Symmetries and Chaos in Modeling Beam Dynamics

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Nonlinear Sciences > Chaotic Dynamics

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Isochronous and period-doubling diagrams for symplectic maps of the plane

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Goals and structure of presentation

- Part 1. Short review of reversible dynamics

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PHYSICS REPORTS

Chaos and time-reversal symmetry. Order and chaos in reversible dynamical systems

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- Part 2. Applications to beam dynamics
  - 1. Choice of initial conditions
  - 2. Selection of chaotic indicators
  - 3. Averaging over initial conditions

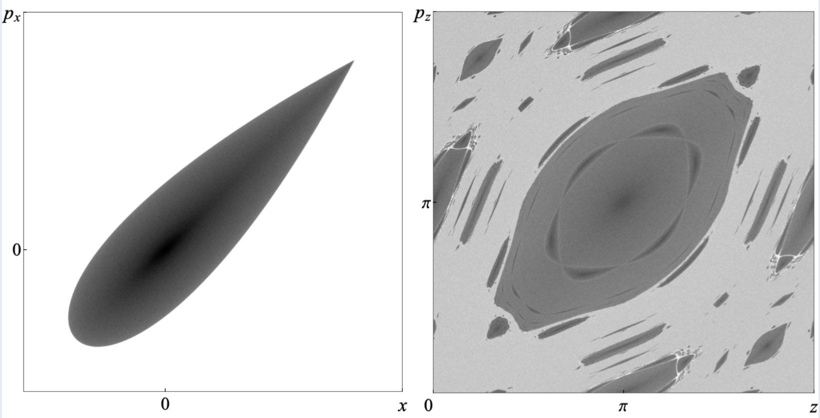
Motivation

1. From J.A.G. Roberts and G.R.W. Quispel

...In comparison, reversible dynamical systems have received far less attention and the treatment that they have received has tended to be less systematic. **This has led to the situation where the literature on the subject is quite scattered, with some authors not being aware of the generality of the mathematical theory underlying their results**. *One of our motivations for writing this report is to try and remedy this situation. For the same reason we provide an extensive list of references on (nonconservative) reversible systems, both on their theory, and on their applications in such diverse fields as condensed matter physics, fluid dynamics, laser physics, molecular dynamics, quasicrystals, chemistry, biology etc. (Sevryuk [1991b])...*

