Isochronous and Period-Doubling Stability Diagrams

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Nonlinear Sciences > Chaotic Dynamics

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Isochronous and period-doubling diagrams for symplectic maps of the plane

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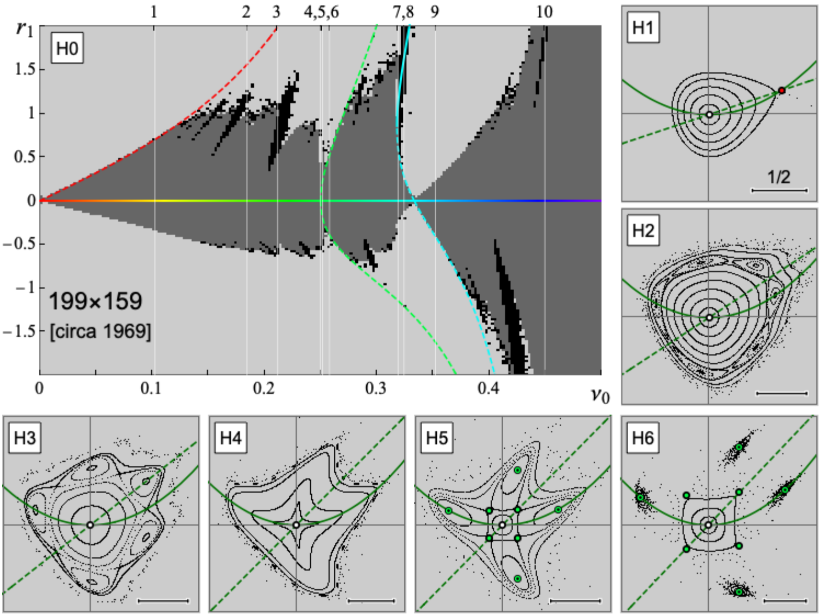
Goals and structure of presentation

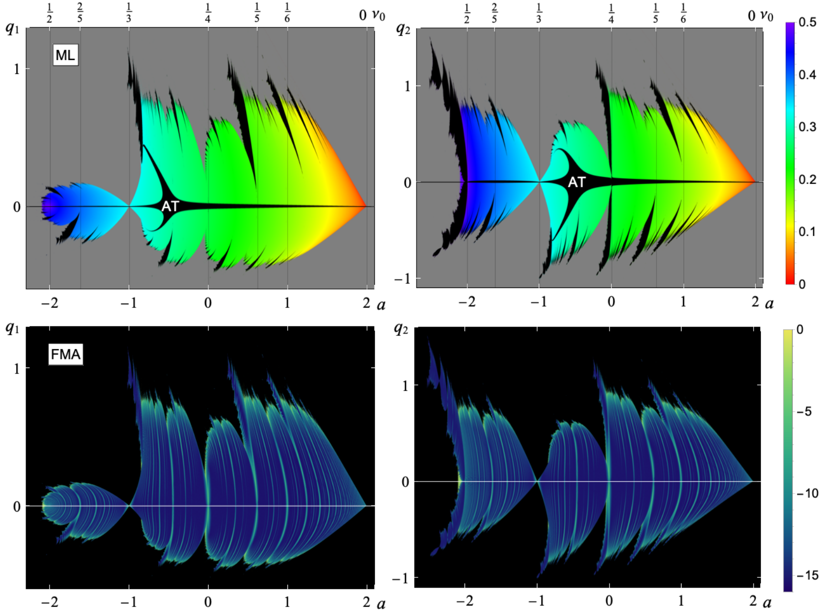
- Part 1. Understanding the diagram
  - Circle map and mode-locking
  - Twisted tongues
  - Seams, cuts, tears and frays
  - Twistless torus
  - Shapes of elementary domains
- Part 2. Results
  - Sextupole
  - Octupole
  - Decapole
  - Duodecapole
  - RF-station

0. Hénon set



1
2
3
4,5,6
7,8
9
10
r_1
H0
1.5
1
0.5
0
−0.5
−1
−1.5
199×159
[circa 1969]
0
0.1
0.2
0.3
0.4
ν_0
H1
1/2
H2
H3
H4
H5
H6

ML vs. FMA vs. SALI/GALI/REM



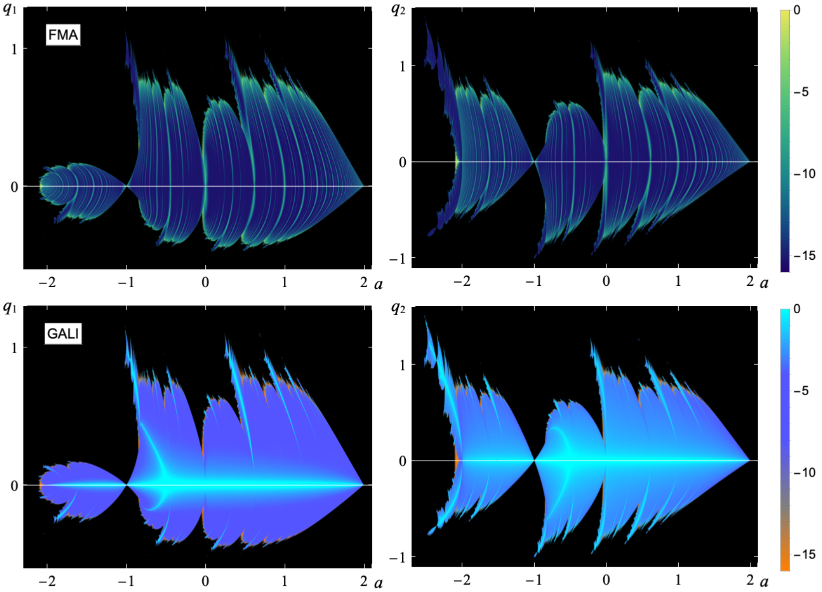


Figure: Isochronous and period-doubling diagrams for the Hénon map

1.1 Circle map and mode-locking

Standard circle map

$$\mathrm{T}_{\nu_0,\epsilon}:\ \theta' = \phi(\theta) \mod 2\pi, \qquad \theta \in \mathbb{S}^1 = [0;2\pi),$$

$$\phi(\theta) = \theta + \Omega + \epsilon \sin\theta, \qquad \Omega = 2\,\pi\,\nu_0.$$

$\nu_0 \in [0;1)$ is the *bare rotation number/tune*, while $\epsilon > 0$ is the *coupling strength*, describing the level of externally applied nonlinearity.

