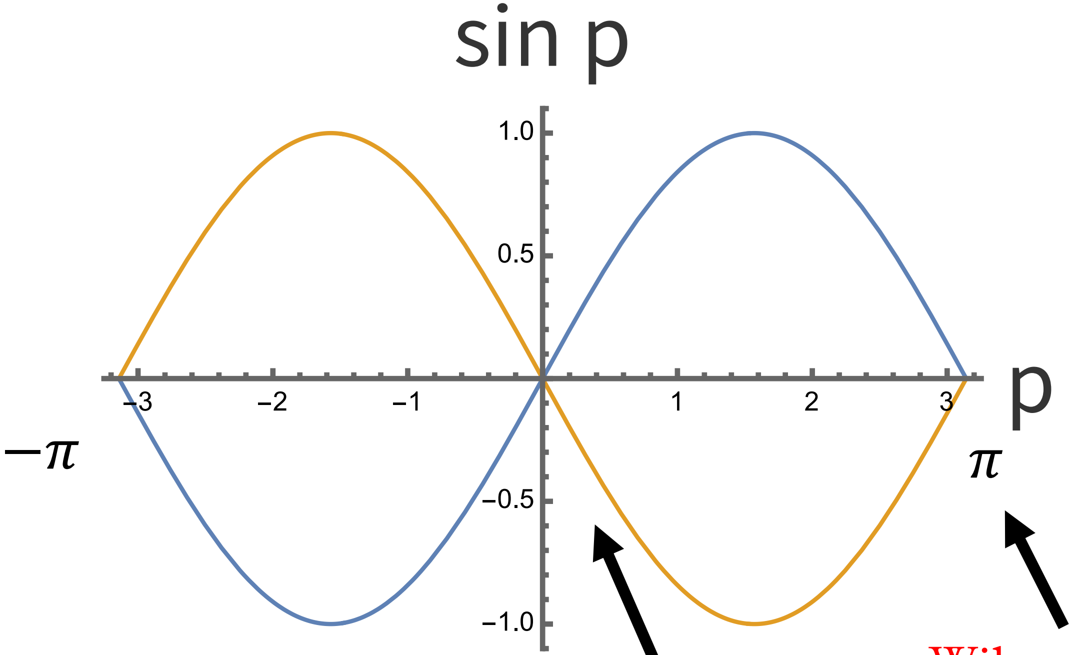
Continuum



Lattice

Even number of zero crossing of periodic functions

Wilson term for Dirac



Lattice in space.

Time not discretized.

Solving the naively discretized Dirac Hamiltonian with eigenvalues $\pm \sin p$

Wilson term removes this.
But kills chiral symmetry

$E = \pm \sin p$

Gaplessness is not protected

Single particle Hamiltonian: $\mathrm{H} = -i\gamma^1\nabla_1 + m + \frac{R}{2}\nabla$

∇_1= Symmetric finite difference in space
∇= symmetric discrete spatial Laplacian

$\nabla \rightarrow (1 - \cos p)$

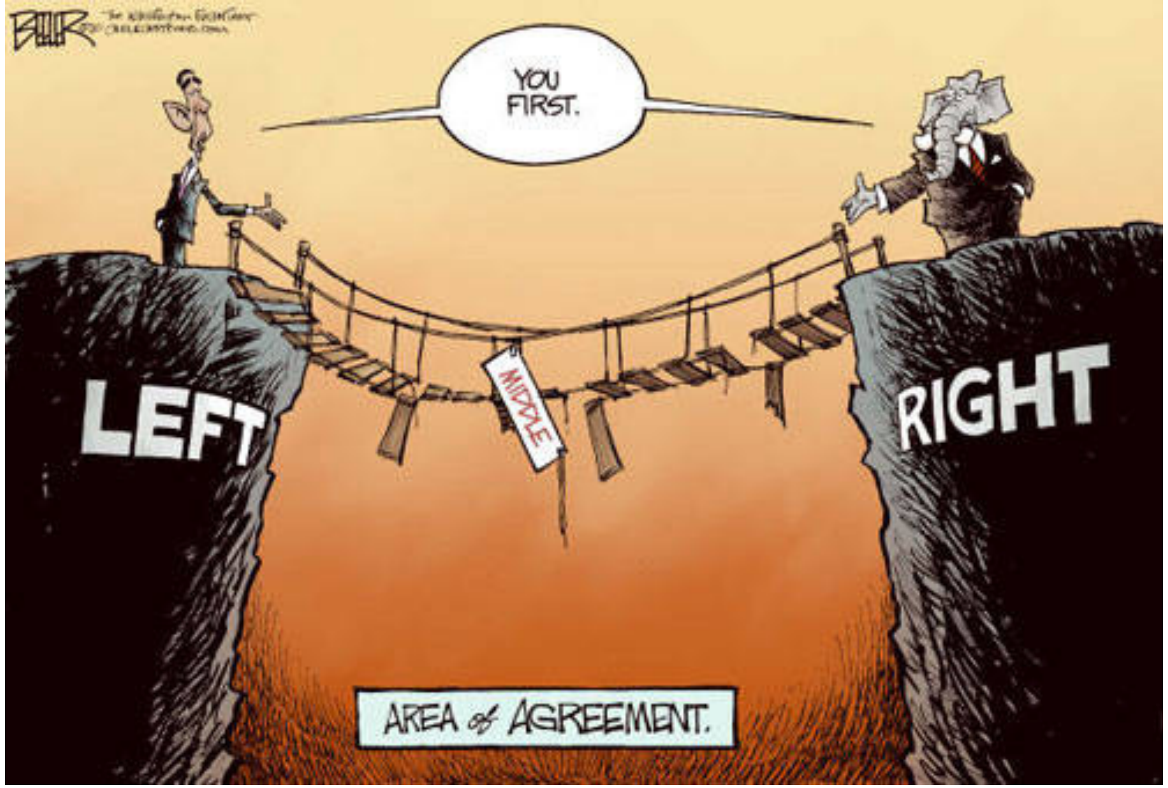
Loss of exact chiral symmetry

Quantum corrections kill massless-ness when interactions are turned on.

Undesirable if we want massless fermions.

Domain wall fermions

Domain wall fermions can solve this:
by separating the two chiralities in space,
just like the edges of a quantum Hall state.
(Kaplan, 1992)

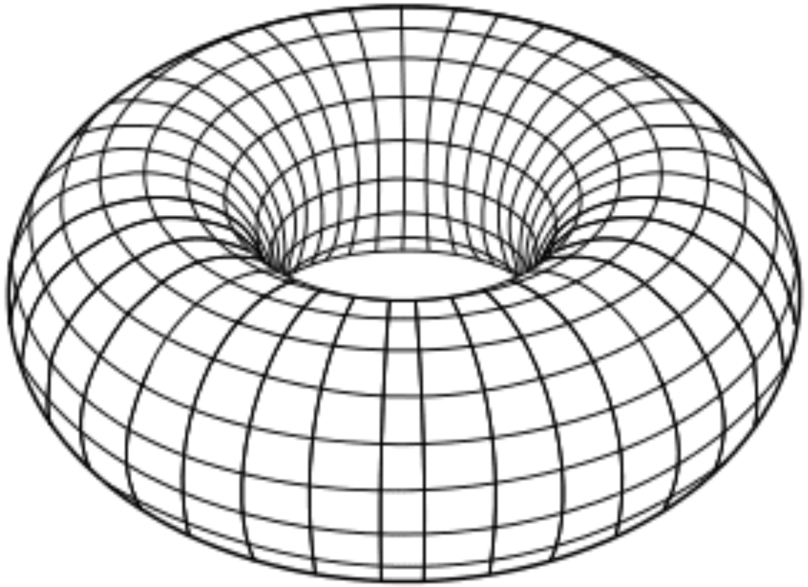


How do you do this?

Go to higher dimension. Consider 2+1 D Dirac fermion. Let's go to finite volume.

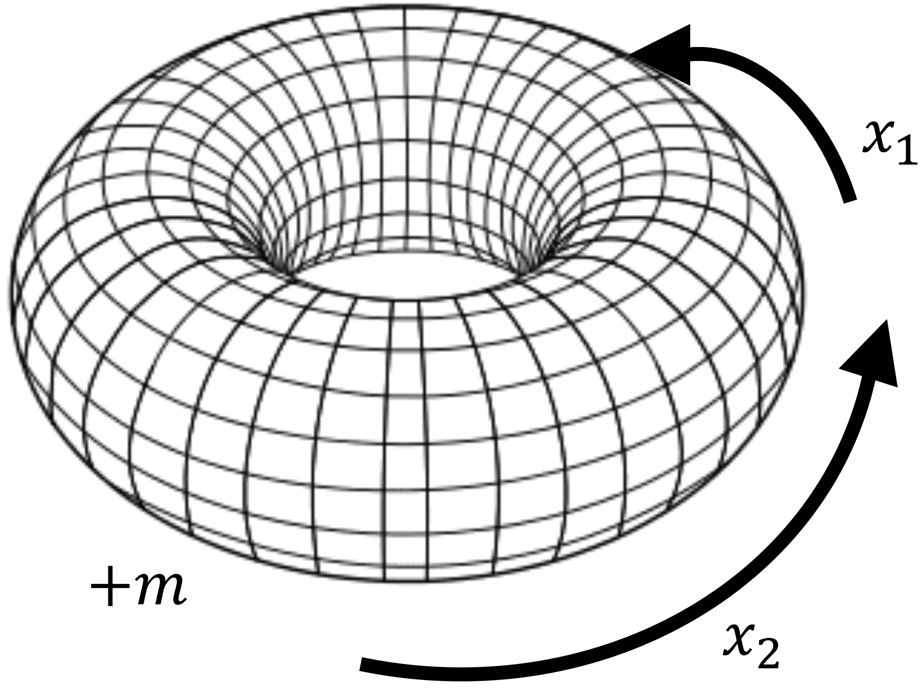
Periodic boundary condition in all directions with uniform Dirac mass.