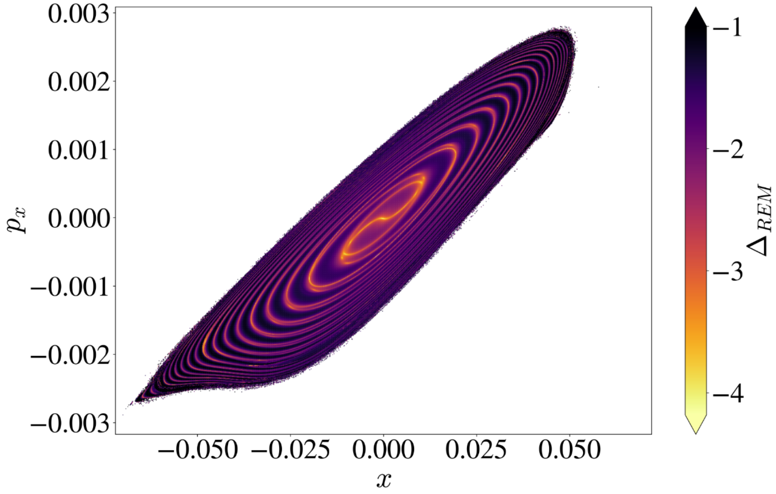
Motivation

2. Thin sextupole magnet and Rf-station. Phase space (q, p)



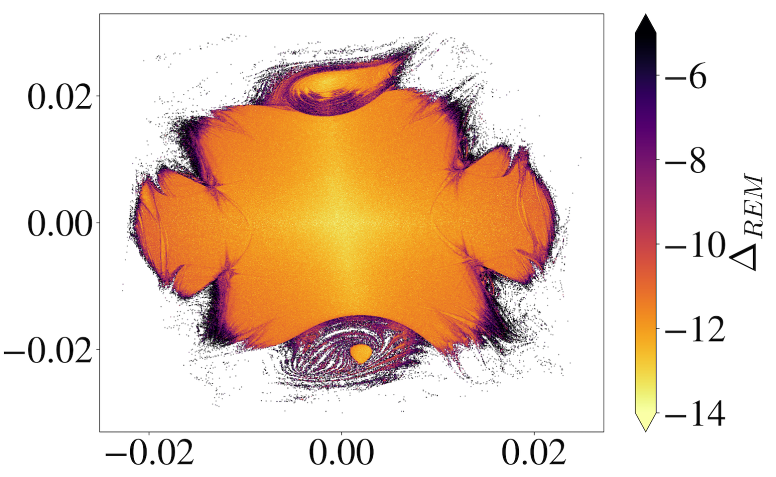
Motivation

3. Recycler, (x, p_x)-plane. Simulation by Cristhian Gonzalez-Ortiz



Motivation

3. Recycler, (x, y)-plane. Simulation by Cristhian Gonzalez-Ortiz



“Classical” concept of time-reversal symmetry

Invariance of equations of motion under the transformation $t \to -t$

- [Loschmidt 1877]. Particles in a velocity-independent force field.
- [Boltzmann 1897,1898]. Maxwell’s equations are reversible if one also reverses the field $B \to -B$.
- [Painlevé 1904]. Newton’s equations of motion for a free-falling body.
- [Penrose 1979, 1989]. The Einstein equations of classical general relativity.
- [Marchal 1990]. Three-body problem.
- [Wigner 1959]. Time-reversal symmetry in quantum mechanics.

A system that is invariant under time reversal is called *reversible*. **Reversibility is not necessarily equivalent to** *invertibility* **or to** *thermodynamic reversibility* **!**

Time-reversal symmetry in ODEs

Example: if $x = \gamma(t)$ is a solution, so is the $x = \gamma(-t)$

- $\ddot{x} = F(x), \qquad F : \mathbb{R} \to \mathbb{R},$
- $\ddot{x} = F(x), \qquad F : \mathbb{R}^n \to \mathbb{R}^n,$
- $\ddot{x} = F(x, \dot{x}^2), \qquad F : \mathbb{R}^2 \to \mathbb{R}.$

Example: $H = \frac{p^2}{2} + V(x)$, or more generally $H[\vec{p}, \vec{x}] = H[-\vec{p}, \vec{x}]$

$$\ddot{x} = F(x) \qquad \to \qquad \begin{aligned} \dot{x} &= p \\ \dot{p} &= F(x, p^2) \end{aligned}$$

If $[x(t), p(t)]$ is a solution, so is the $[x(-t), -p(-t)]$:

$$\boxed{t \to -t \qquad x \to x \qquad p \to -p}$$

Generalized definition [Devaney, 1976]

A dynamical system (not necessarily conservative) is *reversible* if there is an involution in phase space which reverses the direction of time; no restriction to conservative or to even-dimensional systems.

A general system of n coupled first-order ODEs,

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = F(\vec{x}), \qquad \vec{x} \in \mathbb{R}^n,$$

is reversible if there is an involution $G = G^{-1}$ which reverses the direction of time, i.e.,

$$\mathrm{d}(G\,\vec{x})/\mathrm{d}t = -F(G\,\vec{x}), \qquad G \circ G = \mathrm{Id}.$$

- G — *reversing symmetry* of the system;
- trajectories that are left invariant by G are called *symmetric*;
- otherwise they are *asymmetric*.

Time-reversal symmetry in ordinary difference equations

A dynamical system (not necessarily conservative) is *reversible* if there is an involution in phase space which reverses the direction of time; no restriction to conservative or to even-dimensional systems.

A general system of n coupled first-order difference equations,

$$\vec{x}_{i+1} = \mathrm{T}\,\vec{x}_i, \qquad \vec{x}\,' = \mathrm{T}\,\vec{x}, \qquad \vec{x} \in \mathbb{R}^n,$$

is reversible if there is an involution $G = G^{-1}$ which reverses the direction of time, i.e.,

$$\mathrm{T} \circ G\,\vec{x}_{i+1} = G\,\vec{x}_i, \qquad \mathrm{T} \circ G\,\vec{x}\,' = G\,\vec{x}.$$

$$\boxed{\mathrm{T} \circ G \circ \mathrm{T} = G} \qquad \boxed{\mathrm{T} = H \circ G \ \text{ and } \ \mathrm{T}^{-1} = G \circ H}$$

- G, H — *reversing symmetries*: $H \circ H = G \circ G = \mathrm{Id}$.