- **Unperturbed case**, $\epsilon = 0$. The system exhibits a *rigid rotation*:

$$\theta' = \theta + \Omega,$$

where every point moves at a constant angular velocity, Ω.

By analyzing the behavior of $\phi(\theta)$, we identify key scenarios:

• **Small perturbations**, $0 < \epsilon < 1$. In this range, the function $\phi(\theta)$ remains monotonically increasing, meaning all orbits have to move forward. The map $\mathrm{T}_{\nu_0,\epsilon}$ is an analytic *diffeomorphism* — smooth, invertible, and differentiable (along with its inverse) transformation.

• **High perturbations**, $\epsilon > 1$. When ϵ exceeds 1, the function $\phi(\theta)$ is no longer bijective, making the circle map $\mathrm{T}_{\nu_0,\epsilon}$ noninvertible. This opens the possibility to more complex dynamics, such as bistability and subharmonic routes to chaos.

• **Critical case**, $\epsilon = 1$. The value $\epsilon = 1$ is referred to as *critical*, as it marks the boundary between two qualitatively different behaviors seen in the intervals $0 < \epsilon < 1$ and $\epsilon > 1$.

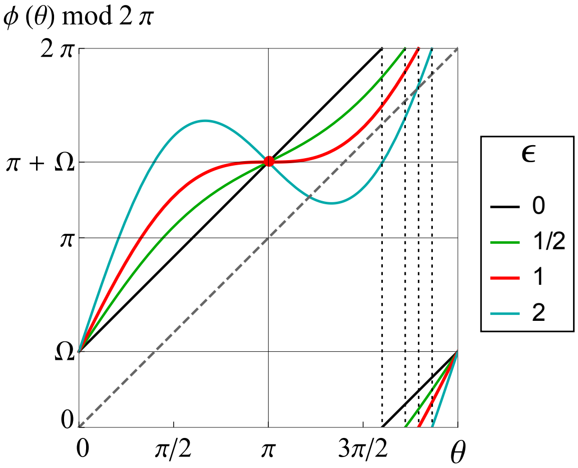


Figure: **Standard circle map's iterative function**. The solid colored curves represent one period of the $\phi(\theta) \bmod 2\pi$ for different values of the coupling strength parameter $\epsilon = 0, 1/2, 1, 2$ as indicated in the legend. The red curve marks the critical case. The black dashed line corresponds to a linear function with a slope of 1, included for reference.

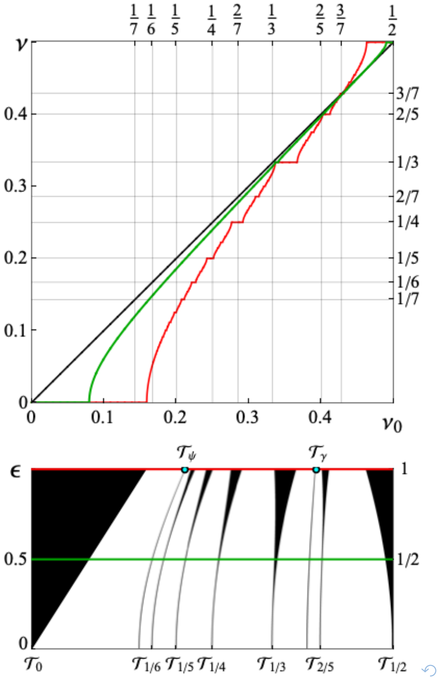
Top plot shows *devil's staircases*:

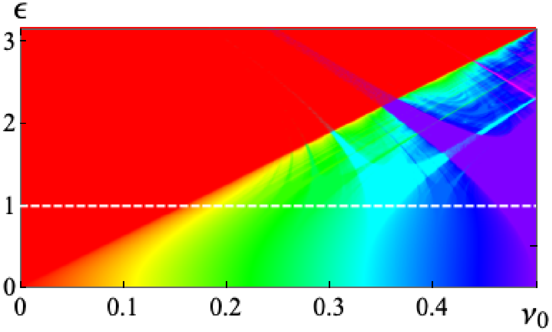
$$\nu = \frac{1}{2\pi} \lim_{n\to\infty} \frac{\phi^n(\theta_0) - \theta_0}{n}.$$

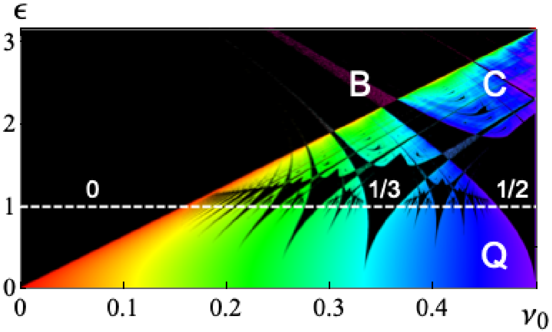
for $\epsilon = 1$ (red) and $1/2$ (green). The bottom plot schematically illustrates several *Arnold tongues*

$$\mathcal{T}_\alpha = \{(\nu_0, \epsilon) | \, \nu = \alpha\}$$

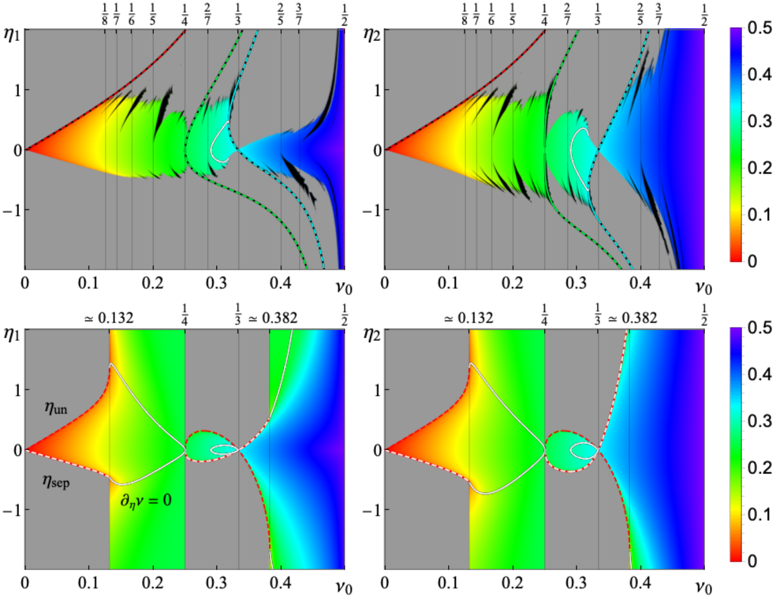
in the (ν_0, ϵ) space. Rational tongues $\mathcal{T}_\alpha$ corresponding to mode-locking are labeled along the horizontal axis, where α matches the value of ν_0. Two curves associated with irrational tongues, $\psi = \sqrt[5]{2} - 1$ and $\gamma = 2 - \phi_{GR}$, are marked at the top.