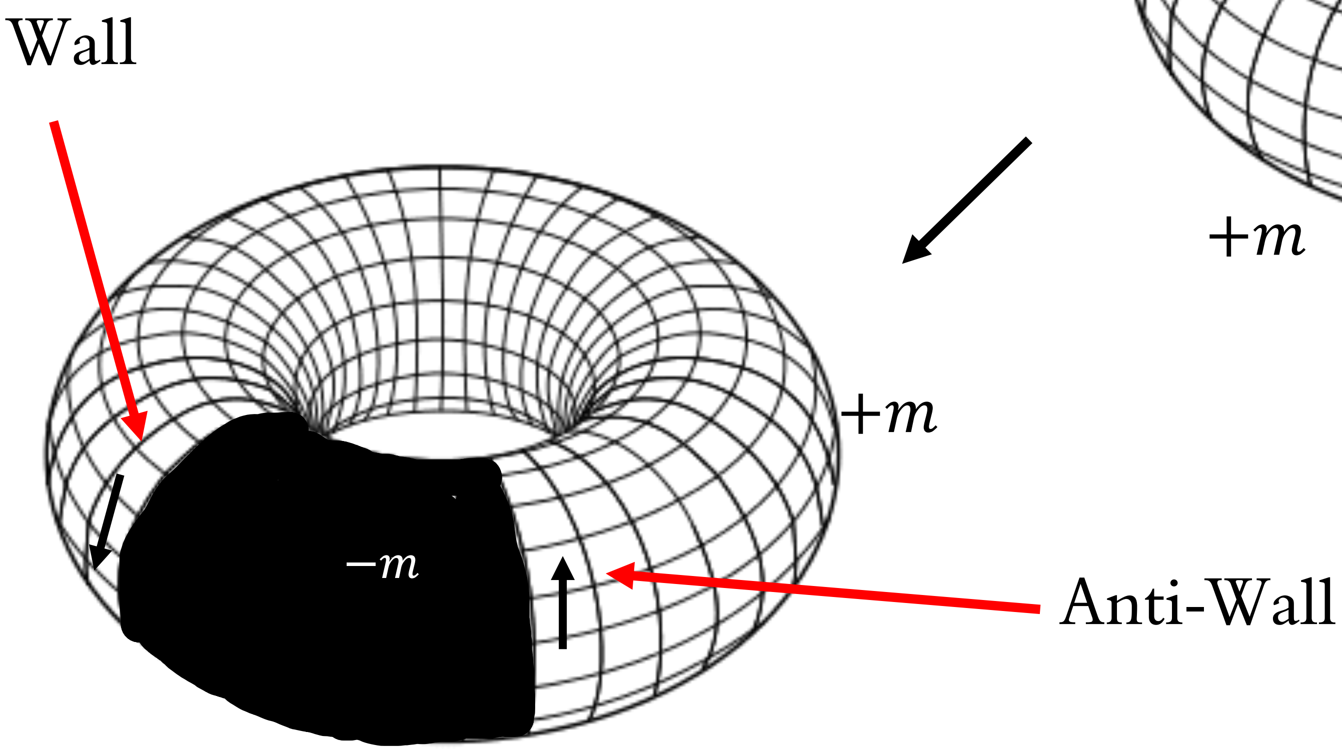
$+m$



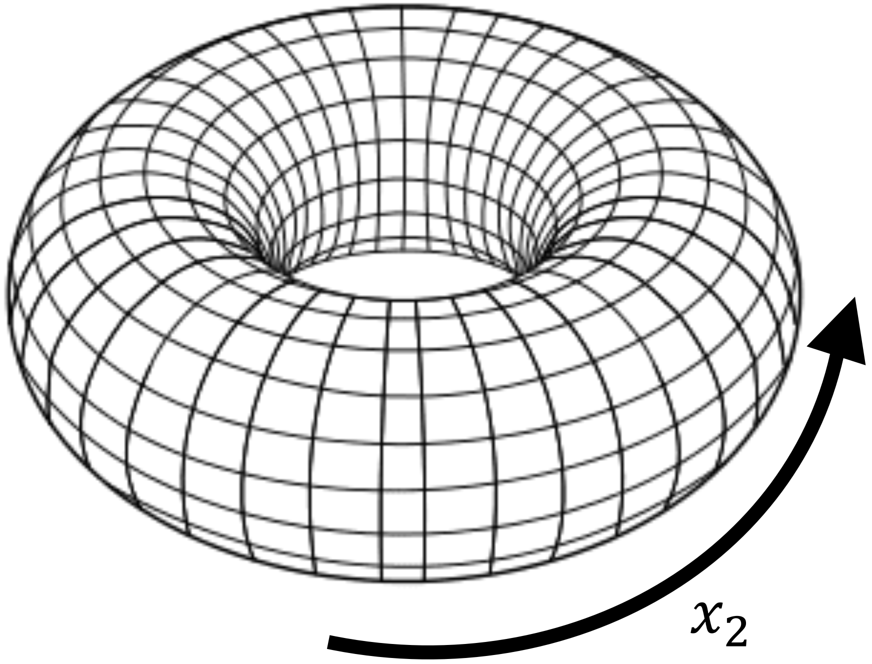
How do you do this?

Introduce domain walls in fermion mass

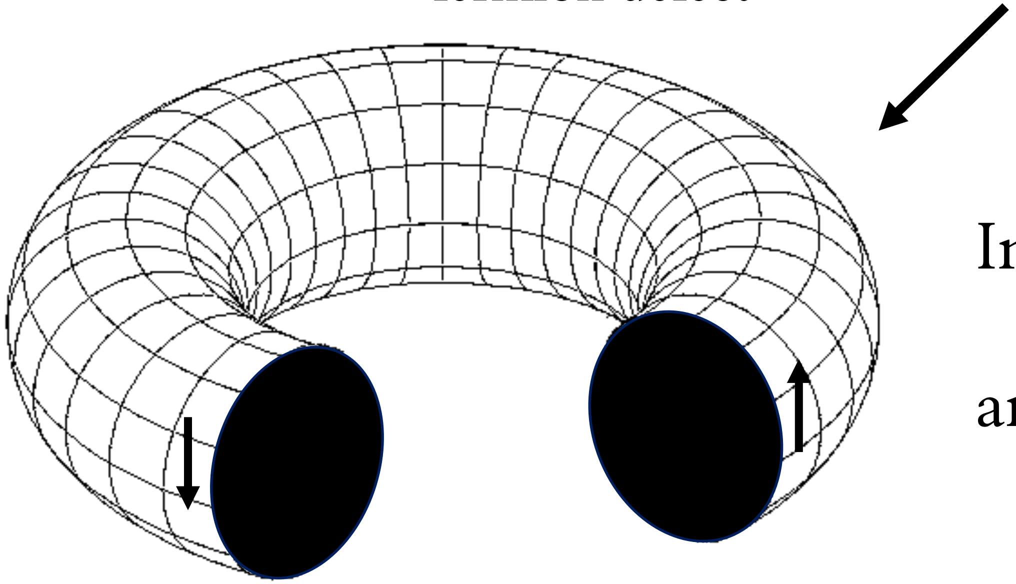




How do you do this?



Standard domain wall
fermion defect



In fact, we can get rid of the negative mass region

and work with open boundary condition in x_2

γ_2 becomes the chirality matrix

On the lattice

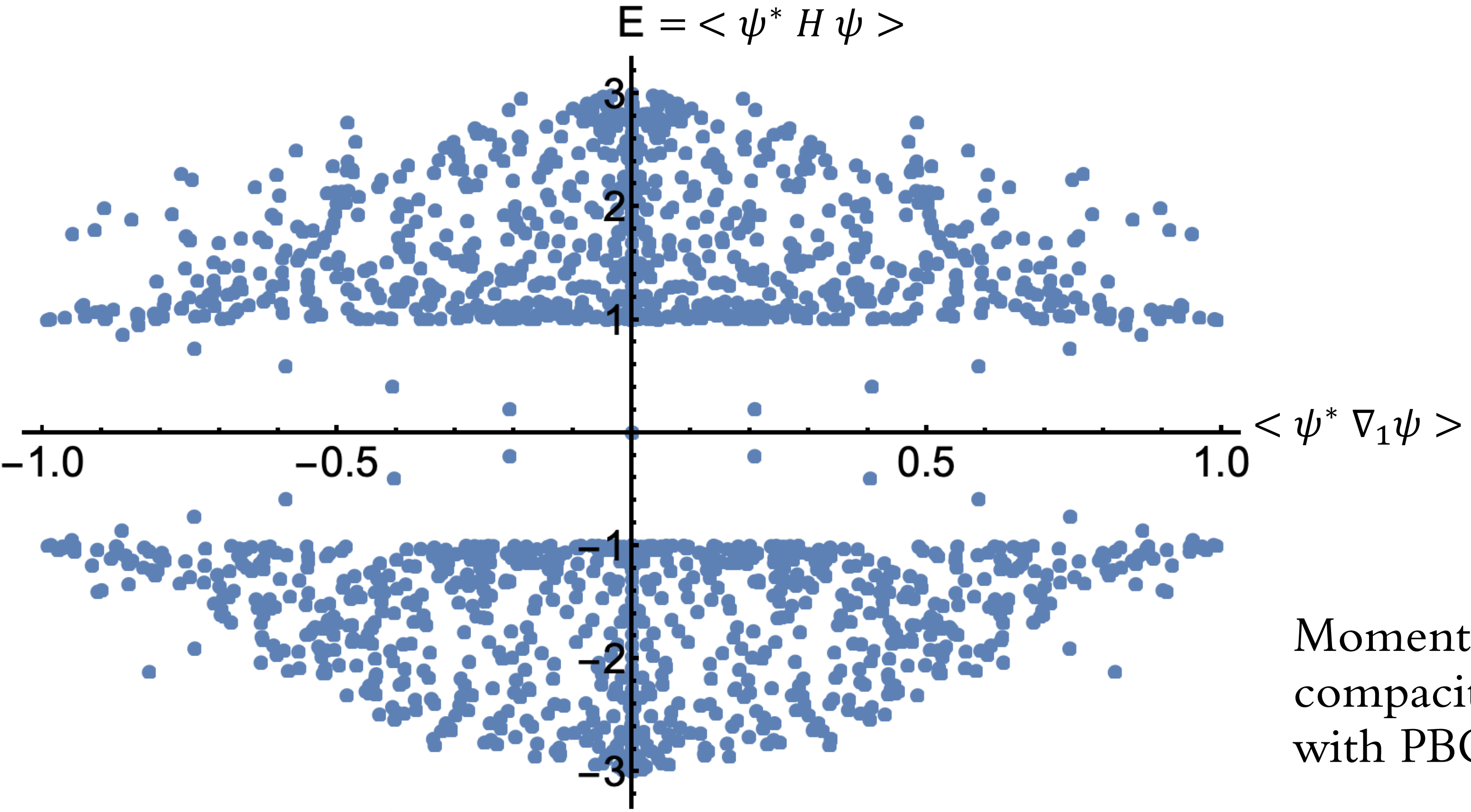
It's the Wilson fermion with no discretization in time.

