Precision Tracking of Beam Dynamics: COSY-INFINITY

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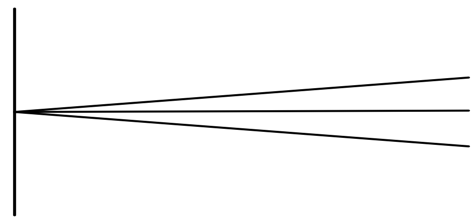
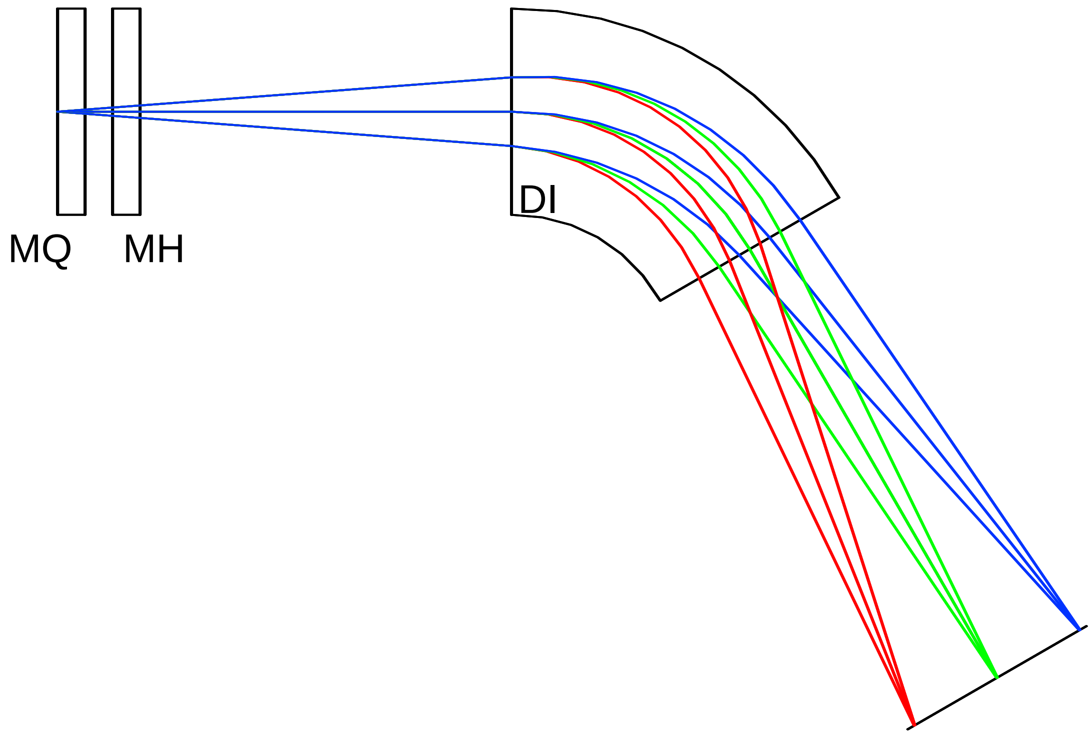
Start Tracking with Various Initial Conditions