ϵ
3
2
1
0
0
0.1
0.2
0.3
0.4
ν_0


ϵ
3
2
1
0
B
C
0
1/3
1/2
Q
0
0.1
0.2
0.3
0.4
ν_0

Back to mode-locking plot



When comparing the ML diagrams with the bifurcation diagram of the circle map, two qualitative differences stand out:

(i) Rational tongues again appear in a sequence similar (but different) to the Farey sequence, for fixed values of $\nu_0 = 0$ and $\nu_0 = 1/3$ with small but nonzero distances along the symmetry lines ($|q_{1,2}| > 0$), we observe instability rather than mode-locked motion — *singular tongues*. For $\nu_0 = 1/2$ (with $a = -2$), a pair of tongues $\mathcal{T}_{1/2}$ forms along the second symmetry line; however, the motion becomes unstable near the origin.

(ii) Tongues respond differently to perturbations by varying slopes as a function of amplitude. For small ϵ in the circle map, the rotation number's derivative $\mathrm{d}/\mathrm{d}(\epsilon^2)$ varies monotonically

$$\lim_{\epsilon\to 0} \mathrm{d}(2\,\pi\,\nu)/\mathrm{d}(\epsilon^2) = -\frac{1}{4}\,\cot[\pi\,\nu_0].$$

Twist coefficients

In *canonical form* where J is the *action* and θ is the *angle*:

$$J' = J,$$
$$\theta' = \theta + 2\,\pi\,\nu(J),$$

the rotation number is often expressed as a power series of J:

$$\nu(J) = \nu_0 + \tau\,J = \nu_0 + \tau_0 J + \frac{1}{2!}\,\tau_1 J^2 + \frac{1}{3!}\,\tau_2 J^3 + \mathcal{O}(J^4),$$

where the derivative of ν with respect to J

$$\tau(J) = \frac{\mathrm{d}\nu}{\mathrm{d}J} = \tau_0 + \tau_1 J + \frac{1}{2}\,\tau_2 J^2 + \mathcal{O}(J^3)$$

known as the *twist*, plays a critical role in nonlinear stability.

McMillan form mappings with a smooth, differentiable force

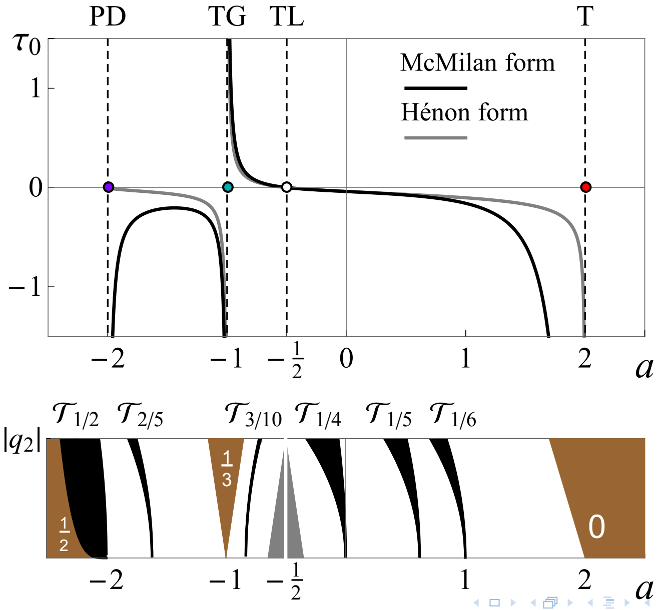
$$f(q) = a\,q + b\,q^2 + c\,q^3 + \ldots$$

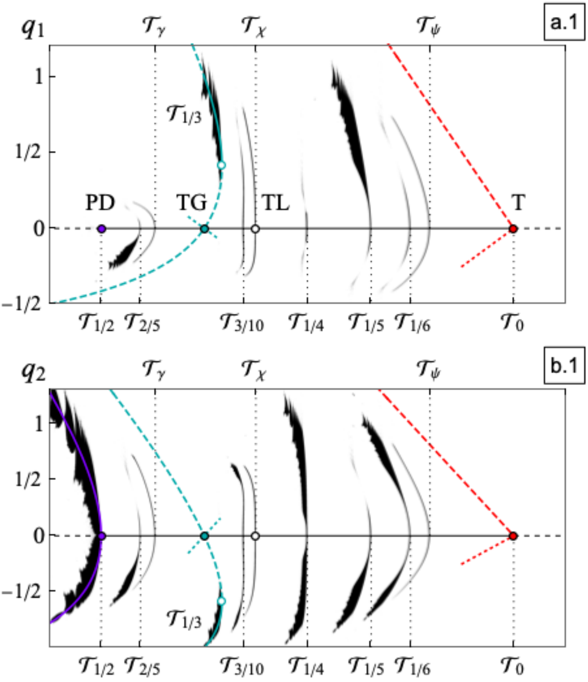
the first twist coefficient, τ_0, is expressed as:

$$2\,\pi\,\tau_0 = \frac{1}{4-a^2}\,\left[4\,b^2\frac{a+1/2}{(a-2)(a+1)} - 3\,c\right].$$

When $b \neq 0$, τ_0 is defined for values of ν_0 excluding $0, 1/2$ and $1/3$ $(a \neq 2, -2, -1)$, where it becomes singular, while τ_1 also requires $\nu_0 \neq 1/4, 1/5, 2/5$ $(a \neq 0, (-1 \pm \sqrt{5})/2)$.