Single particle Hamiltonian: $\mathrm{H} = -i\gamma^i \nabla_i + m + \frac{R}{2}\nabla$ with $i = 1,2$

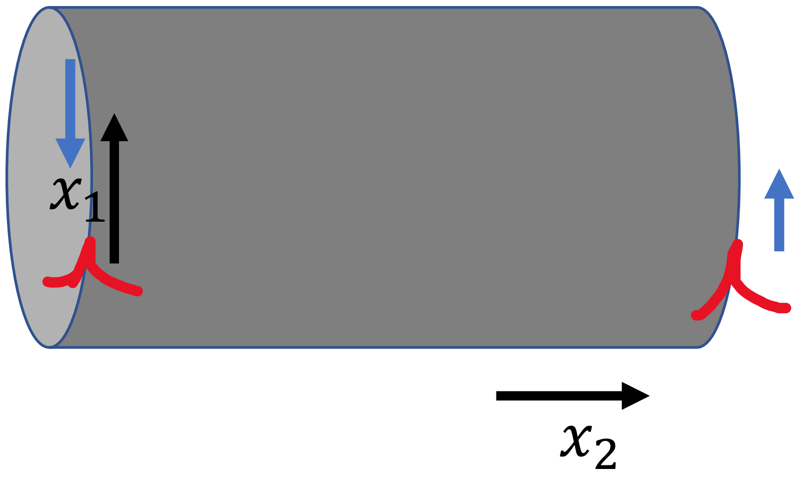
∇_i= Symmetric finite difference in space
∇= symmetric discrete spatial Laplacian

Look at the spectrum

Domain wall fermions on the lattice



Opposite chiralities



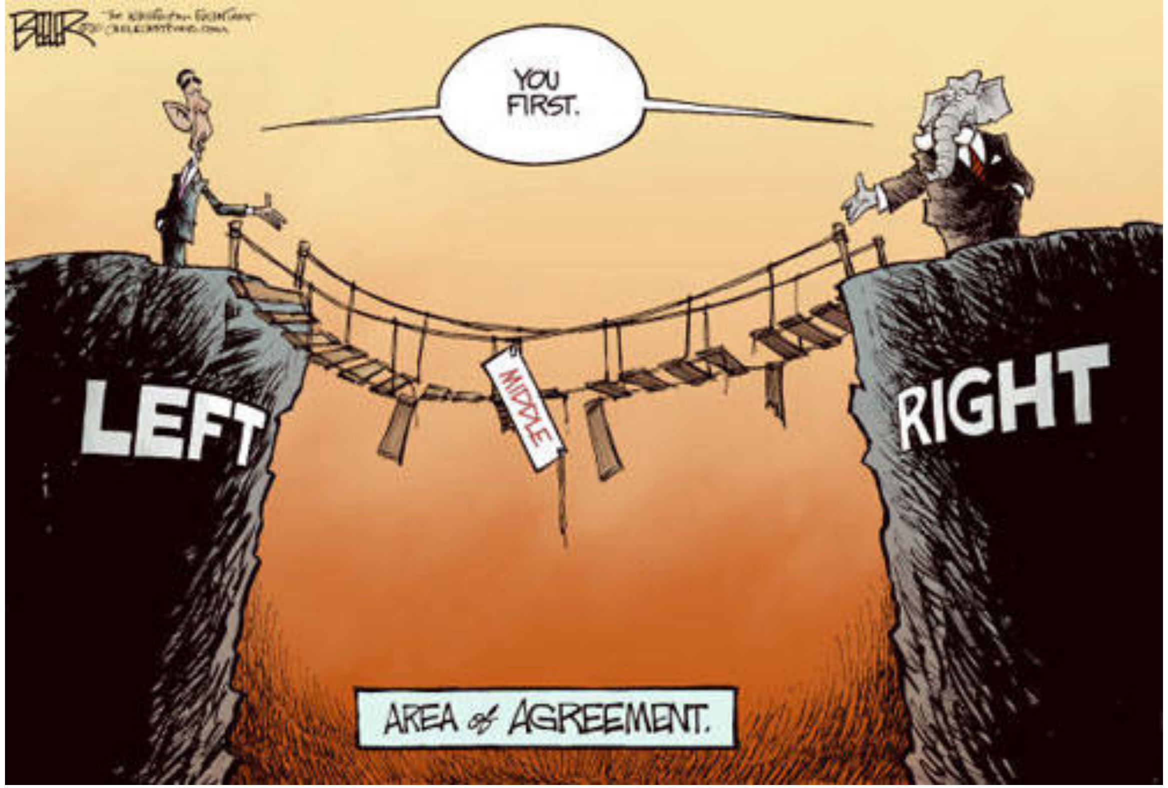
Momentum along the compacitified dimension with PBC

Solved using domain wall fermions

- Right and left moving modes separated in space. So, any quantum correction to mass exponentially suppressed.

- Allow gauge fields to talk to both walls in the same way producing a vector gauge theory.

- Very useful in QCD simulations.

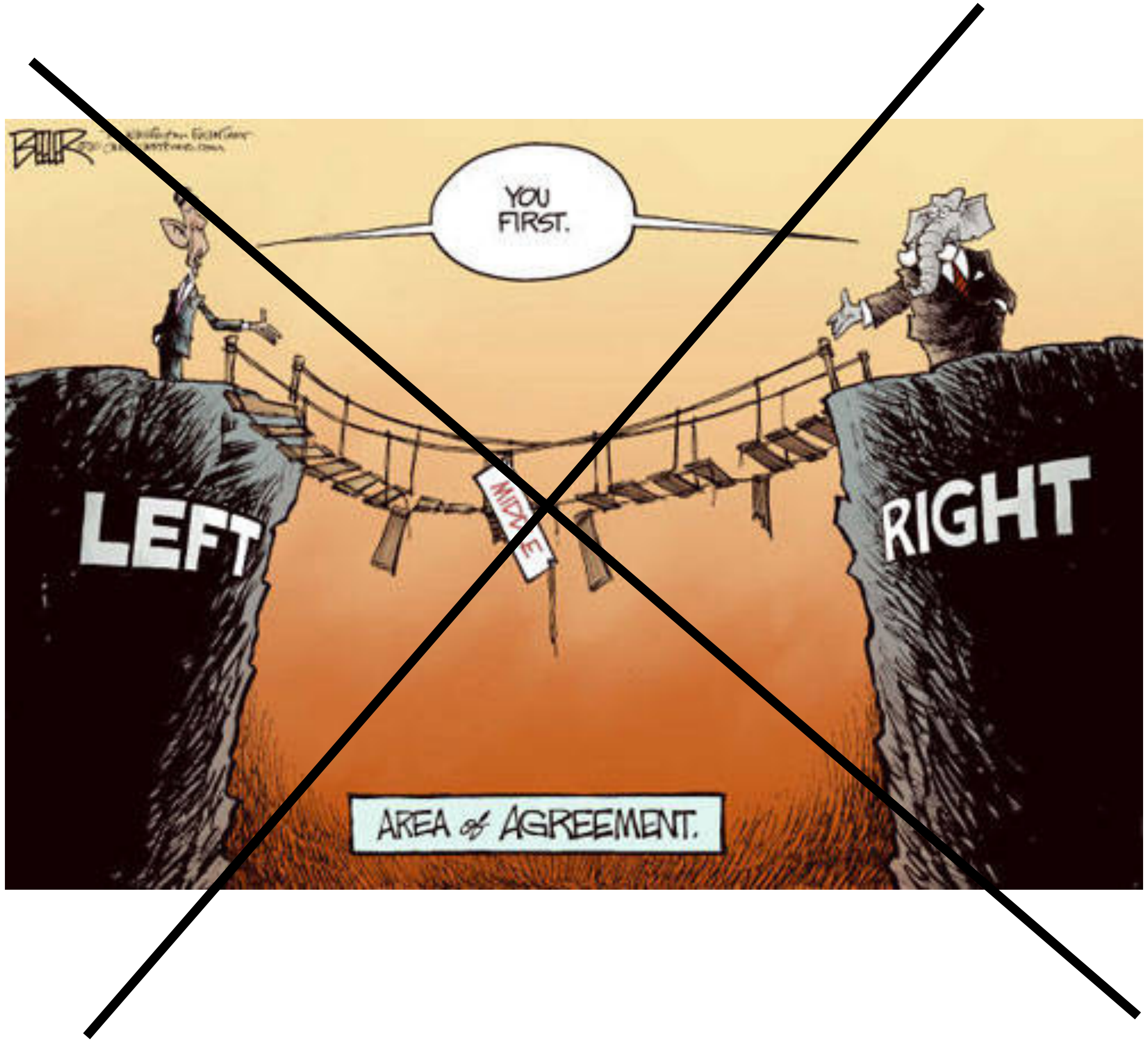


Doesn't work for chiral gauge theories

The idea does not work for chiral gauge theories though.

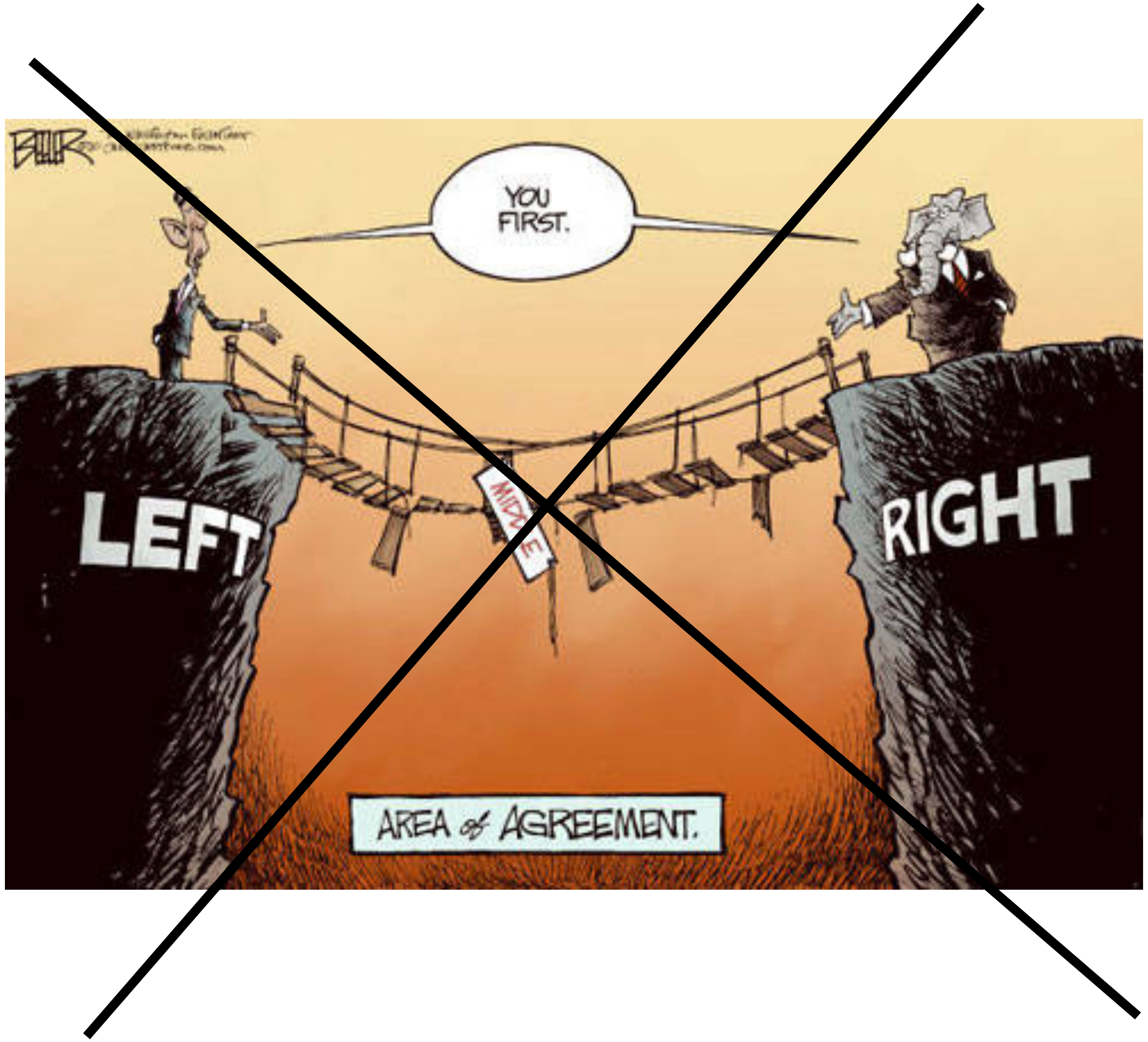
The construction in finite volume necessarily has two defects.

