Entanglement Entropy is Elastic Cross Section

Based on 2405.08056 and 2410.22414 with Ian Low

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Introduction

- Entanglement is a key phenomenon in quantum theories; characterized by entanglement entropy
- The entropy (for many-body systems) is usually an extensive quantity
- Area laws exist for entropy
    - Bekenstein–Hawking entropy for black holes
    - Field theory examples, through AdS/CFT or lattice simulations

Entanglement entropy in particle scattering

Observation of entanglement at LHC

ATLAS, 2311.07288; CMS, 2406.03976

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Maximum or minimum entanglement entropy leads to unique theories

- Maximum entanglement entropy in QED

  Cervera-Lierta, Latorre, Rojo, Rottoli, 1703.02989; Fedida, Serafini, 2209.01405; ...

- Minimum entanglement entropy in classical BH scattering, strong interactions, and BSM theories

  Beane, Kaplan, Klco, Savage, 1812.03138; Aoude, Chung, Huang, Machado, Tam, 2007.09486; Low, Mehen, 2104.10835; Liu, Low, Mehen, 2210.12085; Carena, Low, Wagner, Xiao, 2307.08112; ...

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We consider entanglement entropy in general $2 \to 2$ scatterings

Peschanski, Seki, 1602.00720; Cheung, He, Sivaramakrishnan, 2304.13052; Aoude, Elor, Remmen, Sumensari, 2402.16956; Kowalska, Sessolo, 2404.13743; ...

Wave packets

Ingredient 1: Distinguishable particles

Consider the **elastic** scattering of AB → AB. The initial state:

$$|\text{in}\rangle = |\psi_\text{A}\rangle \otimes |\psi_\text{B}\rangle$$

The density matrix: $\rho^\text{i} = |\text{in}\rangle\langle\text{in}|$: We need $\text{Tr}\,\rho^\text{i} = 1$

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$$|\psi\rangle = \int_p \psi(p)|p\rangle,\ \langle\psi|\psi\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\ |\psi(p)|^2 = 1$$

with $\langle p|q\rangle = (2\pi)^3\, 2E_p\, \delta^3(\vec{p}-\vec{q})$.

We are not using volume regularization

The final state

Ingredient 2: Unitary S-matrix

- $|\text{out}\rangle = S|\text{in}\rangle$, $S = 1 + iT$. $S^\dagger S = 1$, $2\,\text{Im}\,T = T^\dagger T$.

$$\langle\{k_\text{f}\}|T|\{k_\text{i}\}\rangle = (2\pi)^4\delta^4\left(\sum k_\text{f} - \sum k_\text{i}\right) M(\{k_\text{i}\};\{k_\text{f}\})$$

- The "complete" final state density matrix is $|\text{out}\rangle\langle\text{out}|$

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Probabilities:

$$\mathcal{P}_\text{tot} = \langle\text{in}|T^\dagger T|\text{in}\rangle,\ \mathcal{P}_\text{el} = \langle\text{in}|T^\dagger P_\text{AB} T|\text{in}\rangle,\ \mathcal{P}_\text{inel} = \langle\text{in}|T^\dagger(1 - P_\text{AB})T|\text{in}\rangle,$$

where P_AB selects states with exactly one type-A particle and one type-B particle

The plane wave limit

Ingredient 3: The plane wave limit

We require $\psi_{\mathsf{A/B}}(p)$ to

- Be centered around $\vec{k}_{\mathsf{A/B}}$. In the CoM frame, $\vec{k}_{\mathsf{A}} = -\vec{k}_{\mathsf{B}} \equiv \vec{k}$
- Widths in momentum space characterized by δ_{p} in the direction of motion, and δ_{T} in the transverse, with $1/\delta_{\mathsf{p}} \lesssim 1/\delta_{\mathsf{T}} = L$
- Take the plane wave limit: $\delta_{\mathsf{p}} \ll |\vec{k}|$, and expand around small $\delta_{\mathsf{p}}/|\vec{k}|$