Hénon twists




a.1
q_1
$\mathcal{T}_\gamma$
$\mathcal{T}_\chi$
$\mathcal{T}_\psi$
1
1/2
0
−1/2
$\mathcal{T}_{1/3}$
PD
TG
TL
T
$\mathcal{T}_{1/2}$ $\mathcal{T}_{2/5}$ $\mathcal{T}_{3/10}$ $\mathcal{T}_{1/4}$ $\mathcal{T}_{1/5}$ $\mathcal{T}_{1/6}$ $\mathcal{T}_0$
b.1
q_2
$\mathcal{T}_\gamma$
$\mathcal{T}_\chi$
$\mathcal{T}_\psi$
1
1/2
0
−1/2
$\mathcal{T}_{1/3}$
$\mathcal{T}_{1/2}$ $\mathcal{T}_{2/5}$ $\mathcal{T}_{3/10}$ $\mathcal{T}_{1/4}$ $\mathcal{T}_{1/5}$ $\mathcal{T}_{1/6}$ $\mathcal{T}_0$

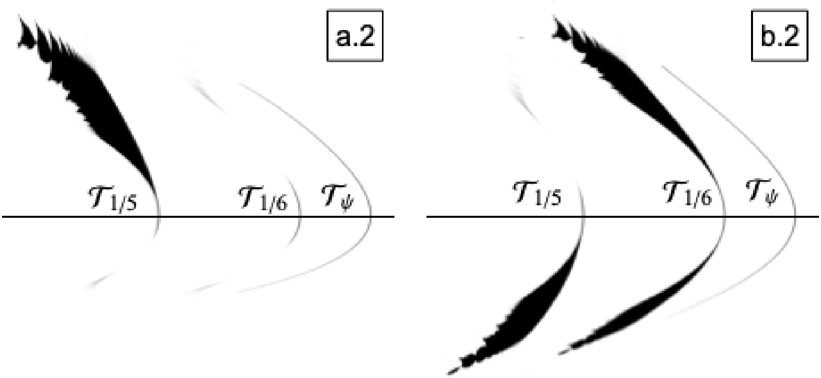


Figure: Magnified vicinity of resonances at $\nu_0 = 1/5$ and $\nu_0 = 1/6$, highlighting the typical structures for odd and even island chains.

Lets get into details...

1.3 Seam, cut, fray and tear

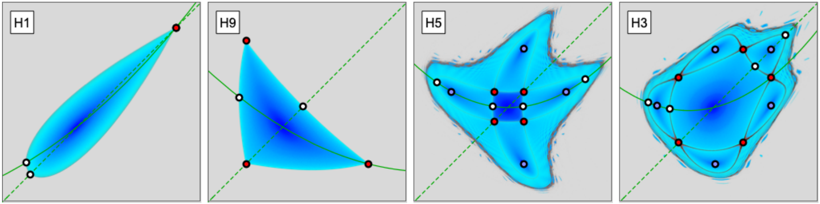


Figure: **Symmetry lines and tongues**. The first two plots show phase-space portraits near singular tongues corresponding to transcritical [H1] and touch-and-go [H9] bifurcations, where ν_0 is close to 0 and $1/3$. The following two plots represent typical scenarios for regular tongues, illustrating even [H5] and odd [H3] island bifurcations near $\nu_0 = 1/4$ and $\nu_0 = 1/5$. Stable cycles are shown in blue, unstable cycles in red, and intersections of symmetry lines with stable/unstable manifolds are highlighted in white. These phase-space portraits are recreations of Hénon's original palettes, presented here in McMillan form, with REM parameters used for coloring.

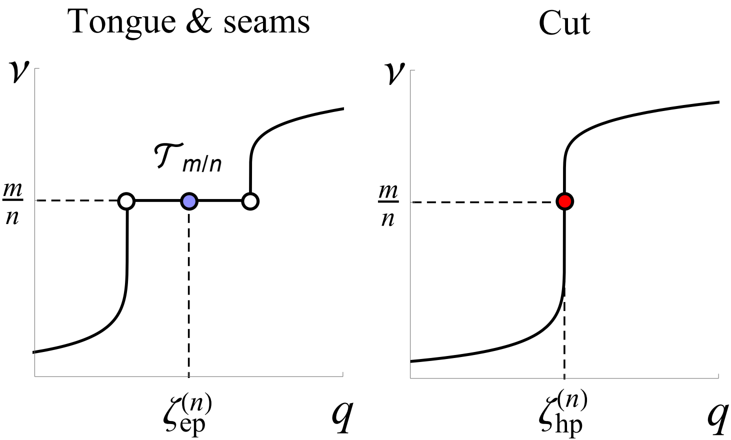
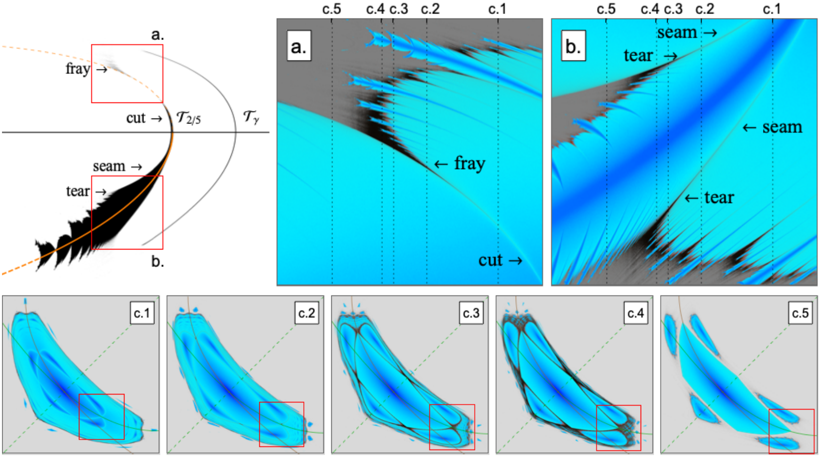
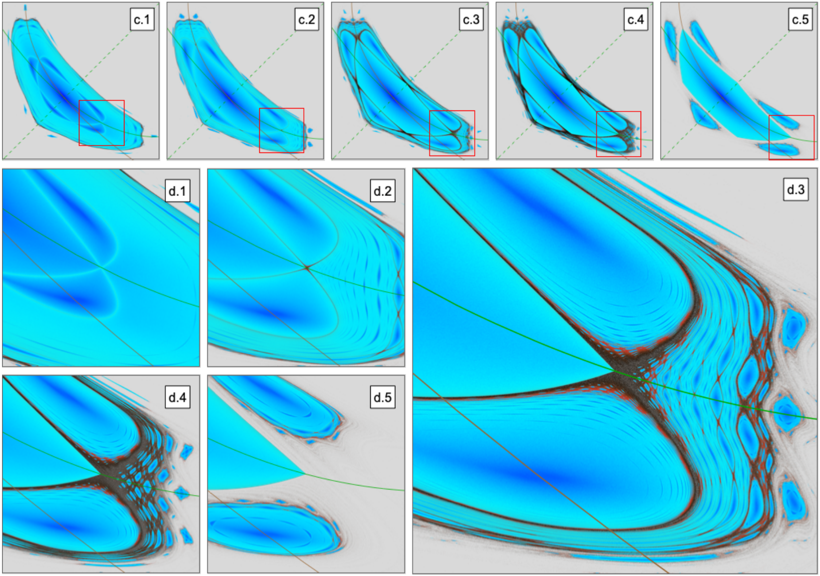


