Universal Model of Thin Nonlinear Lens: Solving the Symmetric McMillan Map

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With Deepest Respect to the Memory of Slava Danilov
(21 Jan, 1966 – 30 Dec, 2014)

Structure of presentation

- I. Introduction/Brief historical overview
- II. Symmetric McMillan map
- III. Fixed points and n-cycles
- IV. Regimes with stable motion
- V. Dynamical properties
- VI. Summary

Cornell University

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Nonlinear Sciences > Exactly Solvable and Integrable Systems

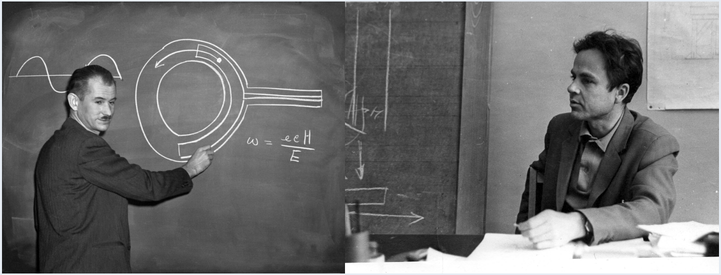
[Submitted on 14 Oct 2024]

Dynamics of McMillan mappings III. Symmetric map with mixed nonlinearity

Tim Zolkin, Sergei Nagaitsev, Ivan Morozov, Sergei Kladov, Young-Kee Kim

Fundamental contributions to chaotic and integrable dynamics

Chirikov map [1969] and McMillan map [1971]



Left: Edwin Mattison McMillan (18 Sep, 1907 – 7 Sep, 1991)
Right: Boris Valerianovich Chirikov (6 Jun 1928 – 12 Feb 2008)

From McMillan map to IOTA

- Series of works by E. Perevedentsev [1997] and V. Danilov [1997,1999,1999,2008].
- V. Danilov & S. Nagaitsev, "Nonlinear accelerator lattices with one and two analytic invariants," (IOTA) [2010].

Some important results

- R.I. McLachlan, "Integrable four-dimensional symplectic maps of standard type," [1993].
- A. Iatrou, J.A. Roberts, "Integrable mappings of the plane preserving biquadratic invariant curves I, II, III," [2001,2002, 2003].
- S. Nagaitsev, T. Zolkin, "Betatron frequency and the Poincaré rotation number," (Danilov Theorem) [2020].

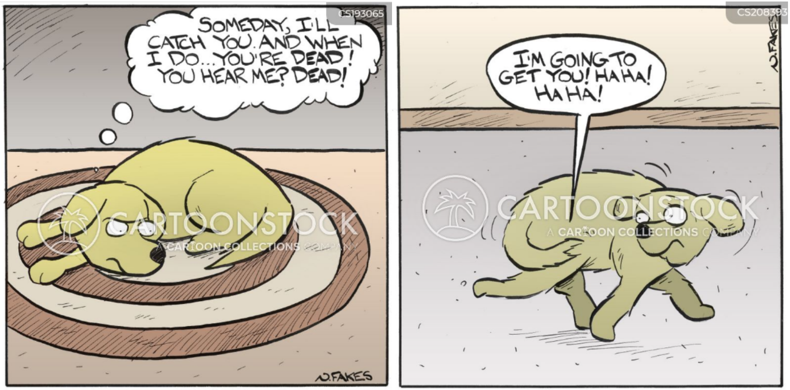
Quadrature — a process of drawing a square with the same area as a given plane figure

Solved systems (thanks to Danilov Theorem)

- Axially symmetric configuration for IOTA (no log r term) [2012-2014]
- McMillan octupole (canonical McMillan map) [2016 + 2023]
- McMillan sextupole (first order approximation to sextupole) [2022 + 2023]
- Linear and nonlinear mappings with polygon invariants [2016,2023-2024]
- 4D axially symmetric McMillan map (e-lens) [2020 + 2024]
- **Symmetric and asymmetric McMillan mappings** [2024]
- ... (to be continued)

with help of S. Nagaitsev, I. Morozov, Y. Kharkov, I. Lobach, B. Cathey, E. Stern, S. Kladov and others.

The tables have turned...



Left: Lucci starts to have very violent thoughts towards his tail.
Right: To Lucci's surprise, one day, the tables turned.

One crucial aspect of integrable systems often overlooked is their utility as approximations of real-world systems. Though idealized, integrable models often provide highly accurate descriptions of physical phenomena. A historical example is the ancient model of planetary motion, where the concept of deferent and epicycle predicted planetary positions from a geocentric perspective. Although modern celestial mechanics acknowledges the complexity of the *n*-body problem, the search for accurate approximations persists...