Two defects lead to opposite chiralities producing vector theory.



Doesn't work for chiral gauge theories

We need to isolate Weyl fermions of a particular chirality --- impossible with the standard domain wall setup.



Two tasks for CGT

- Isolate Weyl fermion content

- Gauge it if successful.

We have remained stuck here for quite long

Two tasks for CGT

- Isolate Weyl fermion content

- Gauge it if successful.

We have remained stuck here for quite long

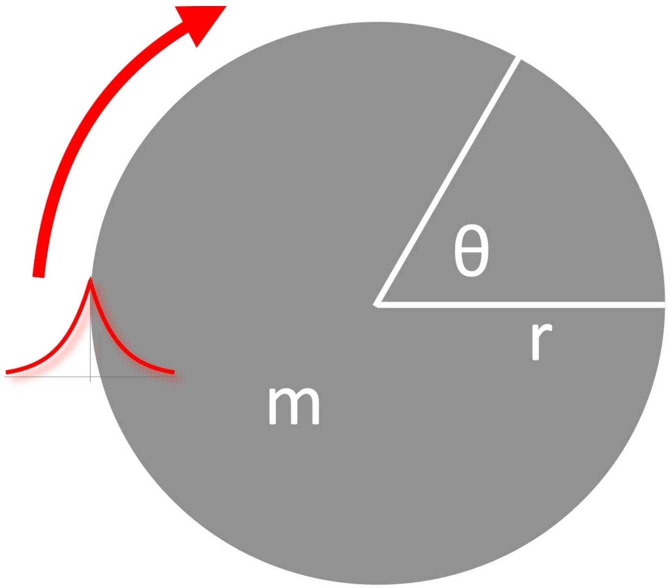
We have a solution!

A different approach

Why have two defects?

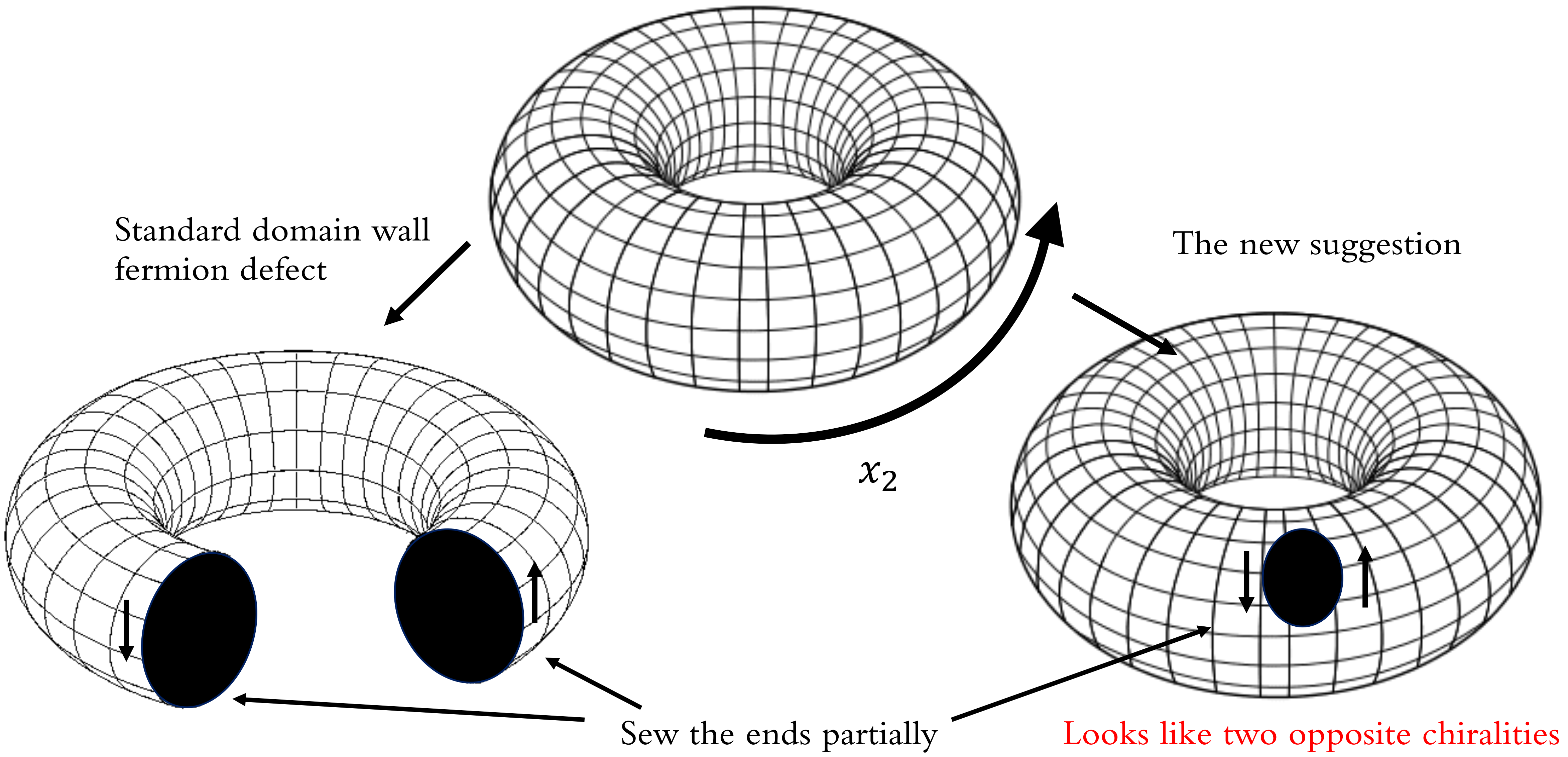
Have one.

Use a manifold with a single defect. E.g. a disc.



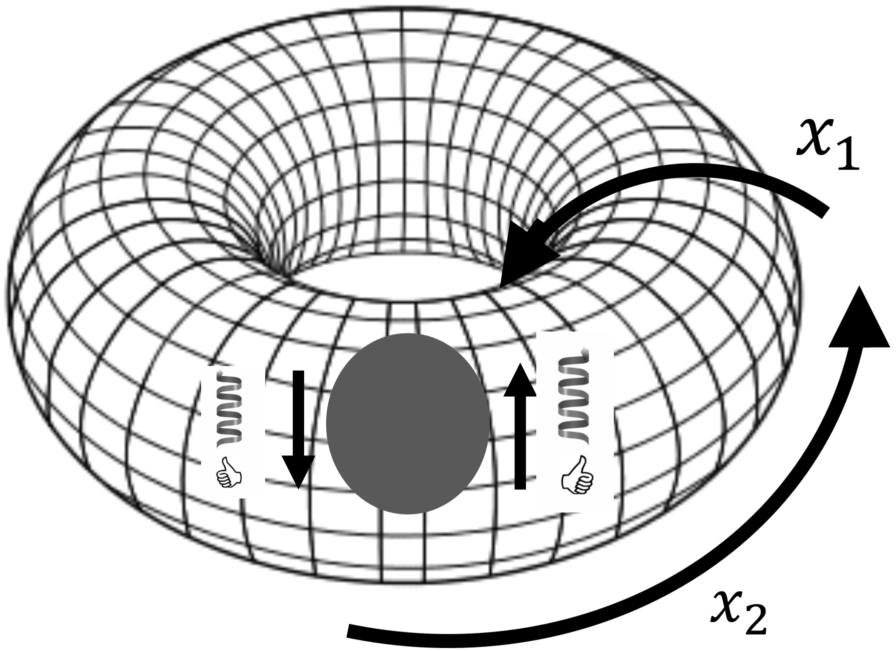
There were reasons to think this might not work

What does the disk mean?



Opposite chirality on the two sides..

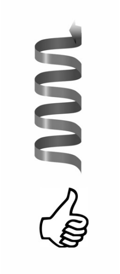
Maybe the problem is that we are keeping the definition of chirality position independent.