Figure: **Seam and cut**. Schematic illustration shows the devil's staircase pattern as the symmetry line crosses through the center of the island (stable n-cycle $\zeta_{\rm ep}^{(n)}$ in blue) and node (unstable n-cycle $\zeta_{\rm un}^{(n)}$ in red). The left plot represents an Arnold tongue — a flat region bordered by intersections with the stable and unstable manifolds of $\zeta_{\rm un}^{(n)}$ (white points). In the right plot, a "cut" appears where the derivative of $\nu(q)$ diverges to infinity.

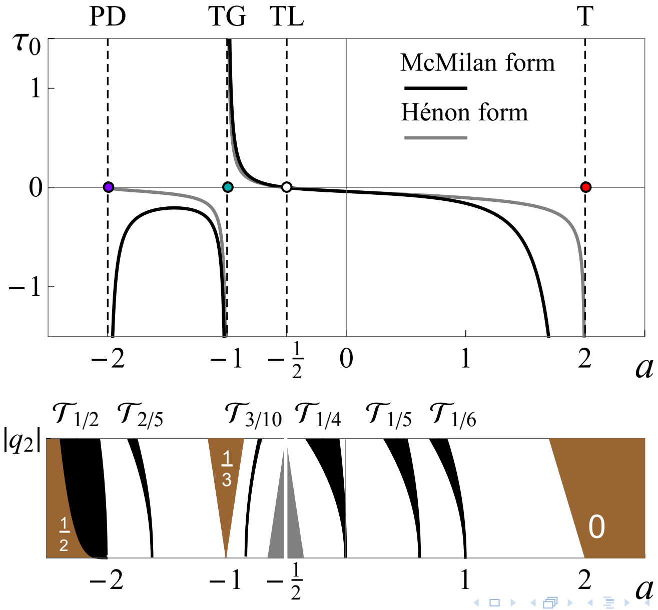
Fray and tear



Fray and tear (phase space)



1.4 Twistless torus



Ribcage and cobras

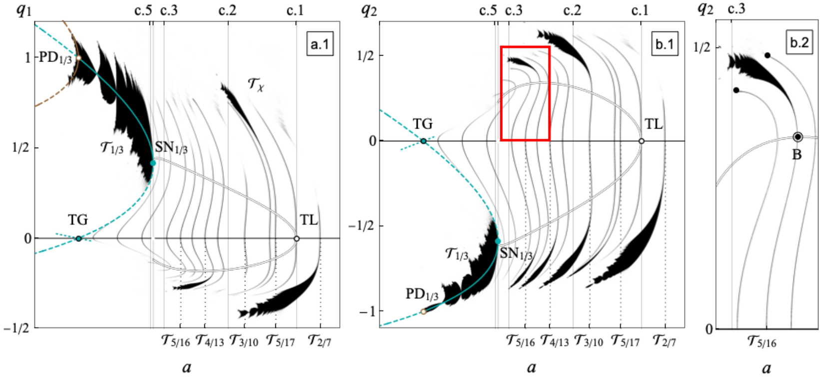
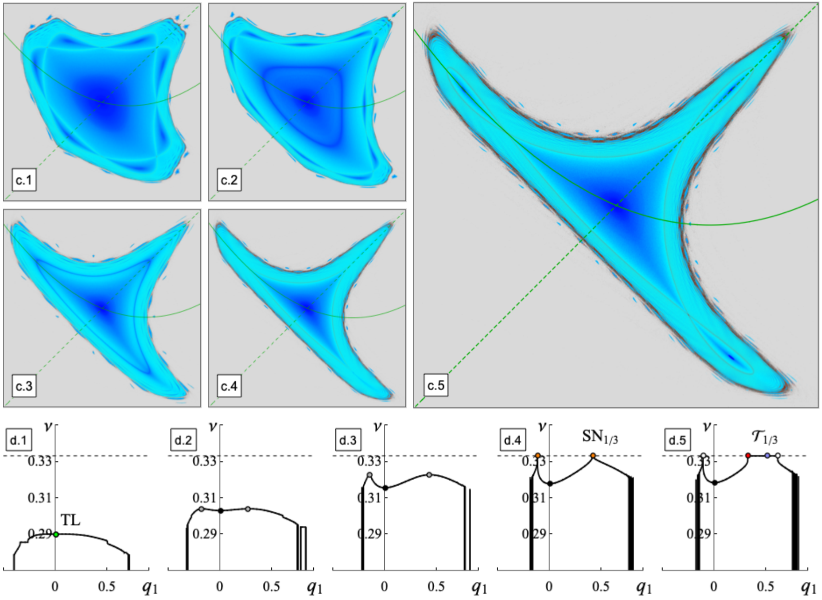


Figure: Structure of Arnold tongues in the Hénon map for the region with positive twist coefficient $\tau_0 > 0$, i.e., $a \in (-1, -1/2)$.