However, Keplers laws of planetary motion, which introduced elliptical orbits, stood out among these methods. From the perspective of modern Hamiltonian dynamics and dynamical systems theory, we now understand that it was not merely a coincidence or a fortunate approximation. By uncovering the integrable "core" of celestial mechanics, Kepler's laws provided a remarkably precise description of planetary dynamics and revealed deeper structure within the chaotic complexity of solar system dynamics.

Similarly, the McMillan map acts as an integrable approximation for a broad class of nonlinear mappings in *standard form* , featuring a *typical* force function (i.e., smooth with at least one nonzero quadratic or cubic coefficient in its Taylor series)...

- The one-turn map of 1D accelerator lattice with thin nonlinear lens can be brought to form:

$$q' = p,$$
$$p' = -q + f(p), \qquad f(\epsilon\, p) = a\,\epsilon\, p + b\,\epsilon^2 p^2 + c\,\epsilon^3 p^3 + \ldots.$$

- Looking for an approximate invariant:

$$\mathcal{K}^{(n)} = \mathcal{K}_0 + \epsilon\,\mathcal{K}_1 + \epsilon^2\mathcal{K}_2 + \ldots + \epsilon^n\mathcal{K}_n : \ \mathcal{K}'^{(n)} - \mathcal{K}^{(n)} = \mathcal{O}(\epsilon^{n+1}),$$

where $\mathcal{K}_m$ are homogeneous polynomials of degree $(m+2)$:

$$\mathcal{K}^{(2)} = \mathcal{K}_0[p,q] - \frac{\epsilon\, b}{a+1}\,(p^2\, q + p\, q^2) + \epsilon^2\left[\frac{b^2}{a\,(a+1)} - \frac{c}{a}\right] p^2 q^2.$$

- Up to a second order $\mathcal{K}^{(n)}[p,q]$ matches McMillan map

$$\frac{\mathrm{A}}{\epsilon^2} = \frac{b^2}{a\,(a+1)} - \frac{c}{a}, \qquad \frac{\mathrm{B}}{\epsilon} = -\frac{b}{a+1}, \qquad \rho = \frac{\mathrm{A}}{\mathrm{B}^2}.$$

II. Symmetric McMillan map

Form of the map

Let $\mathrm{M}_{\mathsf{SF}} : \mathbf{Z} \to \mathbf{Z}'$ be an area-preserving map in *standard form* (SF) from $\mathbf{Z} = (Q, P) \in \mathbb{R}^2$ to itself:

$$\mathrm{M}_{\mathsf{SF}} : \quad Q' = P, \quad P' = -Q + F(P), \qquad \mathrm{M}_{\mathsf{SF}}^{-1} : \quad Q' = -P + F(Q), \quad P' = Q,$$

where $(')$ indicates the application of the map, and $F(P)$ is referred to as the *force function*.

Force function

The most general *symmetric McMillan map* is then defined by a special rational function of degree two:

$$F_{\mathrm{s}}(P) = -\frac{\mathrm{B}_0\, P^2 + \mathrm{E}_0\, P + \Xi_0}{\mathrm{A}_0\, P^2 + \mathrm{B}_0\, P + \Gamma_0}.$$

II.1 Invariant/integral of motion

The map is *integrable*, meaning that there exists an *integral* or *invariant of motion* $\mathrm{K_s}[P,Q]$:

$$\forall\,(Q,P)\in\mathbb{R}^2:\;\; \mathrm{K_s}[P',Q'] - \mathrm{K_s}[P,Q] = 0,$$

that is given by a biquadratic function depending on six parameters:

$$\mathrm{K_s}[P,Q] = \mathrm{A}_0\,P^2\,Q^2 + \mathrm{B}_0\,(P^2 Q + P\,Q^2)$$
$$+\Gamma_0\,(P^2+Q^2) + \mathrm{E}_0\,P\,Q + \Xi_0\,(P+Q) + \mathrm{K}_0$$
$$= \begin{bmatrix} Q^2 \\ Q \\ 1 \end{bmatrix}^{\mathrm{T}} \cdot \left(\begin{bmatrix} \mathrm{A}_0 & \mathrm{B}_0 & \Gamma_0 \\ \mathrm{B}_0 & \mathrm{E}_0 & \Xi_0 \\ \Gamma_0 & \Xi_0 & \mathrm{K}_0 \end{bmatrix} \cdot \begin{bmatrix} P^2 \\ P \\ 1 \end{bmatrix} \right).$$