Short summary on reversibility:

• A map T is *reversible*, if T is as a composition of two involutions:

$$\mathrm{T} = \mathrm{R}_2 \circ \mathrm{R}_1, \qquad \mathrm{T}^{-1} = \mathrm{R}_1 \circ \mathrm{R}_2. \qquad \mathrm{R}_{1,2}^2 = \mathrm{Id}.$$

• Thus, the map and its inverse are *conjugate* to each other, as there exists an invertible *conjugating transformation* P such that

$$\mathrm{T}^{-1} = \mathrm{P} \circ \mathrm{T} \circ \mathrm{P}^{-1}, \qquad (1)$$

since $\mathrm{T}^{-1} = \mathrm{R}_1 \circ \mathrm{T} \circ \mathrm{R}_1$ and $\mathrm{T}^{-1} = \mathrm{R}_2 \circ \mathrm{T} \circ \mathrm{R}_2$.

• Mappings that satisfy (1) with P not necessarily an involution are called *weakly reversible*.

• A map can also possess additional, independent families of reversing symmetries, not necessarily but often weakly reversible, in which case it is called a *multiply reversible* map. A notable case discussed in is that of doubly reversible mappings, such as reversible odd maps, where T commutes with the rotation $\mathrm{Rot}\,(\pi)$.

Example 1.1 Hénon map, $F(x) = x^2$

$$\mathrm{T} = \mathrm{Rot}(\psi) \circ \mathrm{Lens}_{-F} : \quad \begin{aligned} x' &= x \cos\psi - [y - F(x)] \sin\psi, \\ y' &= x \sin\psi + [y - F(x)] \cos\psi, \end{aligned}$$

$$\mathrm{T} = \mathrm{Ref}(\psi/2) \circ \mathrm{R}_F, \qquad \mathrm{T}^{-1} = \mathrm{R}_F \circ \mathrm{Ref}(\psi/2),$$

where

$$\mathrm{Ref}(\psi) : \begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} \cos 2\psi & \sin 2\psi \\ \sin 2\psi & -\cos 2\psi \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \qquad \mathrm{R}_f : \begin{aligned} q' &= q, \\ p' &= -p + f(q). \end{aligned}$$

with corresponding first and second symmetry lines:

$$L_1 : \boxed{y = x \tan(\psi/2)} \quad \text{and} \quad L_2 : \boxed{y = F(x)/2}.$$

Example 1.2 One- and two-lens lattices.

$$q' = p \qquad\qquad q' = -q + f_1(p)$$
$$p' = -q + f(p) \qquad\qquad p' = -p + f_2(q')$$

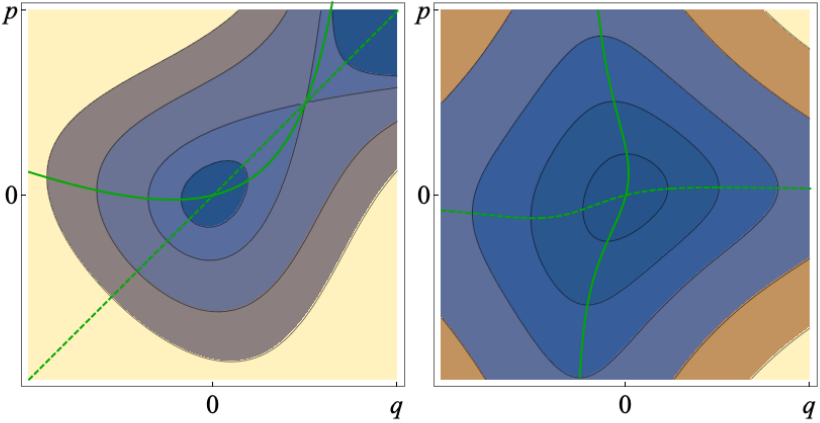


Figure: Symmetric and asymmetric integrable McMillan mappings.