Life of a twistless torus & Stabilization



1.5 Shapes of elementary domains

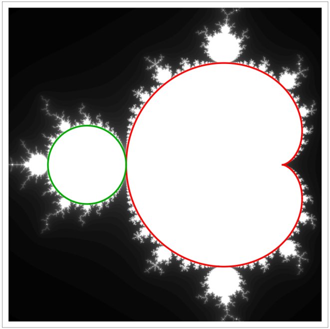
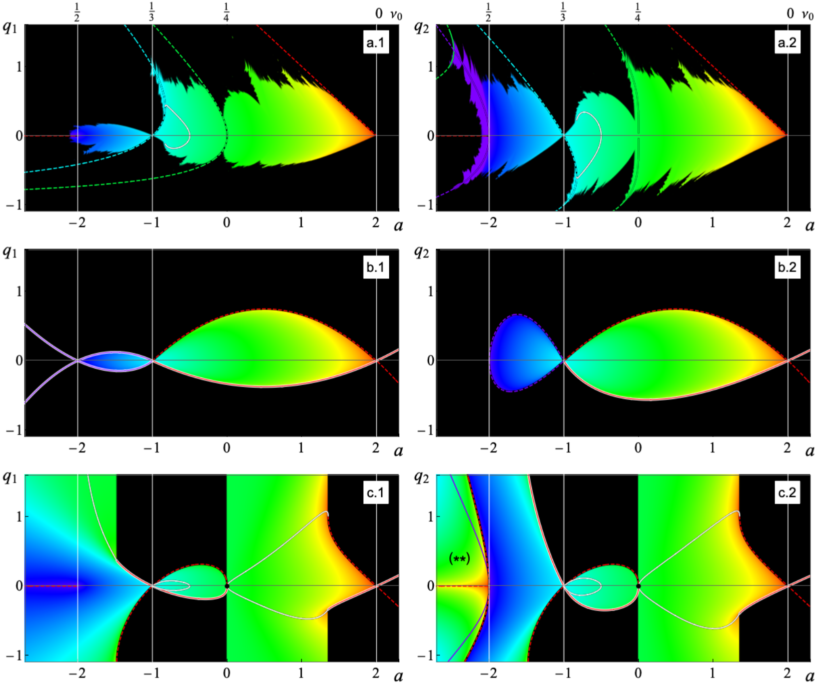


Figure: The MBM set with its central cluster approximated by a cardioid (red) and a cluster at a non-root node approximated by a circle (green).

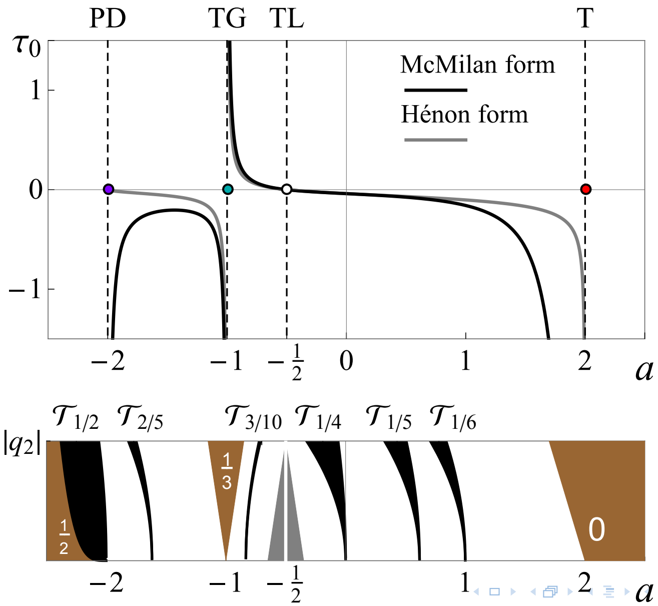
$$c - c_0 = r_0\, e^{i\,\phi}\left(1 - \frac{1}{2}\, e^{i\,\phi}\right) \quad \text{and} \quad c - c_0 = r_0\, e^{i\,\phi}.$$

V. Dolotin and A. Morozov, International Journal of Modern Physics A **23**, 3613 (2008),



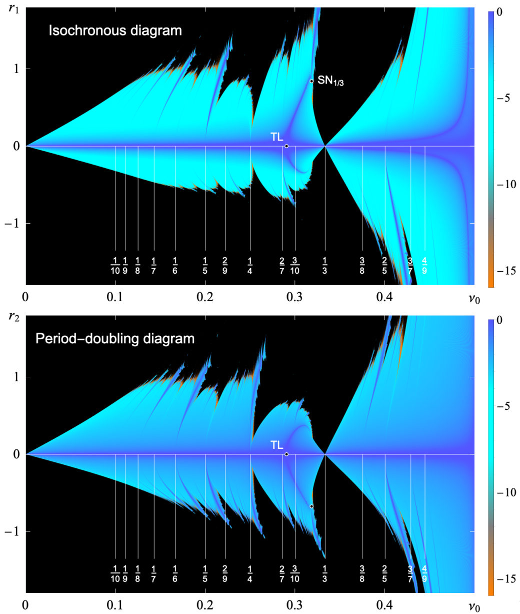
a.1
a.2
b.1
b.2
c.1
c.2
(**)
q_1
q_2
a
ν_0

2. Results

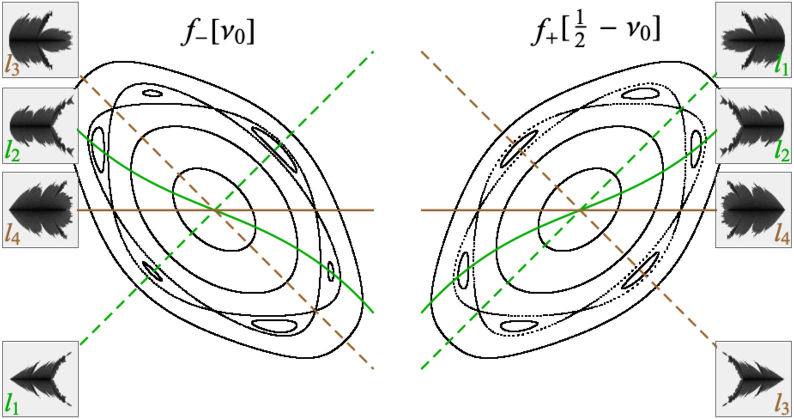


PD
TG
TL
T
τ_0
1
0
−1
McMilan form
Hénon form
−2
−1
$-\frac{1}{2}$
0
1
2
a
$\mathcal{T}_{1/2}$
$\mathcal{T}_{2/5}$
$\mathcal{T}_{3/10}$
$\mathcal{T}_{1/4}$
$\mathcal{T}_{1/5}$
$\mathcal{T}_{1/6}$
$|q_2|$
$\frac{1}{3}$
$\frac{1}{2}$
0
−2
−1
$-\frac{1}{2}$
1
2
a