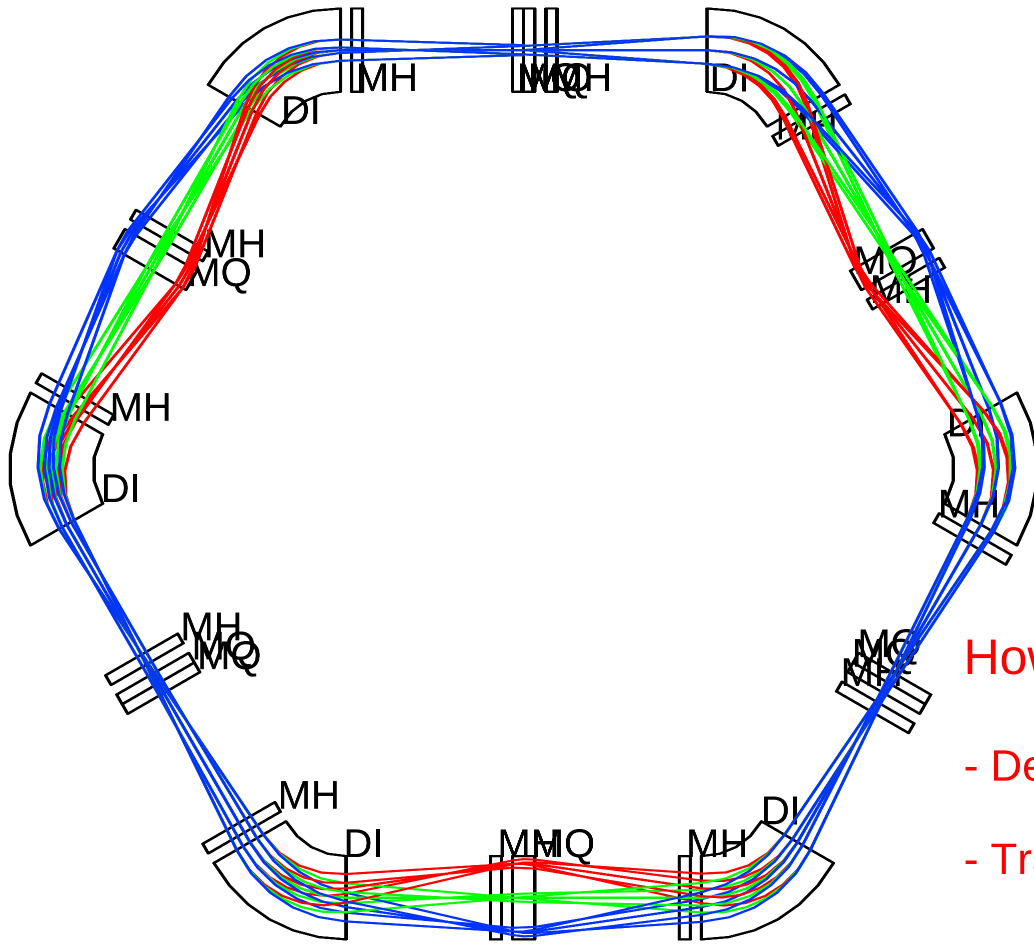


This example shows 27 beams, consisting of 3 x slopes, 3 y sloped, and 3 energies.

Track through a Cell consisting of these Beam Elements.



Forming a Ring.



How efficiently can we

- Design a system?

- Track through?

This example shows a low order achromatic system.

Transfer Map Method and Differential Algebras

- The transfer map $\mathcal{M}$ is the flow of the system ODE.

$$\vec{z}_f = \mathcal{M}(\vec{z}_i, \vec{\delta}),$$

  where $\vec{z}_i$ and $\vec{z}_f$ are the initial and the final condition, $\vec{\delta}$ is system parameters.

- For a repetitive system, only one cell transfer map has to be computed. Thus, it is much faster than ray tracing codes (i.e. tracing each individual particle through the system).
- The Differential Algebraic method allows a very efficient computation of high order Taylor transfer maps.
- The Normal Form method can be used for analysis of nonlinear behavior.

Differential Algebras (DA)

- it works to arbitrary order, and can keep system parameters in maps.
- very transparent algorithms; effort independent of computation order.

The code **COSY Infinity** has many tools and algorithms necessary.