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Probabilities ("$\doteq$" means keeping the leading piece in $\delta_{\mathsf{p}}/|\vec{k}|$):

$$\mathcal{P}_{\mathsf{tot}} \doteq I_0(|\vec{k}|)\sigma_{\mathsf{tot}},\ \mathcal{P}_{\mathsf{el}} \doteq I_0(|\vec{k}|)\sigma_{\mathsf{el}},\ \mathcal{P}_{\mathsf{inel}} \doteq I_0(|\vec{k}|)\sigma_{\mathsf{inel}}$$

where

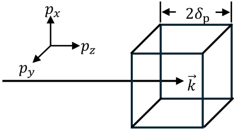
$$I_0(|\vec{k}|) = 4|\vec{k}|\sqrt{s}\int_{p_1,p_2,q_1,q_2} \psi_{\mathsf{A}}(p_1)\psi_{\mathsf{B}}(p_2)\psi^*_{\mathsf{A}}(q_1)\psi^*_{\mathsf{B}}(q_2) \times(2\pi)^4\delta^4(q_1+q_2-p_1-p_2)$$

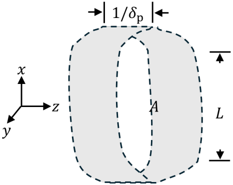
Universal factor

$\mathcal{P} \doteq I_0(|\vec{k}|)\sigma$ is small in the plane wave limit

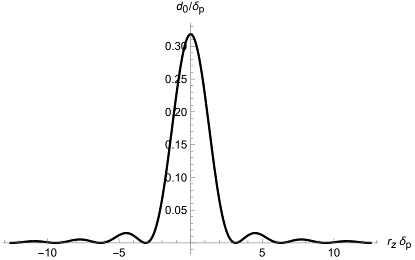
- $I_0(|\vec{k}|) \sim \delta_{\mathsf{T}}^2$ has the dimension of area^{-1}
- The size of $I_0(|\vec{k}|)$ depends on the overlap of the wave functions in the transverse directions in position space

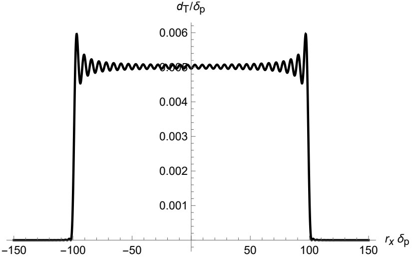
The uniform wave packet





Field density in position space:





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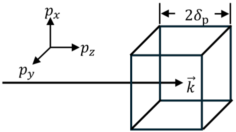

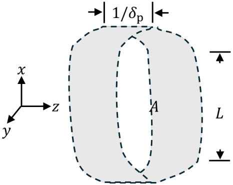

- Roughly uniform in the transverse direction: $\delta_\mathsf{p} \gg 1/L = \delta_\mathsf{T}$, with $\mathcal{O}(1/(\delta_\mathsf{p} L))$ corrections (also, $1/(\delta_\mathsf{p} L) \lesssim \delta_\mathsf{p}/|\vec{k}|$)
- For head-on collisions, $I_0(|\vec{k}|) \doteq 1/A = 1/L^2 = \delta_\mathsf{T}^2$
- $\mathcal{P} \doteq I_0(|\vec{k}|)\sigma \doteq \sigma/A$, c.f. the definition of cross section, e.g.

$$\sigma = \frac{\mathcal{N}}{d_\mathsf{A} l_\mathsf{A} d_\mathsf{B} l_\mathsf{B} A} = \frac{\mathcal{N} A}{(d_\mathsf{A} l_\mathsf{A} A)(d_\mathsf{B} l_\mathsf{B} A)} \to \mathcal{P} A$$

Final state selection

For the elastic scattering AB $\to$ AB, $|\text{out}\rangle_{\text{el}} = P_{\text{AB}}|\text{out}\rangle$