Example 2.1 Doubly reversible odd mappings

For mappings in McMillan form that satisfy $f(p) = -f(-p)$, an additional spatial symmetry arises. Specifically, these mappings commute with the area-preserving involution:

$$\mathrm{T} = \mathrm{Rot}(\pi) \circ \mathrm{T} \circ \mathrm{Rot}^{-1}(\pi).$$

This property generates a distinct class of transformations:

$$\mathrm{Q}_1 = \mathrm{Ref}(\pi/4) \circ \mathrm{Rot}(\pi) = \mathrm{Ref}(3\,\pi/4) : \qquad \begin{array}{l} q' = -p, \\ p' = -q, \end{array}$$

and

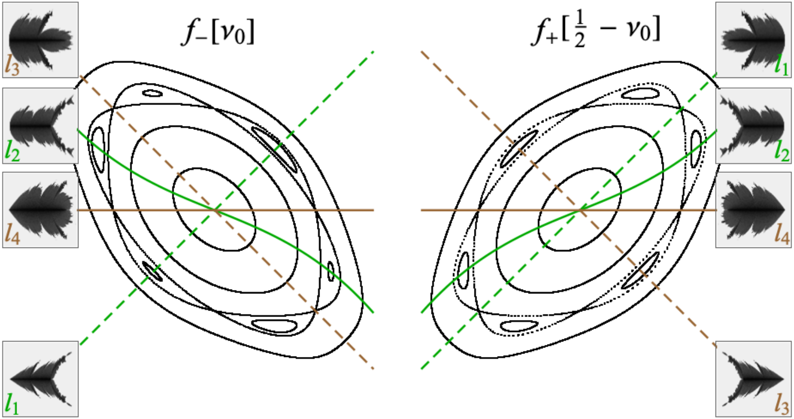
$$\mathrm{Q}_2 = \mathrm{T} \circ \mathrm{Q}_1 : \qquad \begin{array}{l} q' = -q, \\ p' = p - f(q). \end{array}$$

As a result, the system exhibits two additional symmetry lines

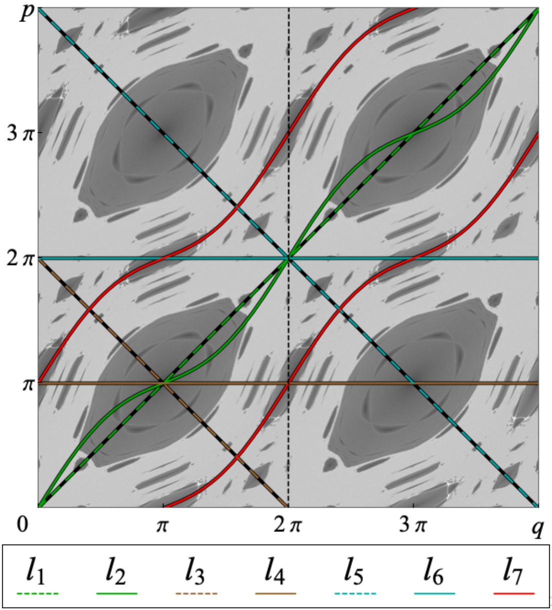
$$l_1:\; p = q, \qquad l_2:\; p = f(q)/2,$$

$$l_3:\; p = -q, \qquad l_4:\; p = 0.$$

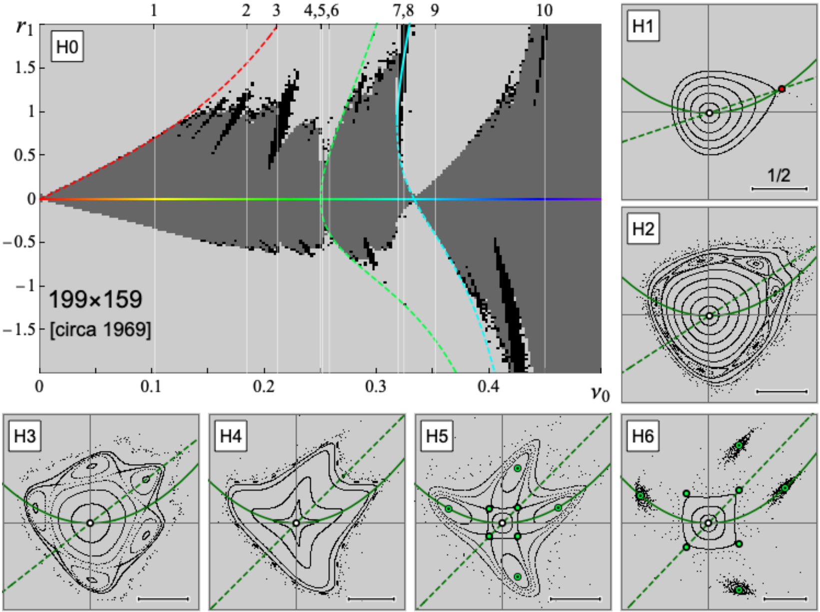
Example 2.2 Thin focusing and defocusing octupoles