II.2 "Normal" form and intrinsic variables

$$q' = p, \qquad f_0(p) = -\frac{q^2 - a\,q}{\rho\,q^2 + q + 1},$$

$$p' = -q + f(p), \qquad f_\pm(p) = -\frac{r\,q^2 - a\,q}{\pm q^2 + r\,q + 1}.$$

$$\overline{\mathrm{B}} \neq 0: \quad \mathcal{K}_{\mathrm{s}}^0[p,q] = \mathcal{K}_0[p,q] + (p^2 q + p\,q^2) + \rho\,p^2 q^2, \quad \rho = \frac{\overline{\Gamma}\,\overline{\mathrm{A}}}{\overline{\mathrm{B}}^2},$$

$$\overline{\mathrm{A}} \neq 0: \quad \mathcal{K}_{\mathrm{s}}^\pm[p,q] = \mathcal{K}_0[p,q] + r\,(p^2 q + p\,q^2) \pm p^2 q^2, \quad r = \frac{1}{\sqrt{\rho}},$$

$$\mathcal{K}_0[p,q] = p^2 - a\,p\,q + q^2.$$

$$\begin{bmatrix} \rho & 1 & 1 \\ 1 & -a & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \pm 1 & r & 1 \\ r & -a & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Linear tune:

$$2\,\pi\,\nu_0 = \arccos[a/2]$$

Nonlinear detuning:

$$2\,\pi\,\mu_0 = \frac{1}{4-a^2}\,\left[3\,a\,\rho - \frac{(a+1)(a+8)}{2-a}\right]$$

III. Fixed points and n-cycles

To do:

- Solve for fixed points ($n = 1$) and n-cycles

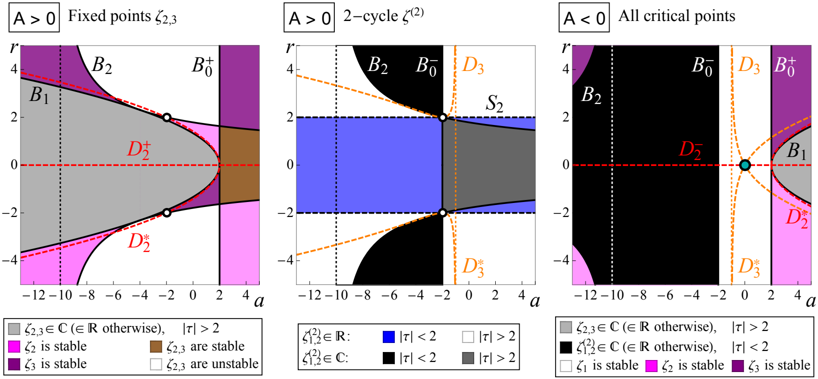
$$\zeta^{(n)} = \{\zeta_0, \zeta_1, \ldots, \zeta_{n-1}\} : \quad \mathrm{M}^n \zeta_0 = \zeta_0, \quad \zeta_0 = (q_0, p_0).$$

- Analyze linear stability, $|\tau[\zeta^{(n)}]| < 2$

$$\tau[\zeta^{(n)}] = \mathrm{Tr}\,\mathrm{J}(\zeta^{(n)}), \qquad \mathrm{J} = \begin{bmatrix} \partial q'/\partial q & \partial q'/\partial p \\ \partial p'/\partial q & \partial p'/\partial p \end{bmatrix}.$$

- Determine real domain, $\zeta^{(n)} \in \mathbb{R}^2$.
- Determine (super-) degeneracies, $\mathcal{K}[\zeta_i^{(n)}] = \mathcal{K}[\zeta_j^{(n)}]$ for $i \neq j$.
- Determine singularities.

Domain/stability diagrams



IV. Regimes with stable motion

Characteristic polynomial

By solving for momentum from the expression for the invariant

$$p = \frac{1}{2}\left(f_\mathrm{s}(q) \pm \frac{\sqrt{\mathcal{D}_4(q)}}{\mathrm{A}\,q^2 + \mathrm{B}\,q + 1}\right),$$