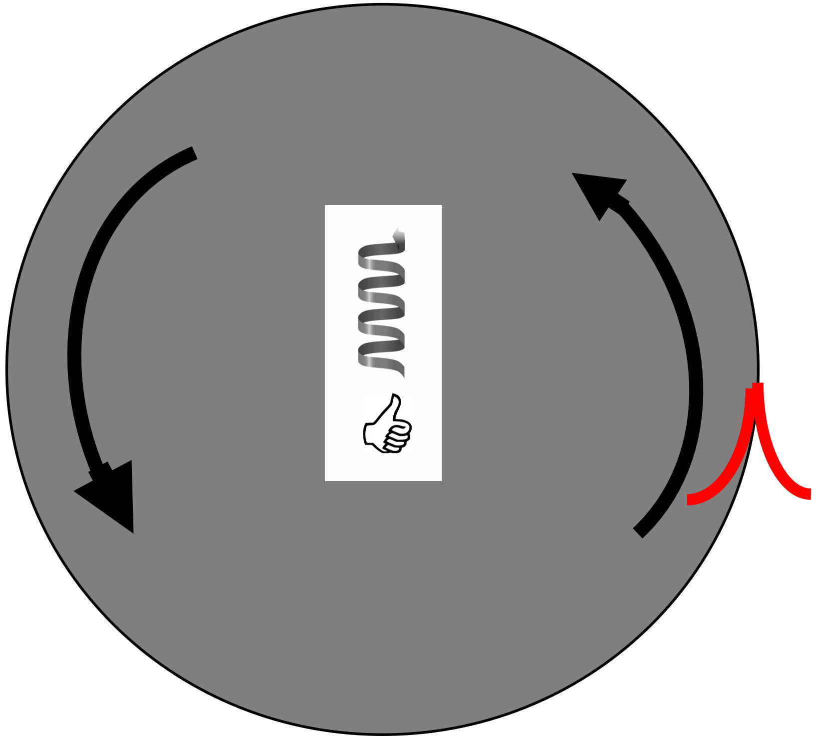
Define chirality in a position dependent manner

Define chirality as clockwise travel vs anticlockwise travel:

counter-clockwise



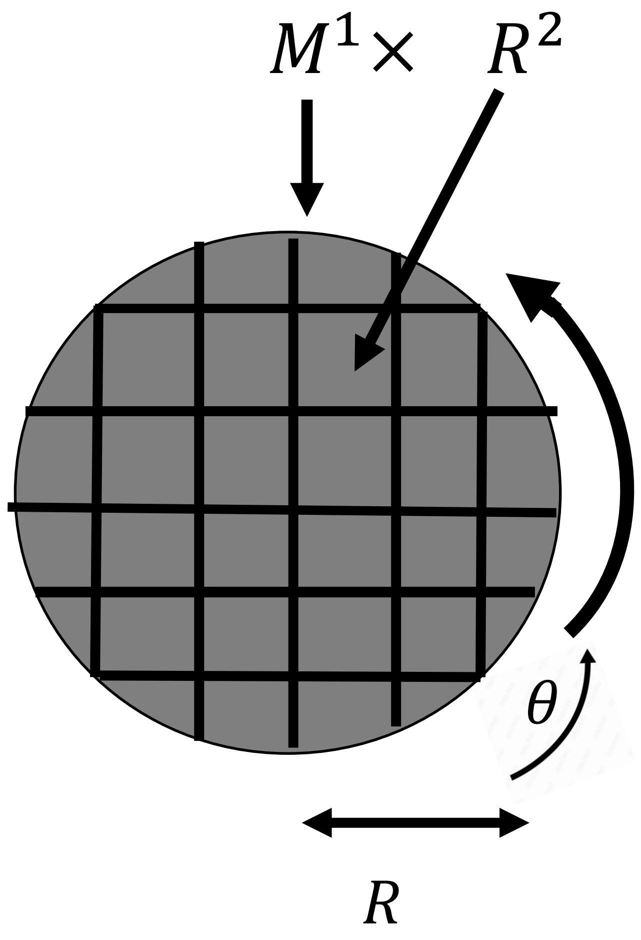
clockwise



Single chirality: Weyl mode

Phys.Rev.Lett. 132 (2024) 14, 141603 (Kaplan)

What do you see on the lattice?



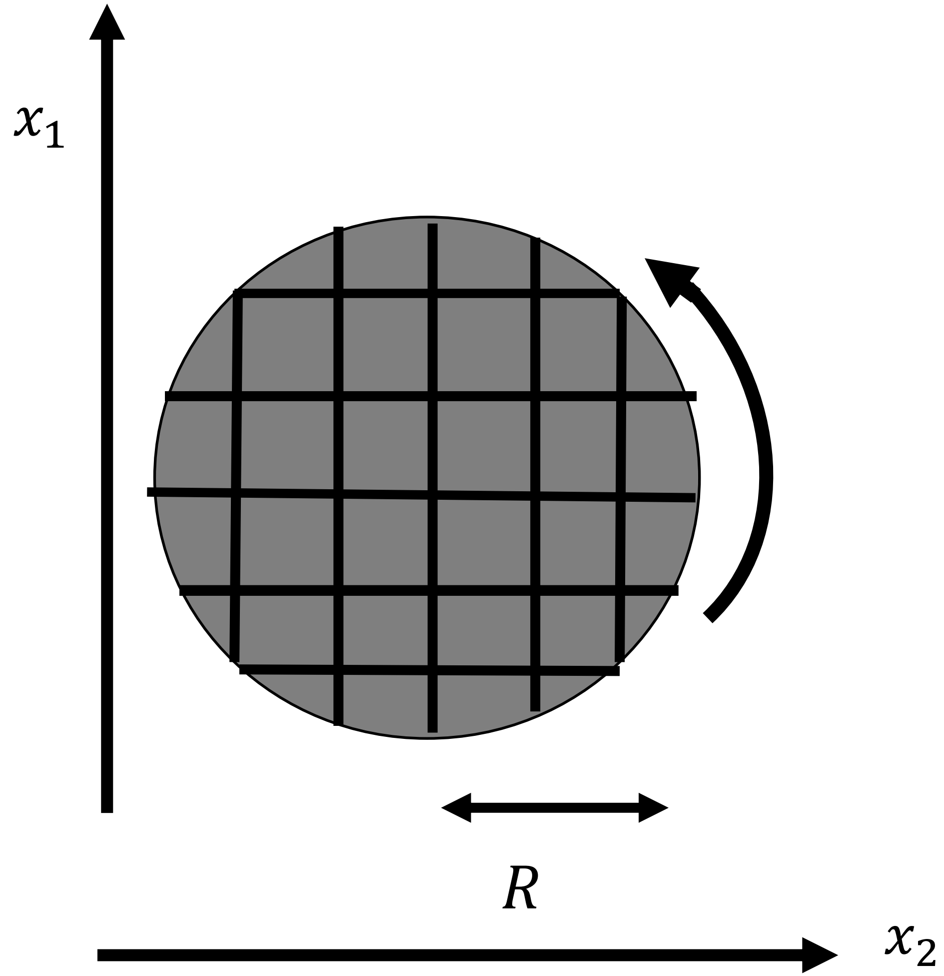
Take M^1 to be **time** and don't discretize it. Discretize r, θ (or x_1, x_2)

Find the eigenvalues of the discrete space Hamiltonian of the full theory.

What do we expect to see?

Kaplan, Sen, *Phys.Rev.Lett.* 132 (2024) 14, 141604

Disc



Check the dispersion.

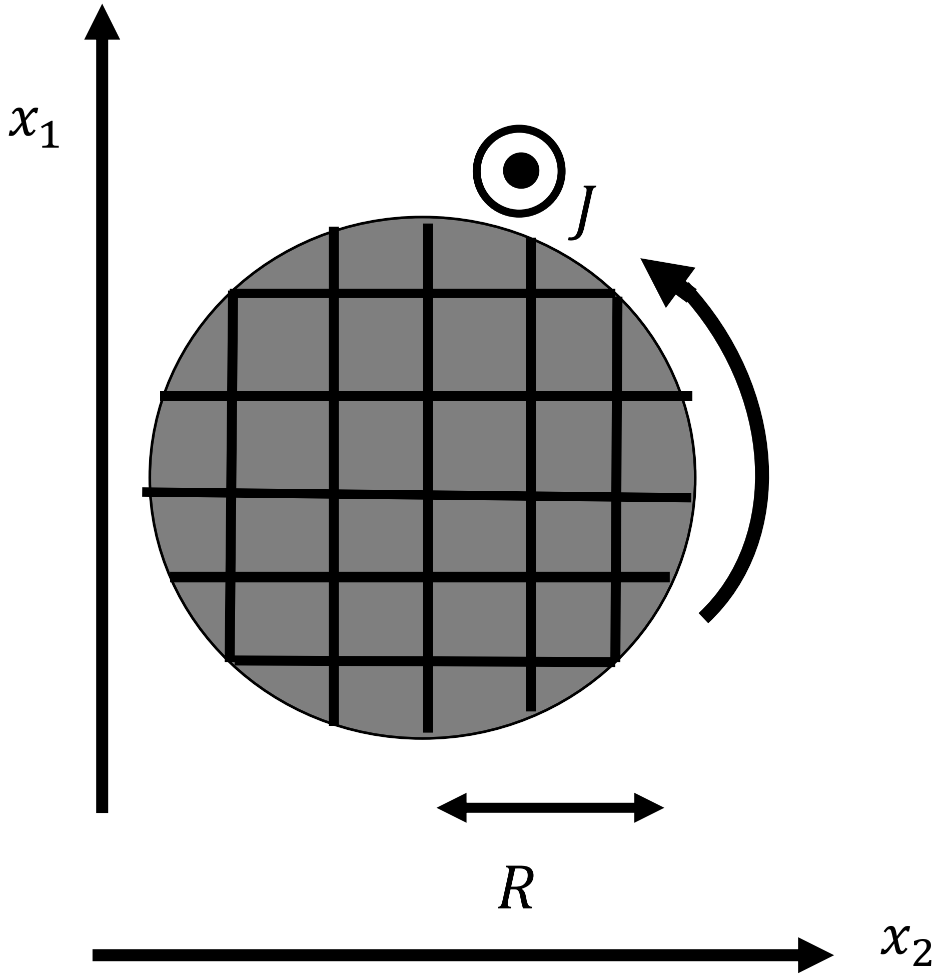
How?

Broken translation invariance along both x_1 and x_2

Does not make sense to plot E vs p_1

Kaplan, Sen, *Phys.Rev.Lett.* 132 (2024) 14, 141604

Disc



We have rotational invariance (approx).