COSY INFINITY

- Arbitrary order
- Maps depending on parameters (even with mass dependence)
- No approximations in motion or field description
- Large library of elements
- Arbitrary Elements (you specify fields)
- Very flexible input language
- Powerful interactive graphics
- Errors: position, tilt, rotation
- Tracking through maps (with/without symplectification. EXPO)
- Normal Form Methods
- Spin dynamics
- Fast fringe field models using SYSCA approach
- Reference manual (70 pages) and Programming manual (100 pages)
- Currently about 2000 registered users

Field Description in Differential Algebra

There are various DA algorithms to treat the fields of beam optics efficiently. For example, **DA PDE Solver**

- requires to supply only
  - the midplane field for a midplane symmetric element.
  - the on-axis potential for straight elements like solenoids, quadrupoles, and higher multipoles.
- treats arbitrary fields straightforwardly.
  - Magnet (or, Electrostatic) fringe fields:
    The Enge function fall-off model

$$F(s) = \frac{1}{1 + \exp(a_1 + a_2 \cdot (s/D) + ... + a_6 \cdot (s/D)^5)}$$

    where D is the full aperture.
    Or, any arbitrary model including the measured data representation.
  - Solenoid fields including the fringe fields.
  - Measured fields: E.g. Use Gaussian wevelet representation.
  - Etc. etc.

DA Fixed Point PDE Solvers

The **DA fixed point theorem** allows to **solve PDEs iteratively** in **finitely many steps** by rephrasing them in terms of a fixed point problem.

Consider the rather general PDE

$$a_1 \frac{\partial}{\partial x}\left(a_2 \frac{\partial}{\partial x} V\right) + b_1 \frac{\partial}{\partial y}\left(b_2 \frac{\partial}{\partial y} V\right) + c_1 \frac{\partial}{\partial z}\left(c_2 \frac{\partial}{\partial z} V\right) = 0,$$

where a_i, b_i, c_i are functions of x, y, z.

The PDE is re-written in **fixed point form** as

$$\begin{aligned} V = V|_{y=0} &+ \int_0^y \frac{1}{b_2}\left(b_2 \frac{\partial V}{\partial y}\right)\Bigg|_{y=0} \\ &- \int_0^y \frac{1}{b_2} \int_0^y \left(\frac{a_1}{b_1}\frac{\partial}{\partial x}\left(a_2 \frac{\partial V}{\partial x}\right) + \frac{c_1}{b_1}\frac{\partial}{\partial z}\left(c_2 \frac{\partial V}{\partial z}\right)\right) dy dy. \end{aligned}$$

Assume the derivatives of V and $\partial V / \partial y$ with respect to x and z are **known in the plane** $y = 0$. Then the right hand side is **contracting** with respect to y (which is necessary for the DA fixed point theorem), and the various orders in y can be **iteratively** calculated by mere iteration.

Fringe Field Effects in Small Rings

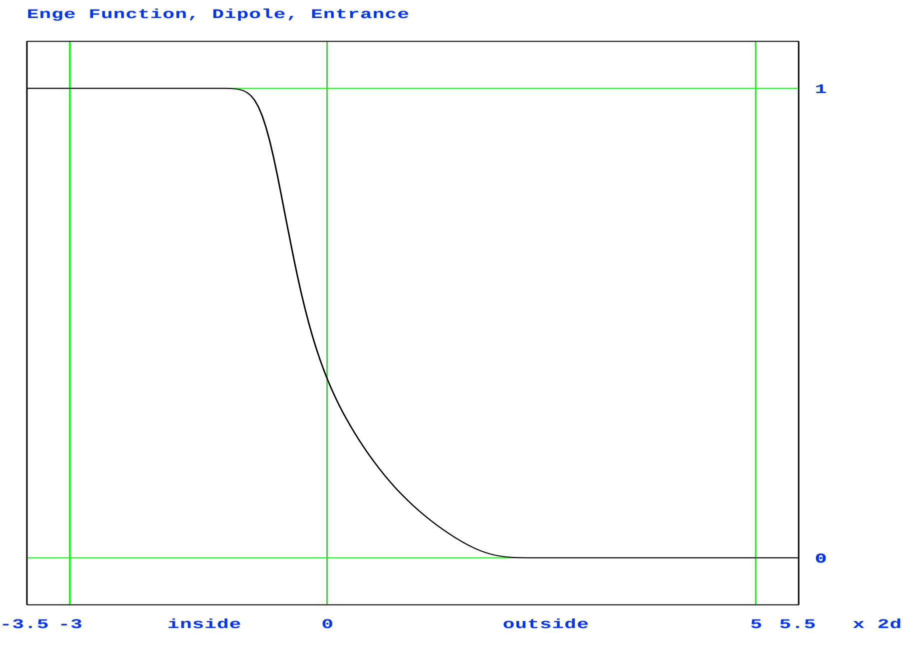
Fringe fields are the regions of the element where the field falls off from being nearly constant with respect to position on reference axis to being nearly zero