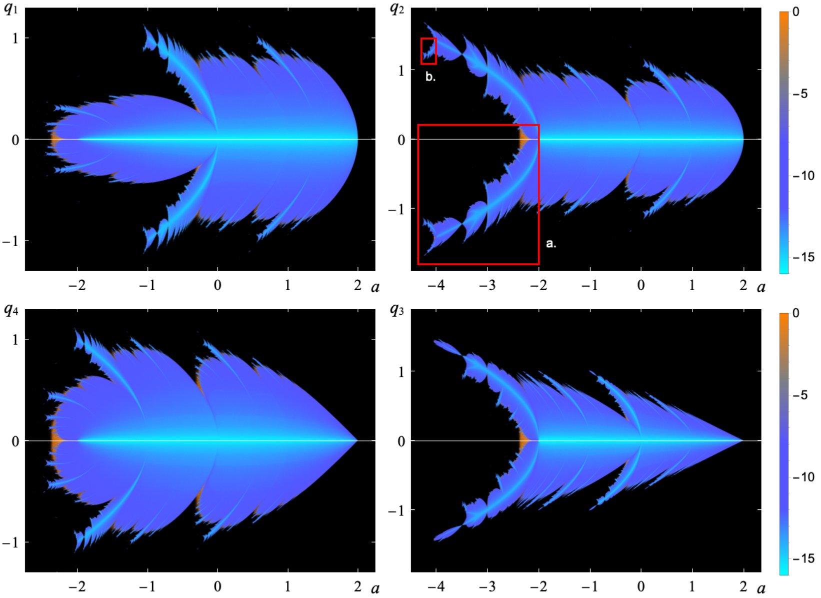
2.1 Thin sextupole in Hénon form of the map



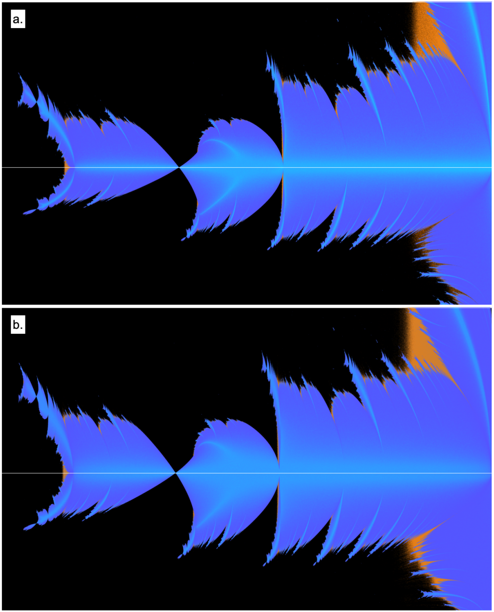
2.2 Thin focusing and defocusing octupoles



Beetles and Bugs



Self similarity



Twist coefficients

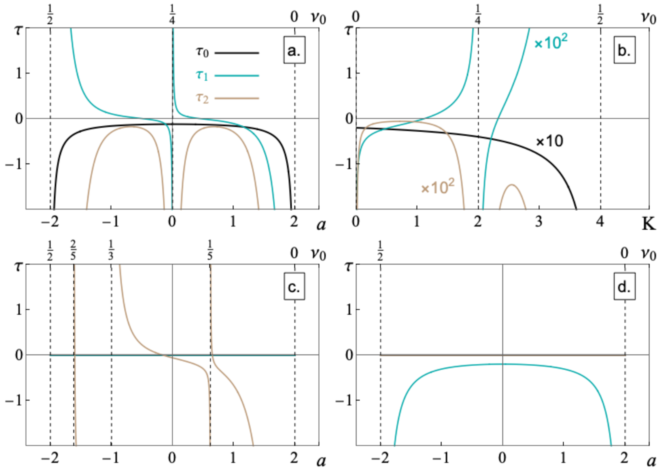
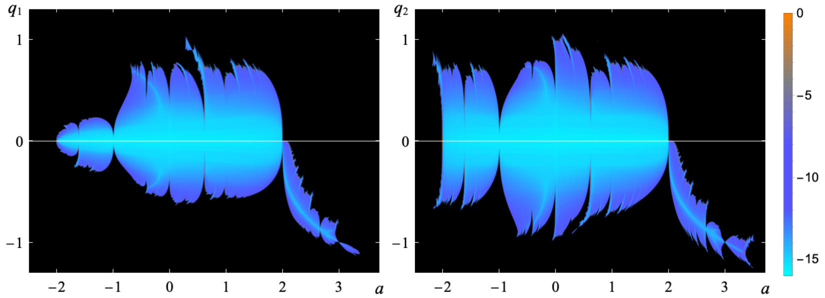
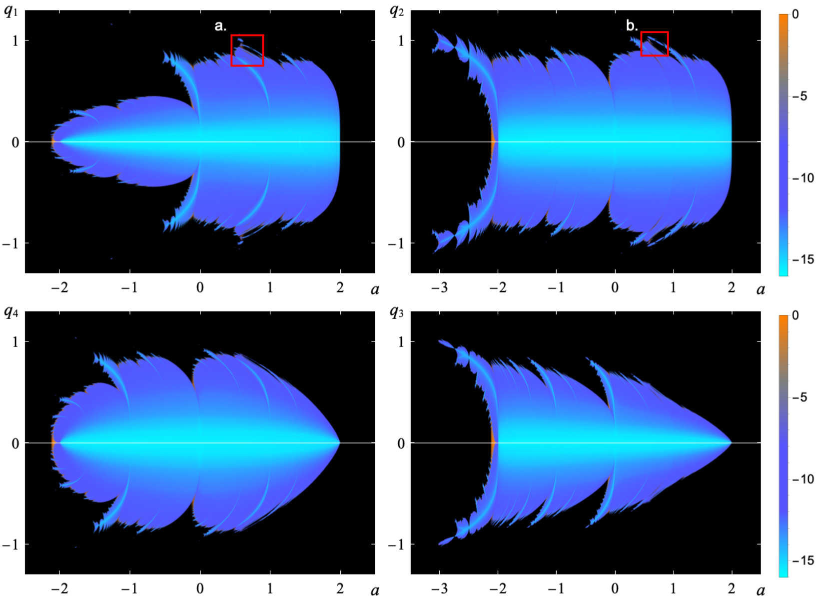


Figure: Twist coefficients for (a.) cubic map, $f_+(p) = a\,p + p^3$, (b.) Chirikov map, $f(p) = 2\,p + \mathrm{K}\sin p$, (c.) fourth-power, $f_+(p) = a\,p + p^4$, and (d.) fifth-power, $f_+(p) = a\,p + p^5$, mappings.