Example 3. Thin RF station



1. Choice of initial conditions

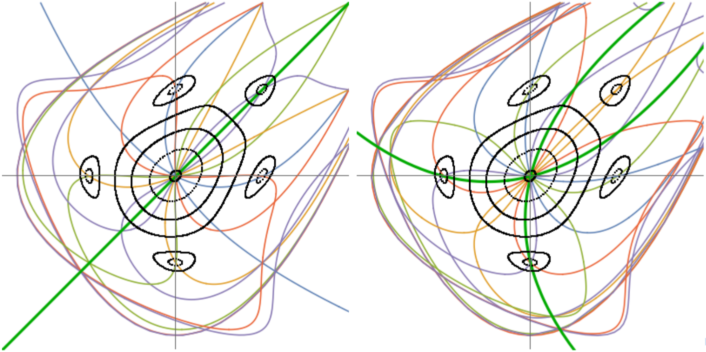


1.1 Symmetry lines and reversibility

Symmetry lines

• If R is a reversing symmetry of T, then so is the entire family of symmetries $\{\mathrm{T}^k \circ \mathrm{R}\}$, forming an infinite group along with the iterates of the map $\{\mathrm{T}^k\}$.

• Lastly, the set of fixed points of R, $\mathrm{Fix}\,\mathrm{R}$, and those of $\mathrm{Fix}\,\mathrm{T}^n\mathrm{R}$, form an infinite family of symmetry lines.



1.2 Invarinat sets

For a stable orbit of a point $\zeta_0 = (q_0, p_0)$, the trajectory typically exhibits one of three behaviors in phase space:

(I) The trajectory forms a zero-dimensional set of n distinct points visited in a unique periodic sequence — an n-cycle:

$$\mathrm{T}^n \zeta_0 = \zeta_0, \qquad n > 0.$$

(II) The trajectory forms a one-dimensional set that lies on an invariant curve, C, in the plane.

- **Chaotic systems**. KAM curves that are densely filled in a quasiperiodic manner.
- **Integrable systems**. Quasiperiodic orbits or groups of *non-isolated* n-cycles.

(III) The trajectory wanders without period or quasiperiodicity, densely covering a region of the phase space. These orbits are *chaotic*, exhibiting exponential sensitivity to initial conditions.

Example. Gingerbreadman map [Devaney]:

$$\mathrm{T}_{\text{Gingerbreadman}}: \qquad \begin{aligned} q' &= p, \\ p' &= -q + |p| + 1. \end{aligned}$$