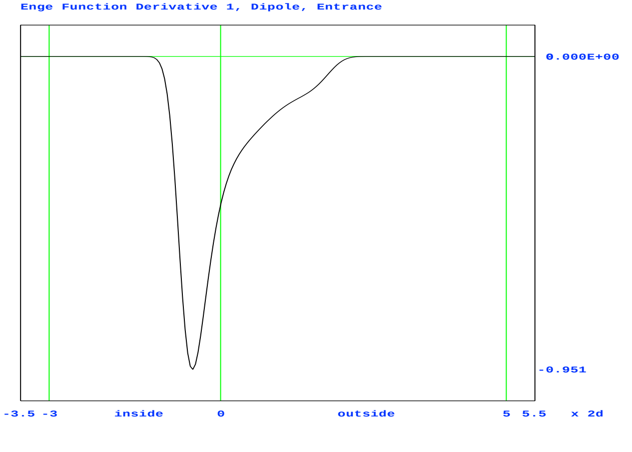
- Fringe fields represent an important part of the ring
  - In small rings, the fringe field region can be about half of the total "length" of the optical element
  - (In LHC or Tevatron, elements are much longer compared to their aperture)
  - In small rings, the change of dynamics (phase advance or linear transfer matrix) in each element is large
  - (In LHC or Tevatron, the effect of each element is much less)
  - In small rings, even the reference orbit is affected
  - (In LHC or Tevatron, carefully finding the "Effective Field Boundary" avoids this problem

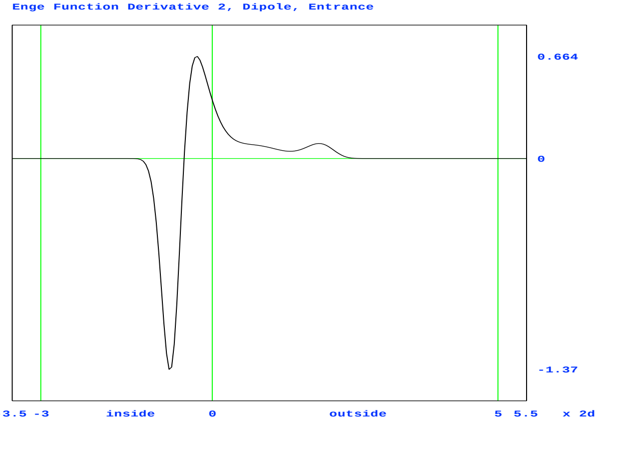
Fringe Fields and Their Nonlinearities

- Fringe fields are often the main source of (non-deliberate) nonlinearities
  - In main fields, one of course attempts very carefully to keep the field constant in direction of reference orbit, and imposes specific axial dependencies
  - In fringe fields, there is natural nonlinearity due to unavoidable curvature of electric or magnetic field lines.
  - These curvatures of fields affect particles at different distances from reference orbit differently, and because of curvature, they do so nonlinearly.
  - All these things are unavoidable; they are a direct consequence of Maxwell's equations.