2.3 Thin decapole

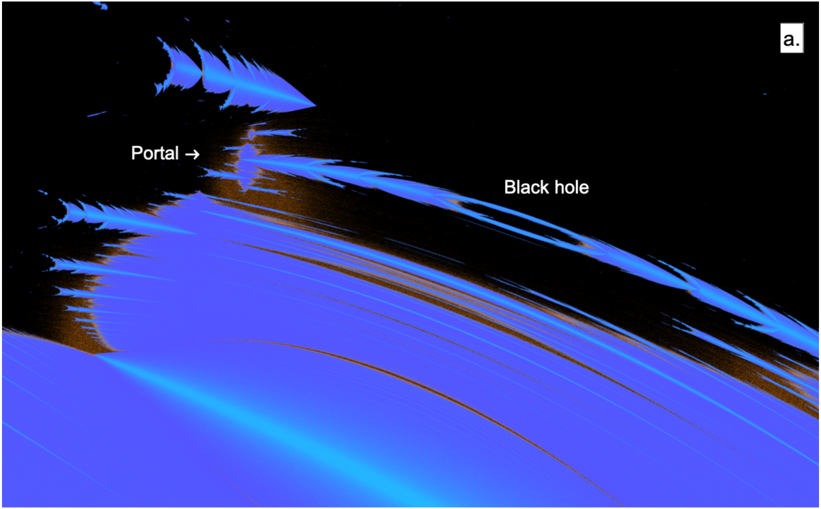


2.4 Thin focusing and defocusing duodecapoles

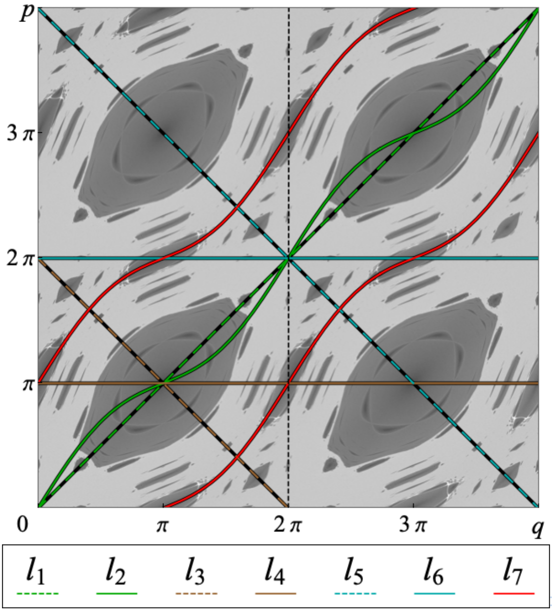


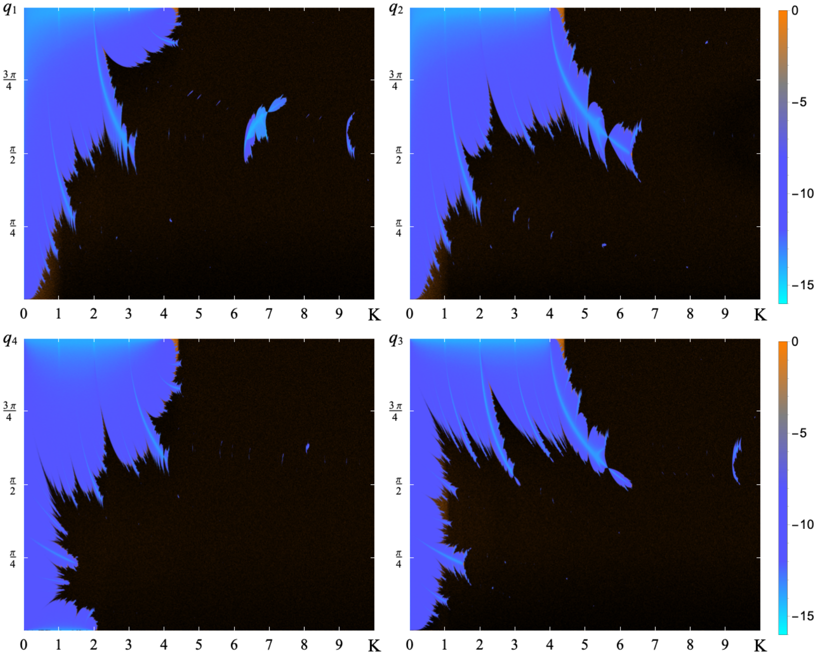
Portal and Black hole

Complex structures can be observed upon the magnification:



2.5 Thin RF station





q_1
q_2
q_4
q_3
$\frac{3\pi}{4}$
$\frac{\pi}{2}$
$\frac{\pi}{4}$
0 1 2 3 4 5 6 7 8 9 K
0
−5
−10
−15

Twist coefficients

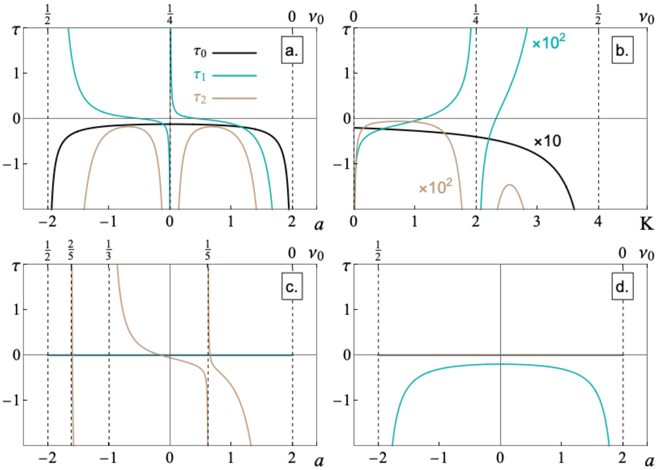


Figure: Twist coefficients for (a.) cubic map, $f_+(p) = a\,p + p^3$, (b.) Chirikov map, $f(p) = 2\,p + \mathrm{K}\sin p$, (c.) fourth-power, $f_+(p) = a\,p + p^4$, and (d.) fifth-power, $f_+(p) = a\,p + p^5$, mappings.

Thank you for your attention.

Questions?