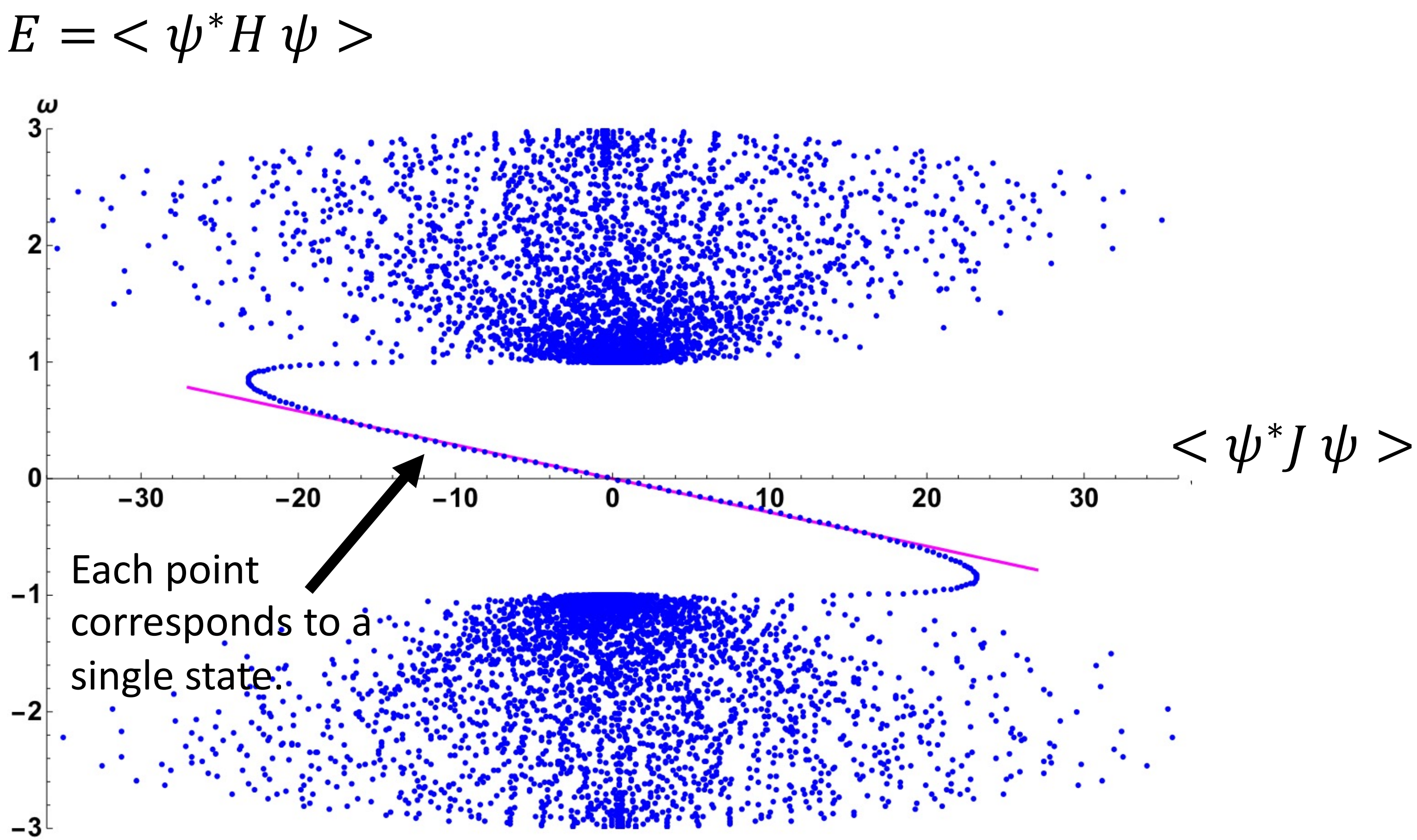
Diagonalize the lattice Hamiltonian.

Compute expectation values of angular momentum J

Plot E vs J

Kaplan, Sen, *Phys.Rev.Lett.* 132 (2024) 14, 141604

Dispersion for the disk



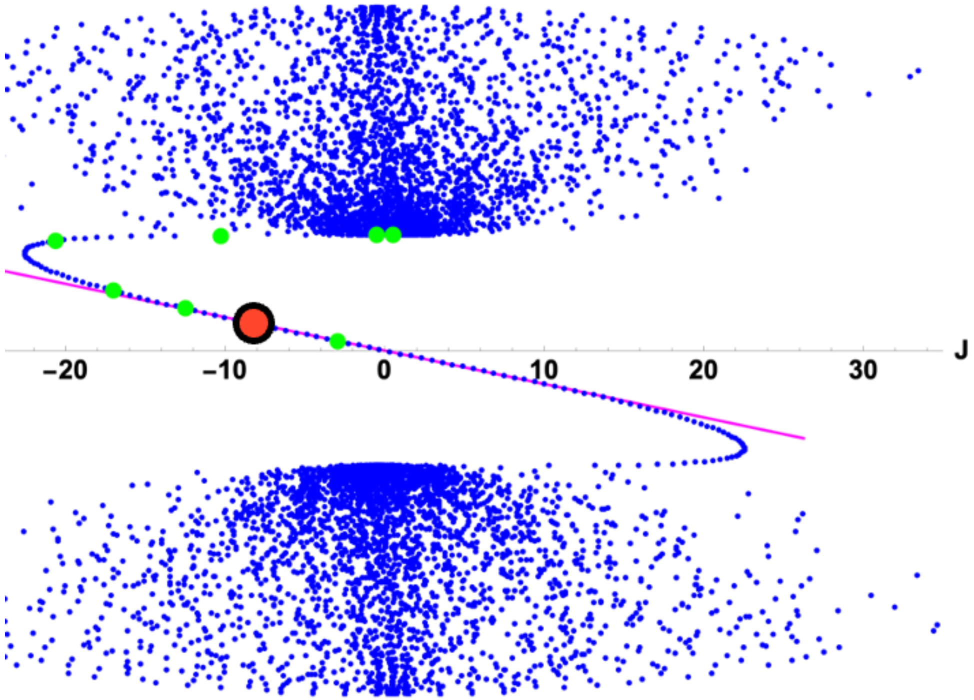
Exactly as expected from the continuum

Disk of radius $R = 34$ in lattice units.

Linear dispersion:

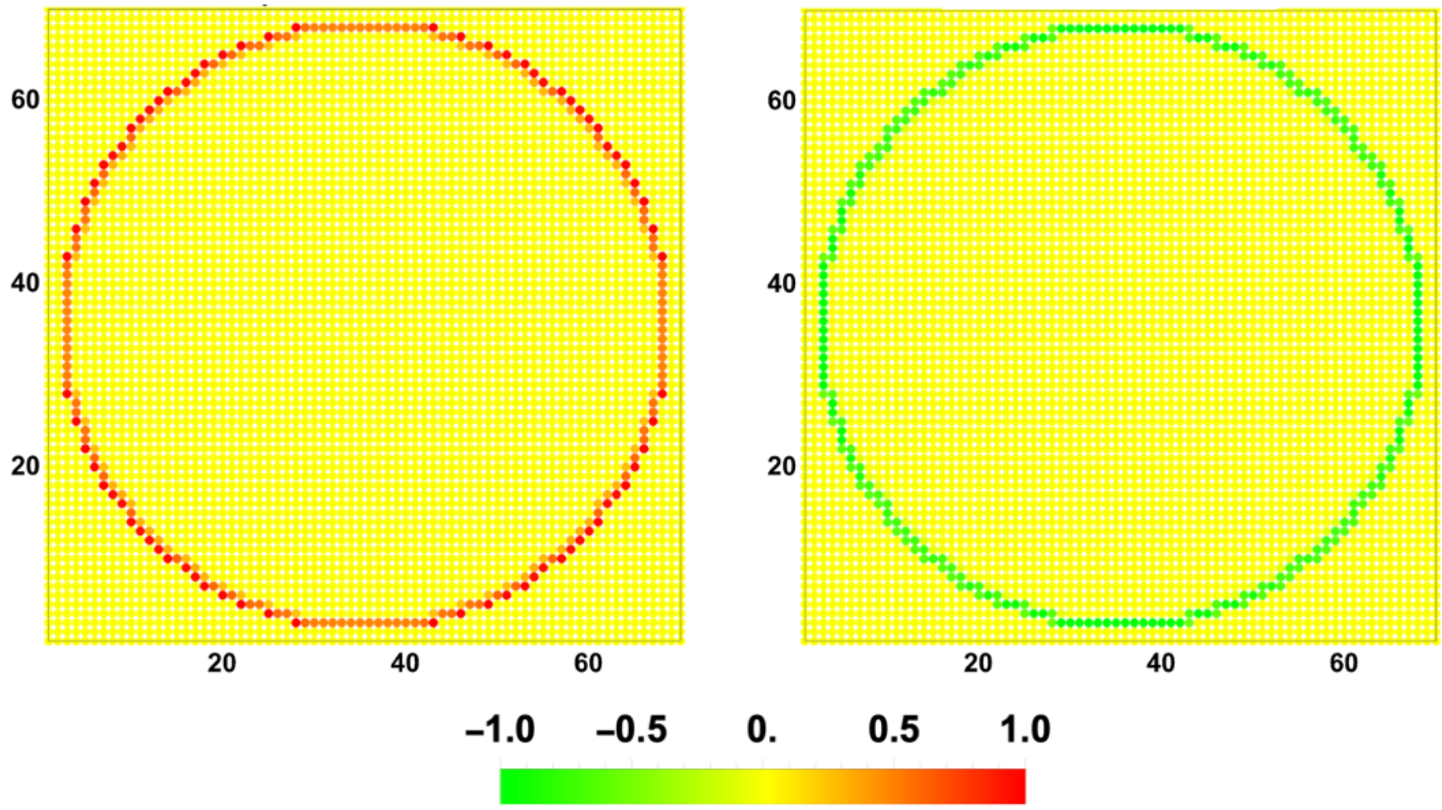
$$E = -J/R$$

Kaplan, Sen, *Phys.Rev.Lett.* 132 (2024) 14, 141604



charge density ρ

current density j_θ



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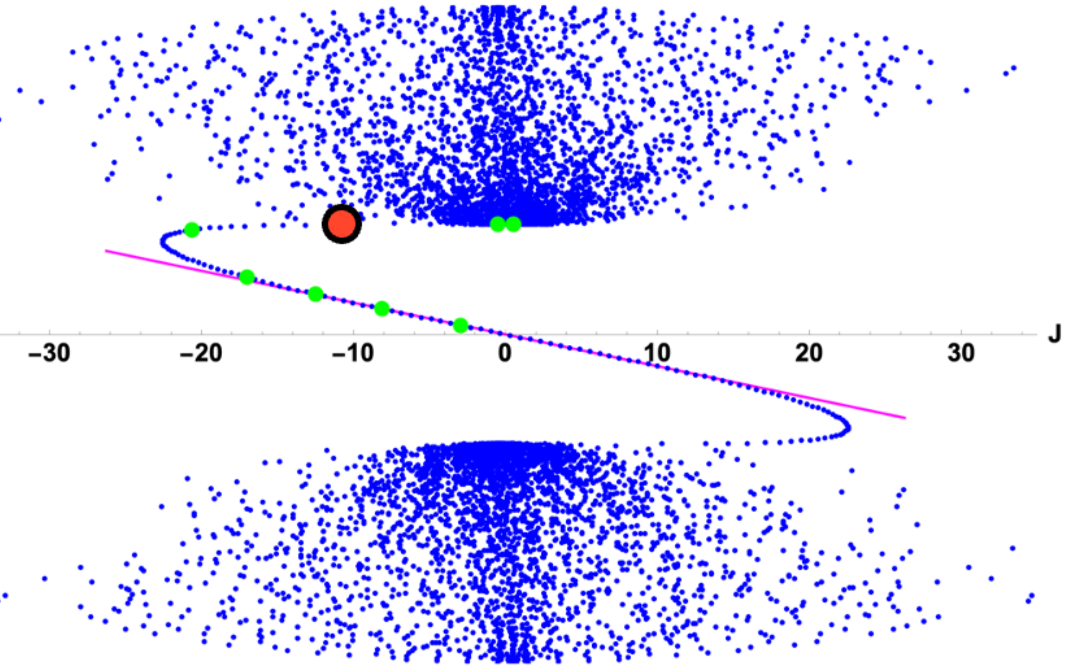