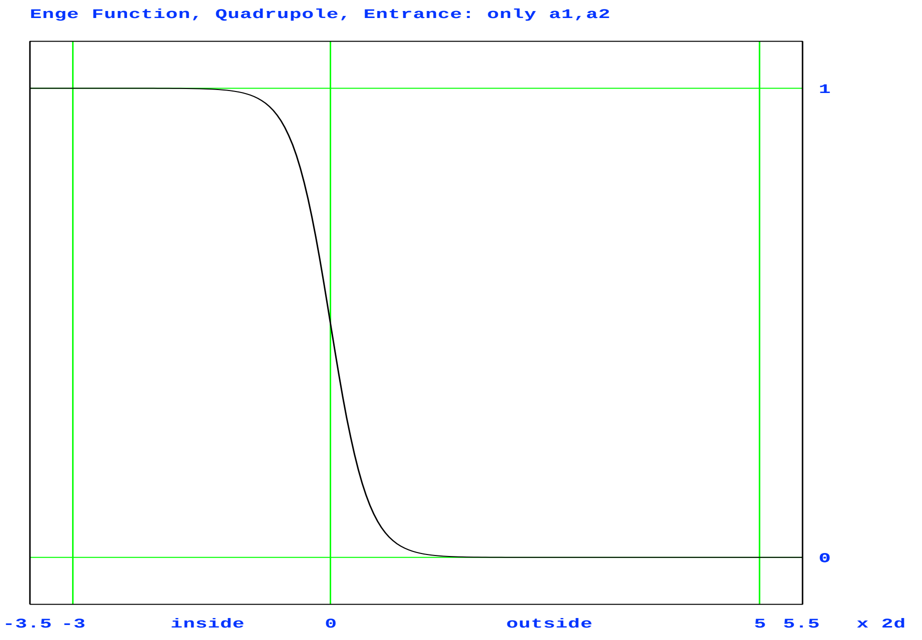
Dipole Enge Function (COSY default)