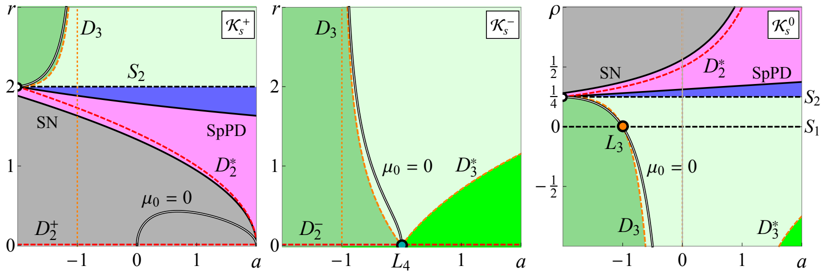
we can classify specific trajectories based on the roots $q_{1,2,3,4}$ of the characteristic polynomial $\mathcal{D}_4(q)$

$$(\mathrm{B}^2 - 4\,\mathrm{A})\,q^4 - 2\,(a+2)\,\mathrm{B}\,q^3 + (a^2 - 4 + 4\,\mathrm{A}\,\mathcal{K}_\mathrm{s})\,q^2 + 4\,\mathrm{B}\,\mathcal{K}_\mathrm{s}\,q + 4\,\mathcal{K}_\mathrm{s},$$

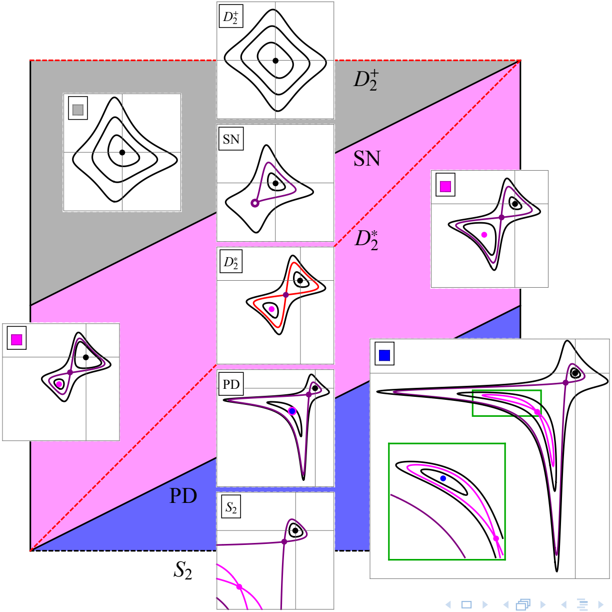
and two roots of the quadratic polynomial in the denominator

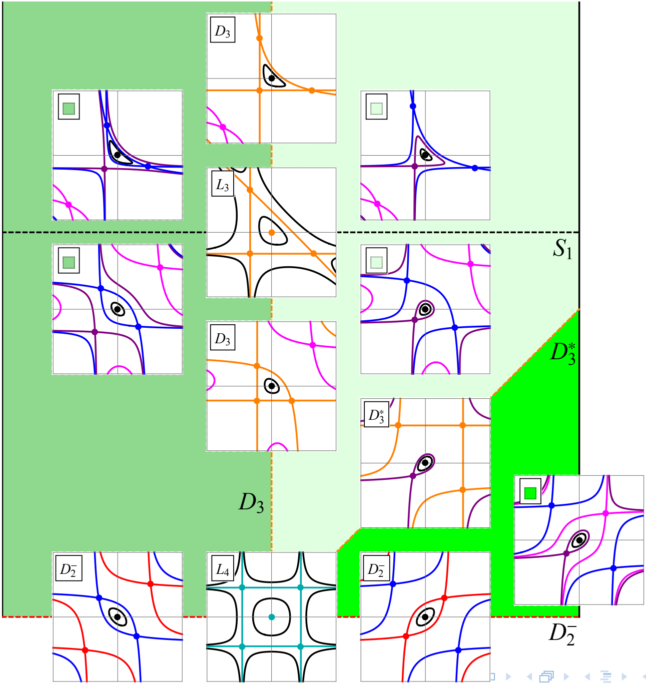
$$q_{5,6} = (-\mathrm{B} \mp \sqrt{\mathcal{R}_0})/(2\,\mathrm{A}).$$

- Unimodal (UM)
- Double-well (DW)
- Double lemniscate (DL)
- Simply connected (SC)



Unimodal, Double-well and double lemniscate





D_3
L_3
D_3
D_3^*
D_2^-
L_4
D_2^-
S_1
D_3^*
D_3
D_2^-

V. Dynamical properties

Canonical action-angle coordinates:

$$\mathrm{M}_s: \quad J' = J, \qquad \{J_n\} = \{J_0\},$$

$$\psi' = \psi + 2\,\pi\,\nu(J), \qquad \{\psi_n\} = \{\psi_0\} + 2\,\pi\,n\,\nu(\{J_0\}).$$