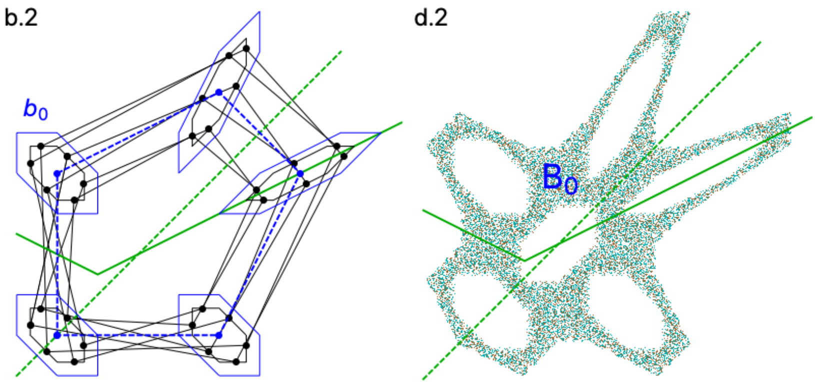
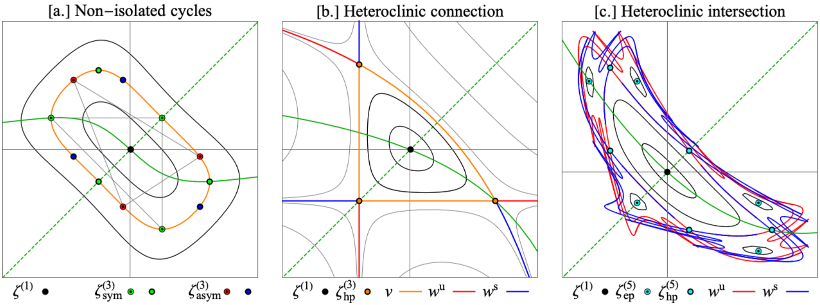


Figure: Phase space diagrams (q, p), showing 5- and 30-cycles (left) as well as few chaotic trajectories for a Gingerbreadman map.

A set of points, Γ, is called *invariant* under the map T, if T Γ = Γ. An invariant set is *symmetric*, if it is invariant under both T and R.

- n-cycles
- Invariant curves
- Stable and unstable manifolds

$$w^{\mathrm{s,u}}\left[\zeta_k^{(n)}\right] = \left\{\zeta_0 \;:\; \lim_{N\to\infty} \mathrm{T}^{\pm(n\cdot N)}\zeta_0 = \zeta_k^{(n)}\right\}$$



1.3 Symmetric groups

- Since for any point in a symmetric orbit, $\zeta \in \Gamma_{\mathrm{sym}}$,

$$\exists\, k \in \mathbb{Z}:\ \mathrm{R}\,\zeta = \mathrm{T}^{k}\zeta \in \Gamma_{\mathrm{sym}},$$

its trajectory must "hop across" the symmetry line if k is odd,

$$\mathrm{T}^{-1} \circ \mathrm{R}\left(\mathrm{T}^{(k-1)/2}\zeta\right) = \mathrm{T}^{(k-1)/2}\zeta \qquad (2)$$

or "cross" it if k is even,

$$\mathrm{R}\left(\mathrm{T}^{k/2}\zeta\right) = \mathrm{T}^{k/2}\zeta. \qquad (3)$$

- If ζ and $\mathrm{T}^{r}\zeta \neq \zeta$ lie on the same symmetry line, the orbit has an even period of $2r$. Moreover, a symmetric periodic orbit has an even period if and only if it includes two points on the same symmetry.

Simple model for symmetric groups

$$\mathrm{Rot} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \qquad \mathrm{Ref} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}.$$

Each iteration, $\mathrm{T} = \mathrm{R}_2 \circ \mathrm{R}_1$, is a sequential application of two involutions, here represented by linear reflections $\mathrm{Ref}(\theta)$. By aligning the first symmetry line with the horizontal axis ($\theta_1 = 0$), and using

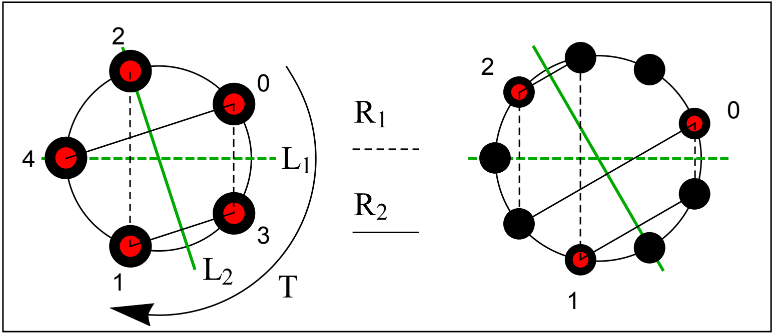
$$\mathrm{Ref}(\theta_2) \circ \mathrm{Ref}(\theta_1) = \mathrm{Rot}(2\,\theta_2 - 2\,\theta_1)$$