charge density ρ

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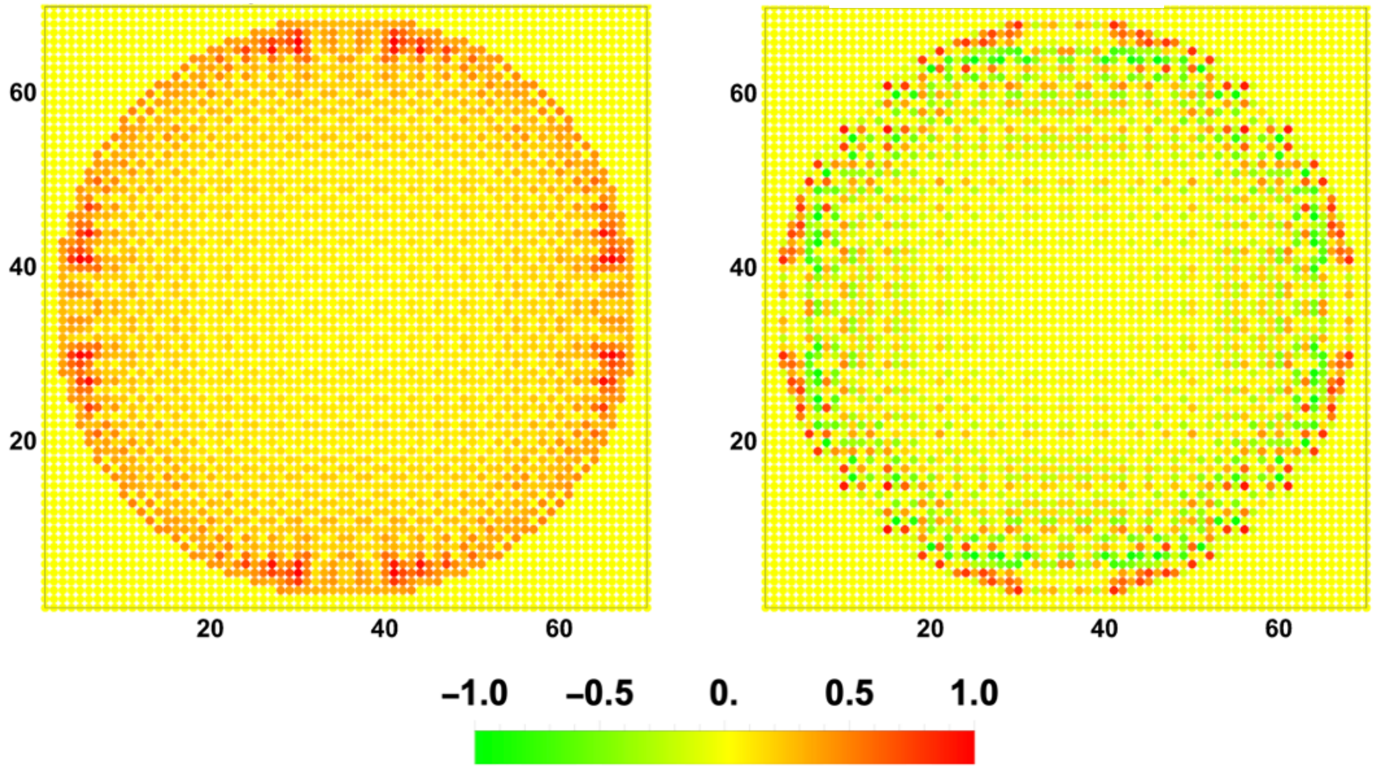


Kaplan, Sen, *Phys.Rev.Lett.* 132 (2024) 14, 141604

Two tasks for CGT

- Isolate Weyl fermion content

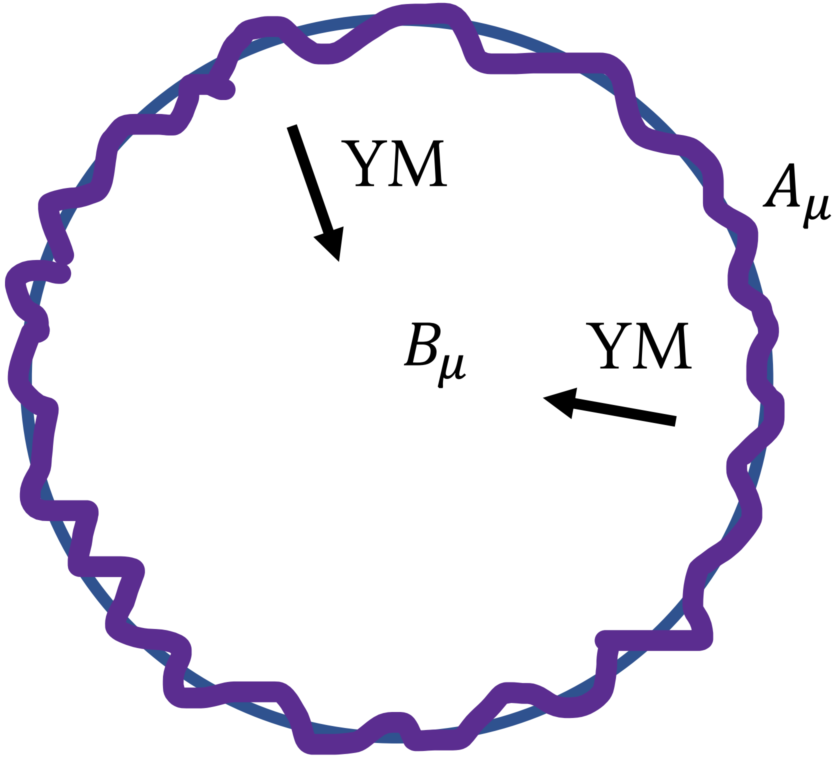
This step is done.

- Proposal for gauging Next.

Gauging

Want a $d = 2, 4$ dimensional gauge field and not $d + 1 = 3, 5$ dimensional one.

$d = 2, 4$ dimensional gauge field B_μ in $d + 1$ dimensional bulk.



Integrate over the boundary gauge field A_μ

Bulk gauge field satisfies equations of motion (e.g. YM) while matching A_μ on the boundary.

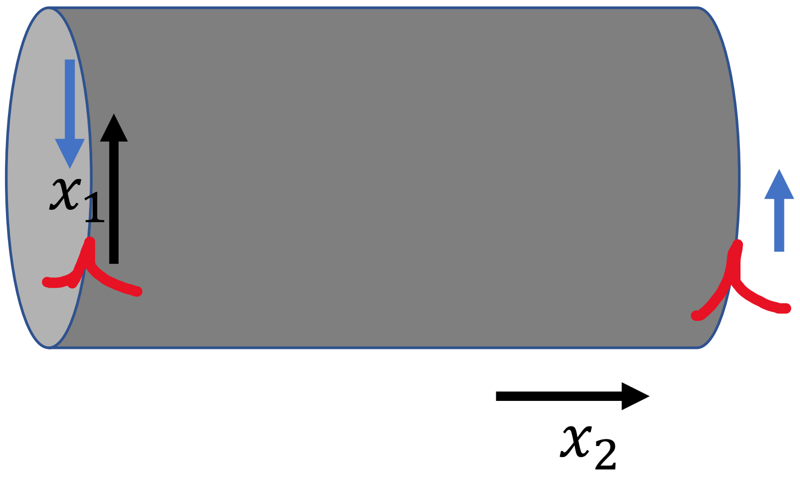
Gauging

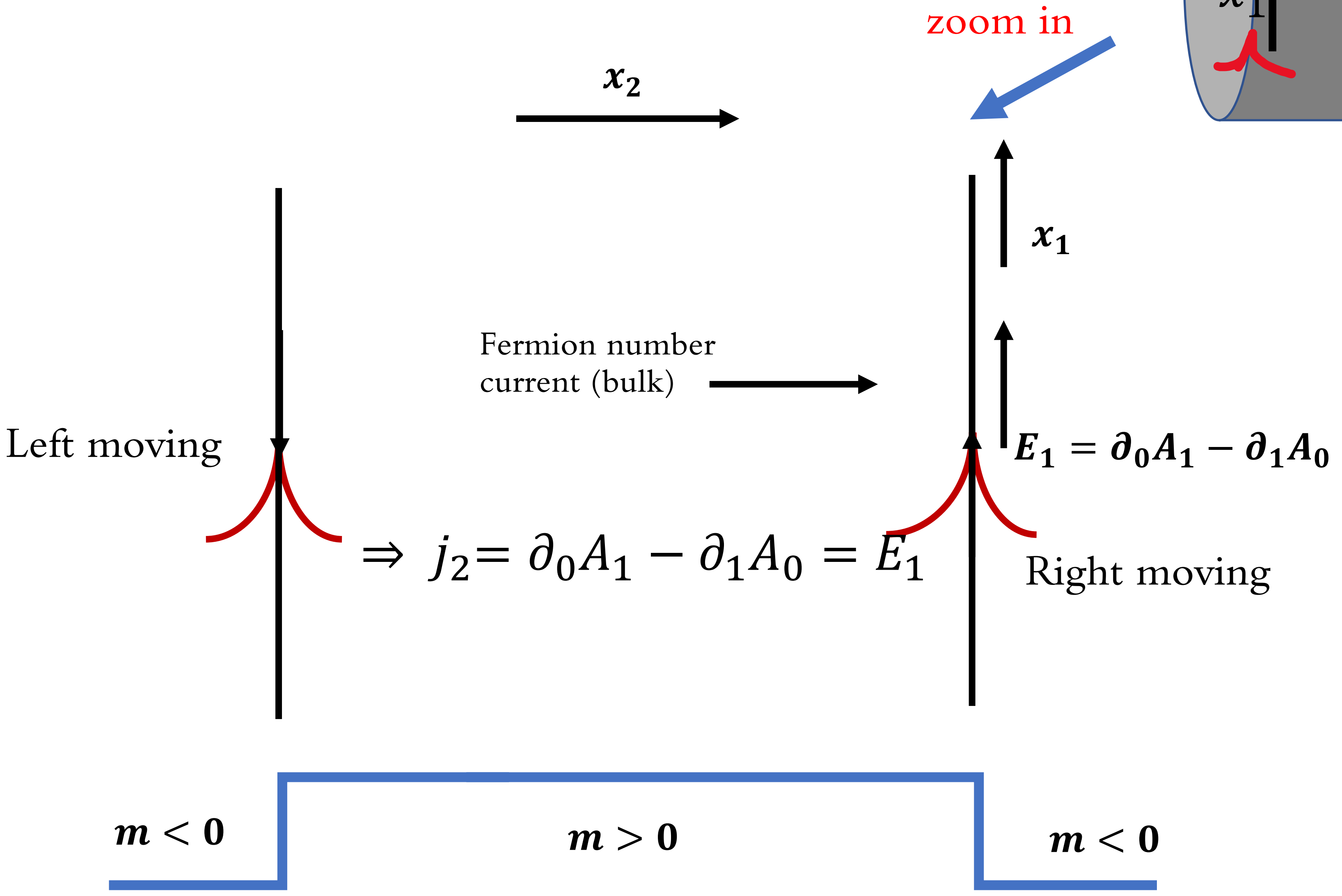
Can engineer any number of Weyl fermions on the boundary.