Use of Danilovs Theorem

$$\nu = \frac{\displaystyle\int_q^{q'} \left(\partial\mathcal{K}_\mathrm{s}/\partial p\right)^{-1}\,\mathrm{d}q}{\displaystyle\oint \left(\partial\mathcal{K}_\mathrm{s}/\partial p\right)^{-1}\,\mathrm{d}q} = \frac{\displaystyle\int_q^{q'} \mathrm{d}q/\sqrt{\mathcal{D}_4(q)}}{\displaystyle 2\int_{q_-}^{q_+} \mathrm{d}q/\sqrt{\mathcal{D}_4(q)}},$$

$$J = \frac{1}{2\,\pi}\oint p\,\mathrm{d}q = \frac{1}{2\,\pi}\int_{q_-}^{q_+} \frac{\sqrt{\mathcal{D}_4(q)}}{\mathrm{A}\,q^2 + \mathrm{B}\,q + 1}\,\mathrm{d}q.$$

V.1 Taking the integrals (S. Kladov)

Use of Danilovs Theorem

$$\nu = \frac{\mathrm{F}[\Phi(q'),\kappa]}{2\,\mathrm{K}[\kappa]}, \qquad J = \frac{\sqrt{|\mathcal{R}_0|}}{2\,\mathrm{A}}\,\Sigma,$$

$$\Sigma = c_{\mathrm{K}}\mathrm{K}[\kappa] + c_{\mathrm{E}}\mathrm{E}[\kappa] + c_0\Pi[\alpha_0,\kappa] + c_1\Pi[\alpha_1,\kappa] + c_2\Pi[\alpha_2,\kappa].$$

	dl-like trajectory	*cn*-like trajectory
m_{K}	$(q_4-q_2)(q_4-q_1)$	$2\,\frac{u\,v}{u-v}\,\frac{(q_2-q_1)^2\left[\lvert q_5-q_1\rvert^2 u-\lvert q_5-q_2\rvert^2 v+(u-v)\,u\,v\right]}{\left[(q_5-q_1)\,u-(q_5-q_2)\,v\right]\left[(q_6-q_1)\,u-(q_6-q_2)\,v\right]}$
m_{E}	$-(q_4-q_2)(q_3-q_1)$	$-2\,u\,v$
m_0	$2\,(q_4-q_1)\left[q_6+q_5-\sum_{i=1}^{4}\frac{q_i}{2}\right]$	$\frac{u+v}{u-v}\left[q_2^2-q_1^2-2\,(q_2-q_1)\,\Re(q_5)-\frac{u^2-v^2}{2}\right]$
m_1	$2\,(q_4-q_1)\,\frac{(q_5-q_3)(q_5-q_2)}{q_6-q_5}$	$\frac{q_1u+q_2v-q_5(u+v)}{q_6-q_5}\,\frac{(q_5-q_1)\,u^2-(q_5-q_2)\,v^2+(q_5-q_2)(q_5-q_1)(q_2-q_1)}{(q_5-q_1)\,u-(q_5-q_2)\,v}$
α_0	$-\frac{q_2-q_1}{q_4-q_2}$	$-\frac{1}{4}\,\frac{(u-v)^2}{u\,v}$
α_1	$-\frac{q_5-q_4}{q_5-q_1}\,\frac{q_2-q_1}{q_4-q_2}$	$-\frac{1}{4}\,\frac{\left[(q_5-q_1)\,u-(q_5-q_2)\,v\right]^2}{(q_5-q_1)(q_5-q_2)\,u\,v}$

V.2 Inverting the integrals (S. Kladov + A. Iatrou et al.)

Lesson 1

$$q_n = \frac{a + b\,\mathrm{sn}^2[\phi_n/2]}{c + d\,\mathrm{sn}^2[\phi_n/2]}, \qquad \phi_n = \frac{\mathrm{K}[\kappa]}{\pi/2}\,\{\psi_n\} = \phi_0 + 4\,n\,\nu\,\mathrm{K}[\kappa].$$