we see that the rotation number must satisfy

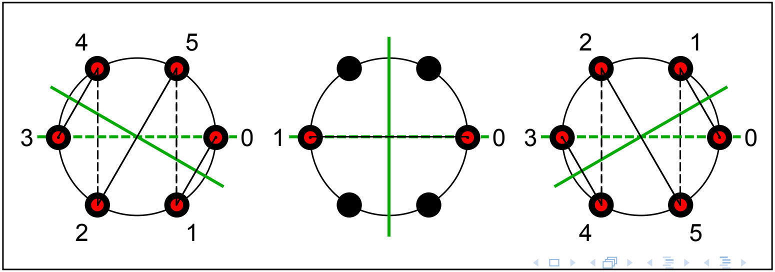
$$\nu = 2\,k/n$$

where k is the angle between symmetries divided over $2\,\pi$.

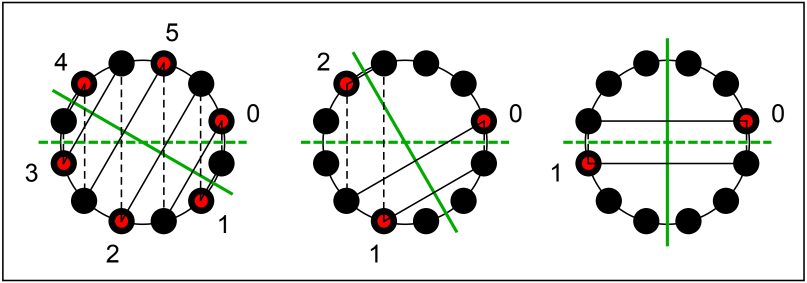
[a.] Odd group, $\Gamma \bigcap L_1 \neq \emptyset$, $\Gamma \bigcap L_2 \neq \emptyset$



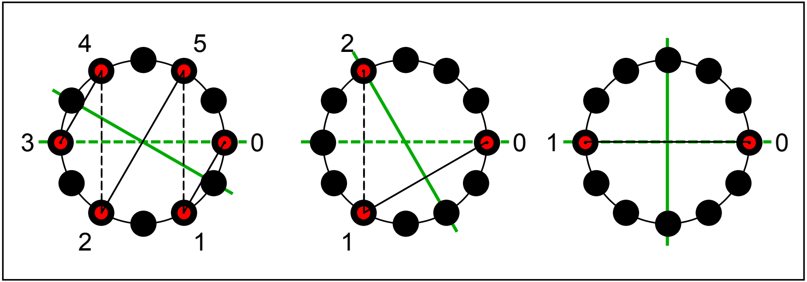
[b.] Even group, $\Gamma \bigcap L_1 \neq \emptyset$, $\Gamma \bigcap L_2 = \emptyset$



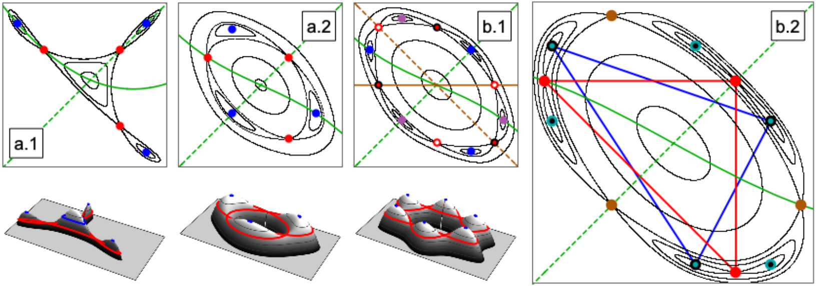
[c.] Even group, $\Gamma \bigcap L_1 = \emptyset$, $\Gamma \bigcap L_2 = \emptyset$



[d.] Even group, $\Gamma \bigcap L_1 \neq \emptyset$, $\Gamma \bigcap L_2 \neq \emptyset$



1.4 Islands and bifurcations



Fixed points:
✓Transcritical (T)
✓Saddle-Node (SN)
✓Pitchfork (PF)
2-cycles:
✓Period-Doubling (PD)

n-**cycles**, $n > 2$:
✓Touch-and-Go (TG)
✓Saddle-Node (SN)
✓*n*-island Chain
✓Doubled *n*-island Chain
– Asymmetric bifurcation