We can gauge any subgroup of the available global symmetry of the free fermion theory. ---- Must make sense only if the theory is anomaly free.

How do we see this?

Domain wall + anomaly



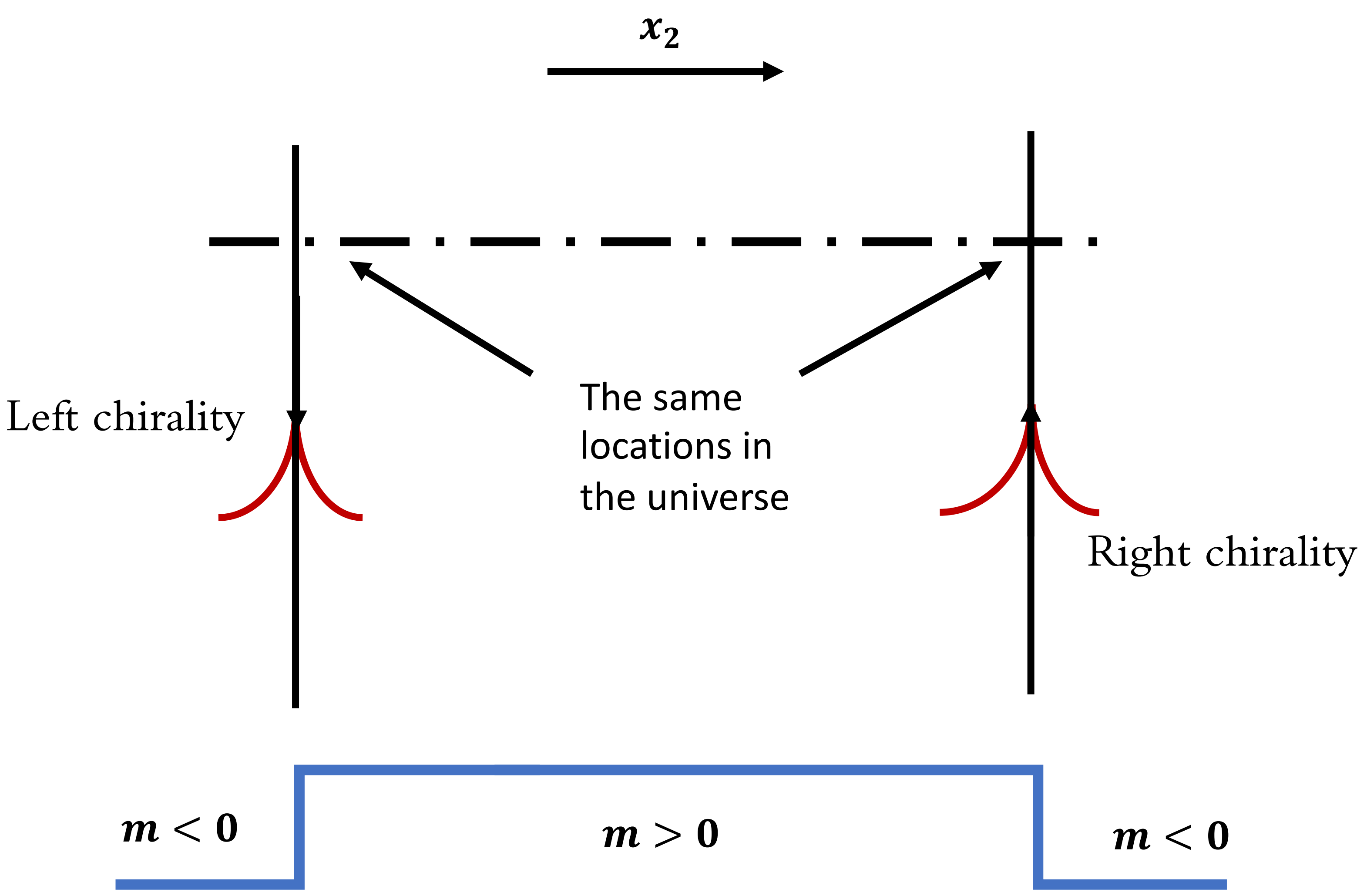


2 + 1 D Dirac fermion setup with domain walls

Gauge the Dirac fermion

Bulk talks to the boundary, great for QCD

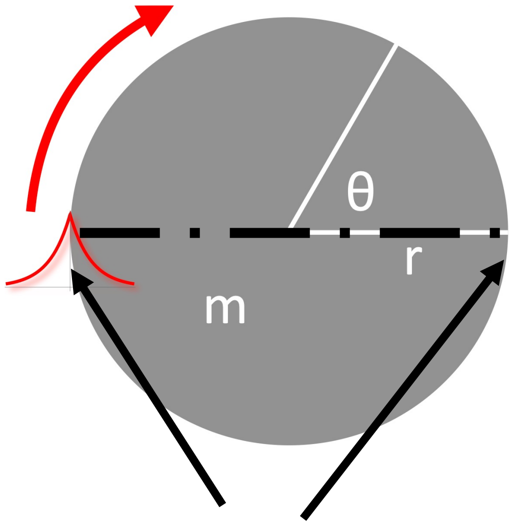
Domain wall + anomaly + QCD



Bulk talks to the boundary, great for QCD

The two walls talk to each other

disk + anomaly



Different locations in the universe, shouldn't communicate across the defect

But they will if the boundary theory has gauge anomaly: nonlocal!!

Thankfully, the standard model is anomaly free.

So, the disk construction makes sense, and the boundary theory is local.

Summary

We have a sensible microscopic theory of a Dirac fermion which at low energy produces a single Weyl fermion on the lattice.

Nielsen Ninomiya is not an obstacle. We were fixated on the wrong kind of defect.

Removes one of the most significant obstacles of realizing a chiral gauge theory.

Criticisms and resolutions

Golterman-Shamir: Phys.Rev.D 109 (2024) 11, 114519

For every Weyl fermion on the boundary, we need a Dirac fermion on the disk.

To get a Dirac fermion on the boundary, we will need two flavors of Dirac on the disk.

Each has its own exact $U(1)$ conserved currents.

Can construct a $4-$dimensional conserved current by integrating along the radial coordinate of the disk (Golterman-Shamir).

Seems to lead to exactly conserved $U(1)$ currents for both vector and axial symmetry.

E.g. for one–flavor QCD on the boundary, there appears to be two exactly conserved $U(1)$ currents, vector and axial.

Is this a bug or a feature?

What could go wrong?

Consider 1- flavor QCD as the target theory.

Expect only a massive η' in the spectrum.

If the $U(1)$ axial symmetry is exact, you may end up with a Goldstone mode instead.

Undesirable.

Easy to see this in EFT

Integrate out the heavy bulk fermion modes:

$$J_\alpha(x,r) = \delta(R-r)\big(1-\delta_{\alpha,r}\big)\bar{\chi}\sigma_\alpha\chi - i\,\kappa\,\theta(R-r)\mathrm{Tr}[\epsilon_{\alpha\beta\gamma\rho\sigma}F_{\beta\gamma}F_{\rho\sigma}\,]$$

5D conserved current

chiral edge states current

bulk gauge field current

GS current: $j_\mu = \int_0^R \frac{r}{R}\,dr\,J_\mu\,,\;\; \mu = 1,2,3,4$

From 5D conservation of J_α, one can show 4D conservation of j_μ

This 4D conservation gives the standard anomalous Ward identity for the edge current:

$$\partial_\mu\big(\bar{\chi}\sigma_\mu\chi\big) = -\frac{1}{16\pi^2}\epsilon_{r\nu\lambda\rho\sigma}\mathrm{Tr}[F_{\nu\lambda}F_{\rho\sigma}]$$

Seems to not be a bug then.

Criticism 2

Aoki, Fukaya: arXiv:2402.09774, arXiv:2404.01002

Gauge field configs with nonzero winding become singular at the center.

This singularity can give rise to zero modes there.

So, integrating out bulk fermions may not be justified in the presence of net winding configs?

This is tied to Golterman-Shamir.

Way out?

- No such zero mode in the bulk if gauge field configs are restricted to zero net winding.

- What does this mean for QCD?

- In QCD, forcing $Q = 0$ will engineer an exact $U(1)$ symmetry without making η' massless (*Leutwyler - Smilga*).

To implement $Q = 0$, write a θ term for QCD, $\theta\, F\tilde{F}$ in the Lagrangian and treat θ as a Lagrange multiplier.

EFT:

η' EFT: $L_\theta = \frac{1}{2}\left(\partial_\mu \eta'\right)^2 + \Lambda^4 \cos\left(\theta + \frac{\eta'}{f}\right)$ and $Z = \int d\theta\, Z_\theta$

Strong scale related to Λ

Expand in small η up to quadratic order:

$$S_{\eta'} = \int d^4x \left(\frac{1}{2}\left(\partial_\mu \eta'\right)^2 - M^2 \eta'^2\right) + \frac{\mathrm{M}^2}{\mathrm{V}} \int d^4x\, \eta'(x) \int d^4y\, \eta'(y)$$

Arxiv: 2412.02024

Way out

The dispersion:

$$S_{\eta'} = \int \frac{d^4k}{(2\pi)^4} \eta'(-k) \left(k^2 - M^2 - \frac{\delta^4(k) M^2}{V}\right) \eta'(k)$$

isolated state at zero momentum

No massless η'

It has been shown that net $Q = 0$ observables, map to $\theta = 0$ QCD up to inverse volume corrections. (*S. Aoki et. al. 2007 and R Brower, S Chandrasekharan, John W Negele 2003*)

Lattice formulation for CGT possible only for $\theta = 0$ QCD.

Big picture

Our conclusion suggests that anomaly cancellation and absence of the strong CP problem are both required to regulate chiral gauge theories non-perturbatively.

Future directions

1. Counter examples, where absence of the CP problem is not required for regulating, if found will be interesting.

2. What happens when $Q = 0$ is relaxed? The GS phenomenology may or may not hold.

3. The lattice analysis of the flow must be done carefully to avoid zeromodes in the bulk.

4. Is it possible to construct a dynamical model of the universe where our world is really on the boundary of a five-dimensional space-time with five-dimensional gauge fields....

...but the gauge fields in the bulk act approximately like non-dynamical fields.

Thank you