Type of trajectory	κ^2	$\Phi(q)$	$\{q_n\}$	ϕ_0
sn-like	$\frac{(q_3-q_2)(q_4-q_1)}{(q_3-q_1)(q_4-q_2)}$	$\arcsin\left[\frac{(q_3-q_1)(q-q_2)}{(q_3-q_2)(q-q_1)}\right]^{1/2}$	$\frac{q_2(q_3-q_1)-q_1(q_3-q_2)\,\mathrm{sn}^2[\phi_n/2,\kappa]}{q_3-q_1-(q_3-q_2)\,\mathrm{sn}^2[\phi_n/2,\kappa]}$	$\pm 2\mathrm{F}[\Phi(\{q_0\}),\kappa]$
		$\arcsin\left[\frac{(q_4-q_2)(q_3-q)}{(q_3-q_2)(q_4-q)}\right]^{1/2}$	$\frac{q_3(q_4-q_2)-q_4(q_3-q_2)\,\mathrm{sn}^2[\phi_n/2,\kappa]}{q_4-q_2-(q_3-q_2)\,\mathrm{sn}^2[\phi_n/2,\kappa]}$	$\mp 2\mathrm{F}[\Phi(\{q_0\}),\kappa]$
dl-like	$\frac{(q_2-q_1)(q_4-q_3)}{(q_3-q_1)(q_4-q_2)}$	$\arcsin\left[\frac{(q_4-q_2)(q-q_1)}{(q_2-q_1)(q_4-q)}\right]^{1/2}$	$\frac{q_1(q_4-q_2)-q_4(q_1-q_2)\,\mathrm{sn}^2[\phi_n/2,\kappa]}{q_4-q_2-(q_1-q_2)\,\mathrm{sn}^2[\phi_n/2,\kappa]}$	$\pm 2\mathrm{F}[\Phi(\{q_0\}),\kappa]$
		$\arcsin\left[\frac{(q_3-q_1)(q_2-q)}{(q_2-q_1)(q_3-q)}\right]^{1/2}$	$\frac{q_2(q_3-q_1)-q_3(q_2-q_1)\,\mathrm{sn}^2[\phi_n/2,\kappa]}{q_3-q_1-(q_2-q_1)\,\mathrm{sn}^2[\phi_n/2,\kappa]}$	$\mp 2\mathrm{F}[\Phi(\{q_0\}),\kappa]$
dr-like	$\frac{(q_2-q_1)(q_4-q_3)}{(q_3-q_1)(q_4-q_2)}$	$\arcsin\left[\frac{(q_4-q_2)(q-q_3)}{(q_4-q_3)(q-q_2)}\right]^{1/2}$	$\frac{q_3(q_4-q_2)-q_1(q_4-q_3)\,\mathrm{sn}^2[\phi_n/2,\kappa]}{q_4-q_2-(q_4-q_3)\,\mathrm{sn}^2[\phi_n/2,\kappa]}$	$\pm 2\mathrm{F}[\Phi(\{q_0\}),\kappa]$
		$\arcsin\left[\frac{(q_3-q_1)(q_4-q)}{(q_4-q_3)(q-q_1)}\right]^{1/2}$	$\frac{q_4(q_3-q_1)-q_1(q_4-q_3)\,\mathrm{sn}^2[\phi_n/2,\kappa]}{q_3-q_1-(q_4-q_3)\,\mathrm{sn}^2[\phi_n/2,\kappa]}$	$\mp 2\mathrm{F}[\Phi(\{q_0\}),\kappa]$
cn-like	$\frac{(q_2-q_1)^2-(u-v)^2}{4\,u\,v}$	$\arccos\frac{(q_2-q)v-(q-q_1)u}{(q_2-q)v+(q-q_1)u}$	$\frac{q_1u+q_2v+(q_1u-q_2v)\,\mathrm{cn}[\phi_n,\kappa]}{u+v+(u-v)\,\mathrm{cn}[\phi_n,\kappa]}$	$\pm \mathrm{F}[\Phi(\{q_0\}),\kappa]$
		$\arccos\frac{(q-q_1)u-(q_2-q)v}{(q_2-q)v+(q-q_1)u}$	$\frac{q_1u+q_2v-(q_1u-q_2v)\,\mathrm{cn}[\phi_n,\kappa]}{u+v-(u-v)\,\mathrm{cn}[\phi_n,\kappa]}$	$\mp \mathrm{F}[\Phi(\{q_0\}),\kappa]$

TABLE I. Elliptic modulus κ, amplitude function $\Phi(q)$, and the solution of the map $\{q_n\} = \{p_{n-1}\}$, along with the initial phase ϕ_0, are provided for different types of trajectories. $\phi_n = 2\,\mathrm{K}[\kappa]\,\{\psi_n\}/\pi = \phi_0 + 4\,n\,\nu\,\mathrm{K}[\kappa]$ is the rescaled angle variable such that the sign of ϕ_0 is selected based on the initial conditions $\{p_0\} \gtrless 0$. The parameters u and v are defined as $u = \sqrt{(q_2-q_r)^2+q_i^2}$ and $v = \sqrt{(q_1-q_r)^2+q_i^2}$, respectively.