- $\text{Tr}(|\text{out}\rangle\langle\text{out}|) = \langle\text{in}|S^\dagger S|\text{in}\rangle = 1$
- Properly normalized ρ^{f} (Lüders rule):

$$\rho^{\text{f}} = \frac{1}{1-\mathcal{P}_{\text{inel}}}|\text{out}\rangle_{\text{el}\ \text{el}}\langle\text{out}|$$

- $\mathcal{P}_{\text{inel}}$ is small, thus we can expand around $\mathcal{P}_{\text{inel}} = 0$

Final state entropy

$$\rho^{\mathsf{f}} = \frac{1}{1 - \mathcal{P}_{\mathsf{inel}}} |\mathsf{out}\rangle_{\mathsf{el}\ \mathsf{el}}\langle \mathsf{out}|$$

For the linear entropy $\mathcal{E}_2(\rho) = 1 - \mathsf{Tr}\,\rho^2$, the final entanglement entropy is

$$\mathcal{E}_2^{\mathsf{f}} \doteq I_0(|\vec{k}|) \left(\frac{\mathsf{Im}\, M^{\mathsf{F}}}{|\vec{k}|\sqrt{s}} - 2\sigma_{\mathsf{inel}} \right)$$

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- σ_{inel} comes from $\mathcal{P}_{\mathsf{inel}}$ in the normalization factor
- Terms higher order in M are higher order in $\delta_{\mathsf{p}}/|\vec{k}|$ (c.f. $\rho^{\mathsf{f}} \sim S|\mathsf{in}\rangle\langle\mathsf{in}|S^\dagger$ with $S = 1 + iT$)
- $\mathrm{Im}\, M^{\mathsf{F}} = 2|\vec{k}|\sqrt{s}\,\sigma_{\mathsf{tot}}$ from the optical theorem

The area law

The key result

Entanglement entropy in the plane wave limit

$$\mathcal{E}_2^{\mathsf{f}} \doteq 2I_0(|\vec{k}|)\sigma_{\mathsf{el}} \doteq 2\mathcal{P}_{\mathsf{el}}$$

Low, **ZY**, 2405.08056

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$$\mathcal{E}_2^{\mathsf{f}} \doteq 2\frac{\sigma_{\mathsf{el}}}{A}$$

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- For uniform wave packets,

$$\mathcal{E}_2^{\mathsf{f}} \doteq 2\frac{\sigma_{\mathsf{el}}}{A}$$

- For Tsallis and Rényi entropies,

$$\mathcal{E}_{n,\mathsf{T/R}}^{\mathsf{f}} \doteq \frac{n}{n-1} I_0(|\vec{k}|)\sigma_{\mathsf{el}} \doteq \frac{n}{n-1}\mathcal{P}_{\mathsf{el}}$$

Generalization

Consider 3 type of quantum numbers:

- What distinguishes A and B, which we will call "charge"
- Kinematics, described by wave packets
- Other discrete quantum numbers $f_{\mathsf{A/B}}$ ("flavors"), labelled by i and $\bar{i}$

Example

In e^+e^- scattering, the "charge" can be electric charge, and the "flavor" can be spin

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The initial state:

$$|\text{in}\rangle = |\psi_{\mathsf{A}}\rangle \otimes |i\rangle \otimes |\psi_{\mathsf{B}}\rangle \otimes |\bar{i}\rangle,$$

with $\langle i|j\rangle = \delta^{ij}$, $\langle \bar{i}|\bar{j}\rangle = \delta^{\bar{i}\bar{j}}$