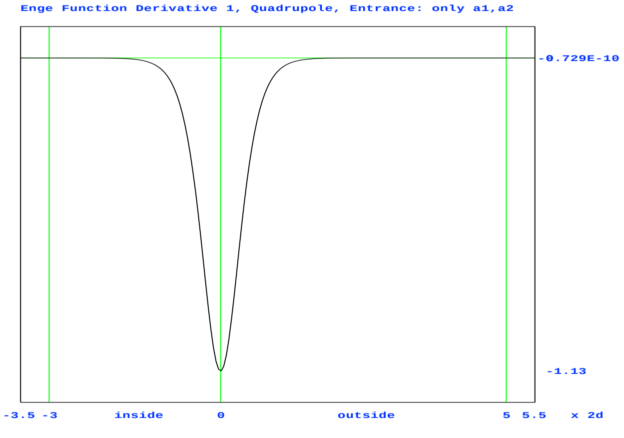


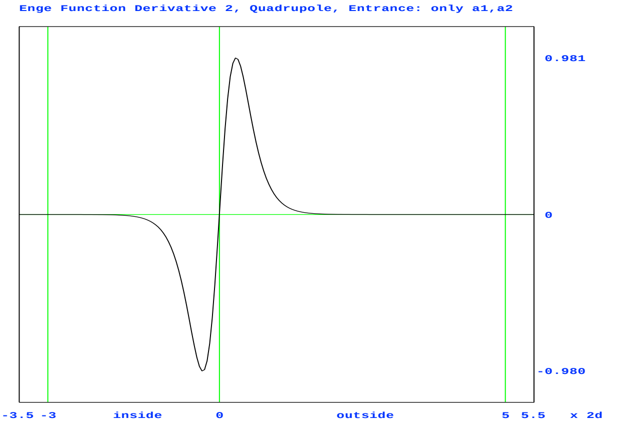