V.3 Detuning

Lesson 2

$$2\,\pi\,(\nu-\nu_0)=\frac{s_1}{1!}\,\frac{J}{4-a^2}-\frac{s_2}{2!}\,\frac{J^2}{(4-a^2)^{5/2}}+\mathcal{O}(J^3),\quad \text{where}$$

$$s_1=3\,a\,\mathrm{A}-\frac{(a+1)(a+8)}{2-a}\,\mathrm{B}^2,$$

$$s_2=a\,(74+7\,a^2)\,\mathrm{A}^2-2\,\frac{208+442\,a+248\,a^2+71\,a^3+3\,a^4}{2-a}\,\mathrm{A}\,\mathrm{B}^2$$
$$+\,(a+1)\,\tfrac{736+626\,a+198\,a^2+7\,a^3-a^4}{(2-a)^2}\,\mathrm{B}^4.$$

Example: Quadratic Hénon map, $f(q) = a\,q + q^2$

The corresponding second-order approximate invariant is:

$$\mathcal{K}^{(2)}_{\text{SX-2}}[p,q] = \mathcal{K}_0[p,q] - \frac{p^2 q + p\,q^2}{a+1} + \frac{p^2 q^2}{a\,(a+1)},$$

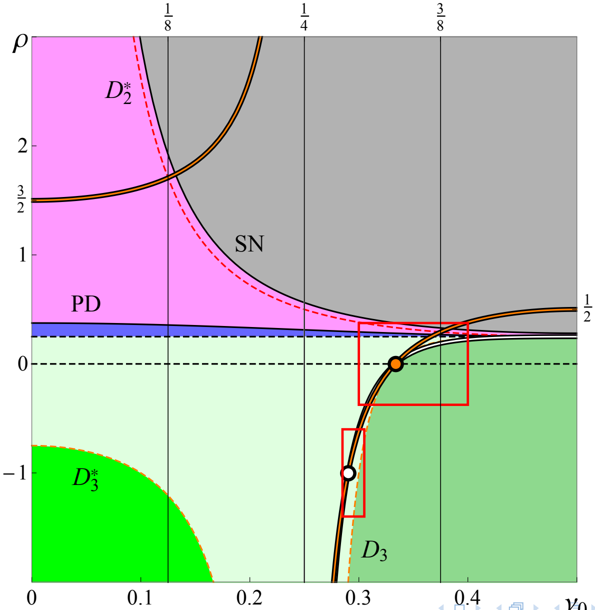
$$f_{\text{SX-2}}(q) = \frac{a\,q^2 + a^2(a+1)\,q}{q^2 - a\,q + a\,(a+1)} = a\,q + q^2 + \mathcal{O}(q^4),$$

with its normal form given by:

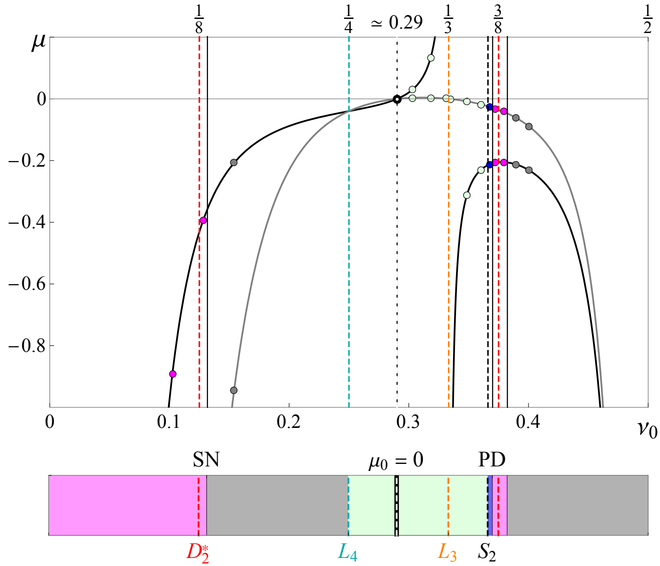
$$\mathcal{K}^{(2n)}_{\text{SX-2}}[p,q] = \mathcal{K}_0[p,q] + p^2 q + p\,q^2 + \rho_n\,p^2 q^2,$$

$$\rho_n = \frac{a+1}{a} = \frac{2\,\cos[2\,\pi\,\nu_0] + 1}{2\,\cos[2\,\pi\,\nu_0]}.$$