Comparison chart

	ML	FMA	GALI/REM
Tongues	$+$	$+$	$+$
Tongue structure	$-$	$\pm$	$+$
Anti-tongue	$\pm$	$-$	$+$

For ML plot we use ν as a color, and black for mode-locked refions

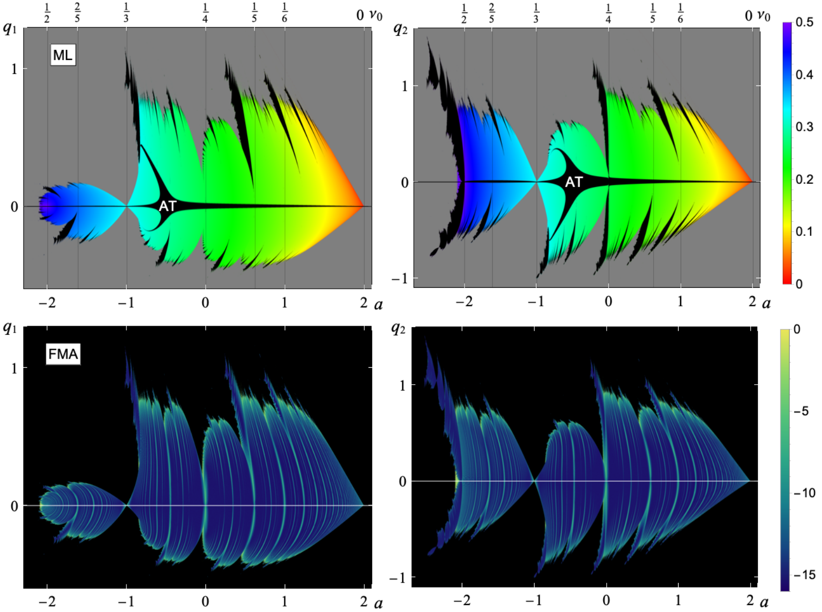
$$\mathrm{d}\nu/\mathrm{d}q = \epsilon \approx 0$$

For FMA and SALI/GALI we use $\log_{10}(10^{-16} + ind)$

$$ind_{\mathsf{FMA}} = |\nu_1 - \nu_2|$$

$$ind_{\mathsf{REM}} = \sqrt{(q_{fin} - q_{ini})^2 - (p_{fin} - p_{ini})^2}$$

ML vs. FMA vs. GALI



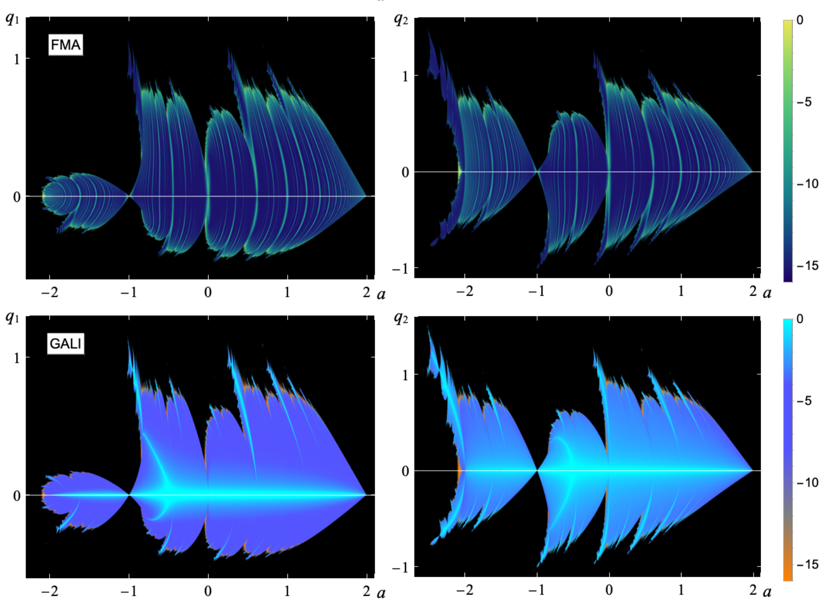


Figure: Isochronous and period-doubling diagrams for the Hénon map

Thank you for your attention.

Questions?