Cutting the cake

	A	B
p	p_A	p_B
f	f_A	f_B

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$$\begin{aligned} & \mathcal{E}^{\mathsf{f}}_{n,\mathsf{T/R,A}} \\ \doteq \quad & \frac{n}{n-1}\mathcal{P}_{\mathsf{el}} \\ \doteq \quad & \frac{n}{n-1} I_0(|\vec{k}|)\sigma_{\mathsf{el}} \\ \to \quad & \frac{n}{n-1}\frac{\sigma_{\mathsf{el}}}{A} \end{aligned}$$

i, $\bar{i}$ to any j, $\bar{j}$

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i, $\bar{i}$ to any $j \neq i$, $\bar{j} \neq \bar{i}$

Low, **ZY**, 2410.22414

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\end{aligned}
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\end{aligned}
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i, $\bar{i}$ to any $j \neq i$, any $\bar{j}$

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Example for mixed states

The density matrix:

$$\rho^{\mathsf{i}} = \rho^{\mathsf{i}}_{f_{\mathsf{A}}} \otimes \rho^{\mathsf{i}}_{p_{\mathsf{A}}} \otimes \rho^{\mathsf{i}}_{f_{\mathsf{B}}} \otimes \rho^{\mathsf{i}}_{p_{\mathsf{B}}},$$

with

$$\rho^{\mathsf{i}}_{f_{\mathsf{A}}} = \sum_{i} \mathbf{f}_i \; |i\rangle\langle i|, \qquad \rho^{\mathsf{i}}_{f_{\mathsf{B}}} = \sum_{\bar{i}} \bar{\mathbf{f}}_{\bar{i}} \; |\bar{i}\rangle\langle \bar{i}|$$

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The subsystem entropy of particle A:

$$\mathcal{E}^{\mathsf{f}}_{2,\mathsf{A}} \doteq 2 I_0(|\vec{k}|) \sum_{i,\bar{i}} \mathbf{f}_i^2 \, \bar{\mathbf{f}}_{\bar{i}} \, (\sigma_{\mathsf{el}})_{i\bar{i}}$$

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Compare with the probability:

$$\mathcal{P}_{\mathsf{el}} \doteq I_0(|\vec{k}|) \sum_{i,\bar{i}} \mathbf{f}_i \bar{\mathbf{f}}_{\bar{i}} (\sigma_{\mathsf{el}})_{i\bar{i}} \equiv I_0(|\vec{k}|) \, \overline{\sigma}_{\mathsf{el}}$$

Condition for $\mathcal{E} \propto \mathcal{P}$

When $(\rho^{\mathsf{i}}_{f_{\mathsf{A}}})^2 \propto \rho^{\mathsf{i}}_{f_{\mathsf{A}}}$, i.e. $\mathbf{f}_i = 0$ or $1/n_{\mathsf{A}}$ with n_{A} being the non-zero entries of $\mathbf{f}_i$,

$$\mathcal{E}^{\mathsf{f}}_{2,\mathsf{A}} \doteq \frac{2}{n_{\mathsf{A}}}\mathcal{P}_{\mathsf{el}} \doteq \frac{2}{n_{\mathsf{A}}} I_0(|\vec{k}|)\overline{\sigma_{\mathsf{el}}} \to \frac{2}{n_{\mathsf{A}}} \frac{\overline{\sigma_{\mathsf{el}}}}{A}$$

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- Same condition for the subsystem entropy to never decrease

  Cheung, He, Sivaramakrishnan, 2304.13052

- Example: Unpolarized beam

Conclusions

Ingredients

1. Distinguishable particles
2. Unitary S-matrix
3. The plane wave limit

- Entanglement entropy as cross section/probability
- General and non-perturbative (in coupling)
- What we know about inclusive cross sections:
  - Bound on growth of total cross section: $\log^2 s$

    Froissart, 1961; Martin, 1963
  - Actual growth saturates the bound, with $\sigma_{\text{el}}/\sigma_{\text{tot}} \to 1/2$

    Cheng, Wu, 1970
  - Measured in colliders

Outlook

- An area law for a 2-body system: What is the boundary? Holography? Relations to black holes?
- Other entropies?
  - Identical particles
  - Specific momenta
  - Other final states
- Beyond the plane wave limit