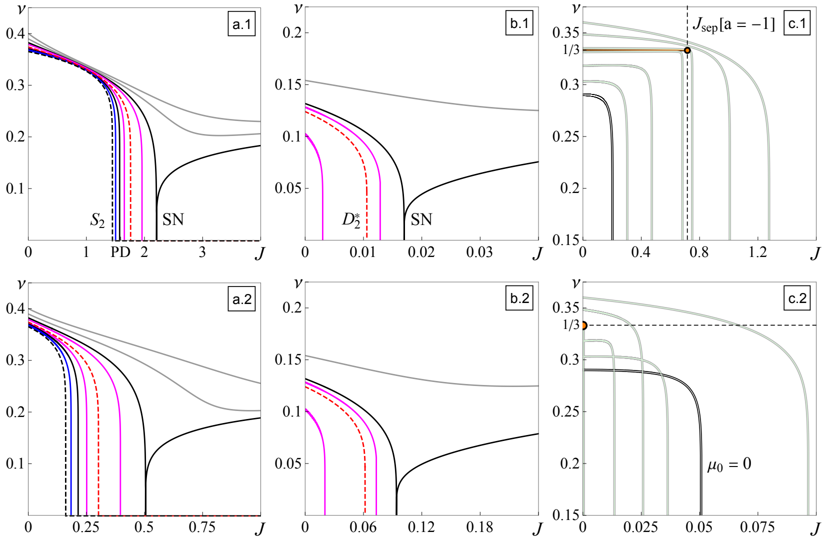
Atlas of intrinsic parameters



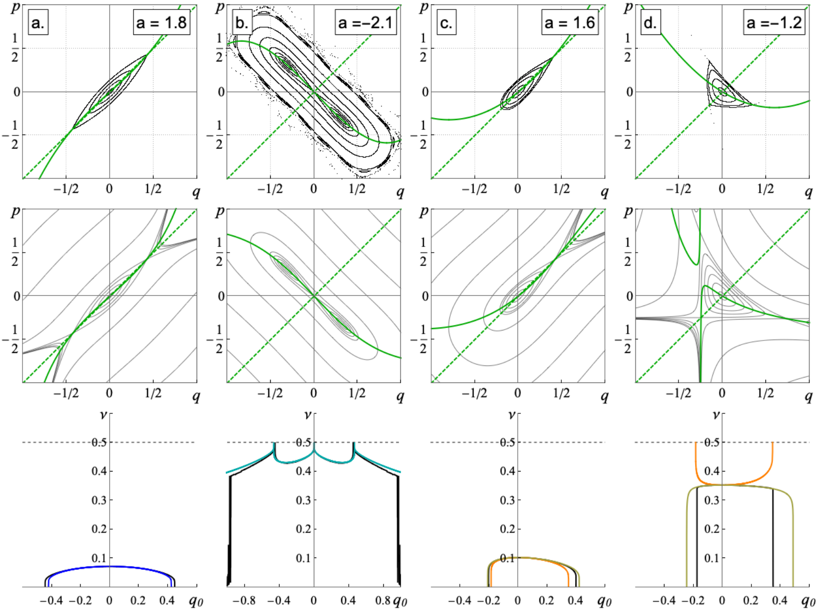
Nonlinear detuning (twist), $\mu_0 = \mathrm{d}\nu/\mathrm{d}J$



Nonlinear emittance vs. nonlinear tune, $\nu(J)$



Sextupole and Octupole



VI. Summary

This article presents a comprehensive study of the most general symmetric McMillan map, emphasizing its role as a universal model for understanding nonlinear oscillatory systems, particularly symplectic/area-preserving mappings of the plane in standard form with typical force functions. By identifying only two irreducible parameters the linearized rotation number at the fixed point and the coefficient representing the ratio of nonlinear terms in the biquadratic invariant the McMillan map is shown to be both relatively simple and compact, yet highly accurate as an integrable approximation for a broad class of standard-form mappings, especially near main resonances. Through an in-depth analysis of the maps intrinsic parameters, we provide a complete solution to the mapping equations and classify regimes of stable motion. This general model offers analytical expressions for the nonlinear tune shift, rotation number, and action-angle variables, and, also serves as a systematic approach to understanding the qualitative behavior of nonlinear systems under various parameter settings.

VI. Summary

In the second part of the study, we focus on specific applications of the symmetric McMillan map to model chaotic systems, specifically the quadratic Hénon map and accelerator lattices with thin sextupole magnet. By establishing a connection between these systems, we demonstrate how the McMillan map extends the linear Courant-Snyder formalism, enabling predictions of dynamic aperture and the nonlinear betatron tune (rotation number) as a function of amplitude. We also provide the expression for the approximated single particle emittance of the beam (the phase space area occupied by particles). This work underscores the importance of using integrable systems to accurately model complex nonlinear interactions under certain conditions, reinforcing the relevance of such models in both theoretical research and practical applications.

Thank you for your attention.

Questions?