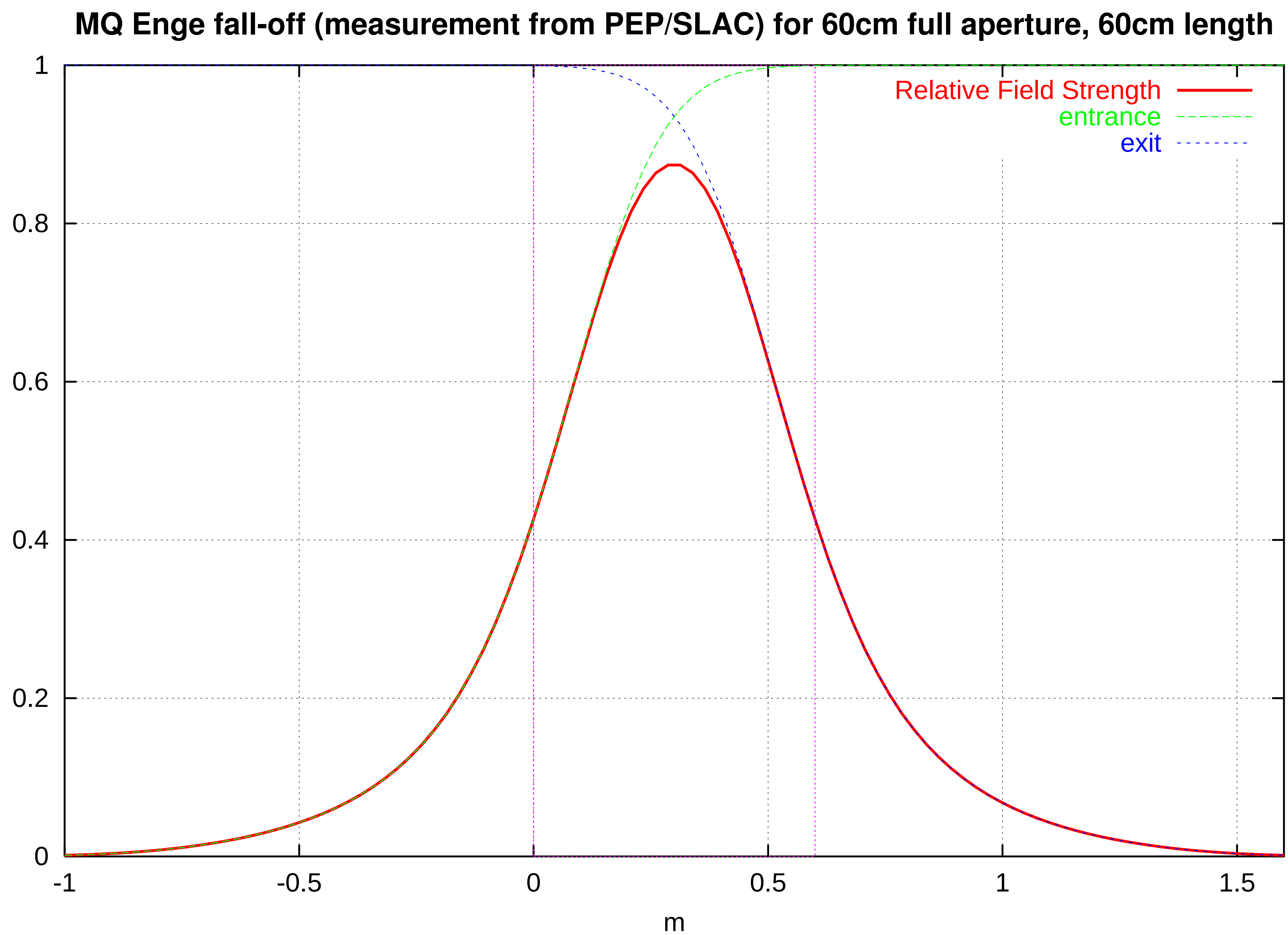


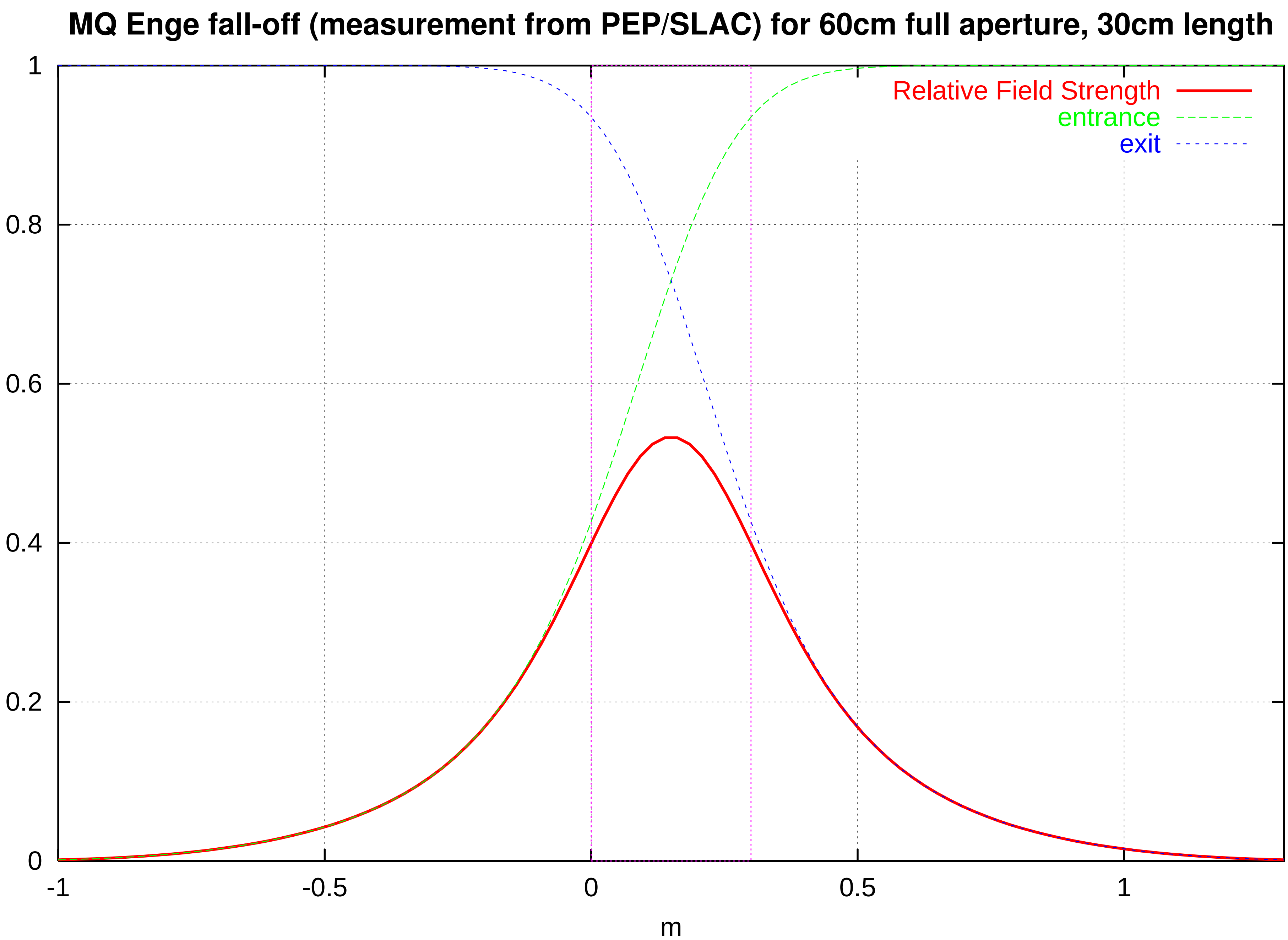
Quadrupole Enge Function (only a_1, a_2)

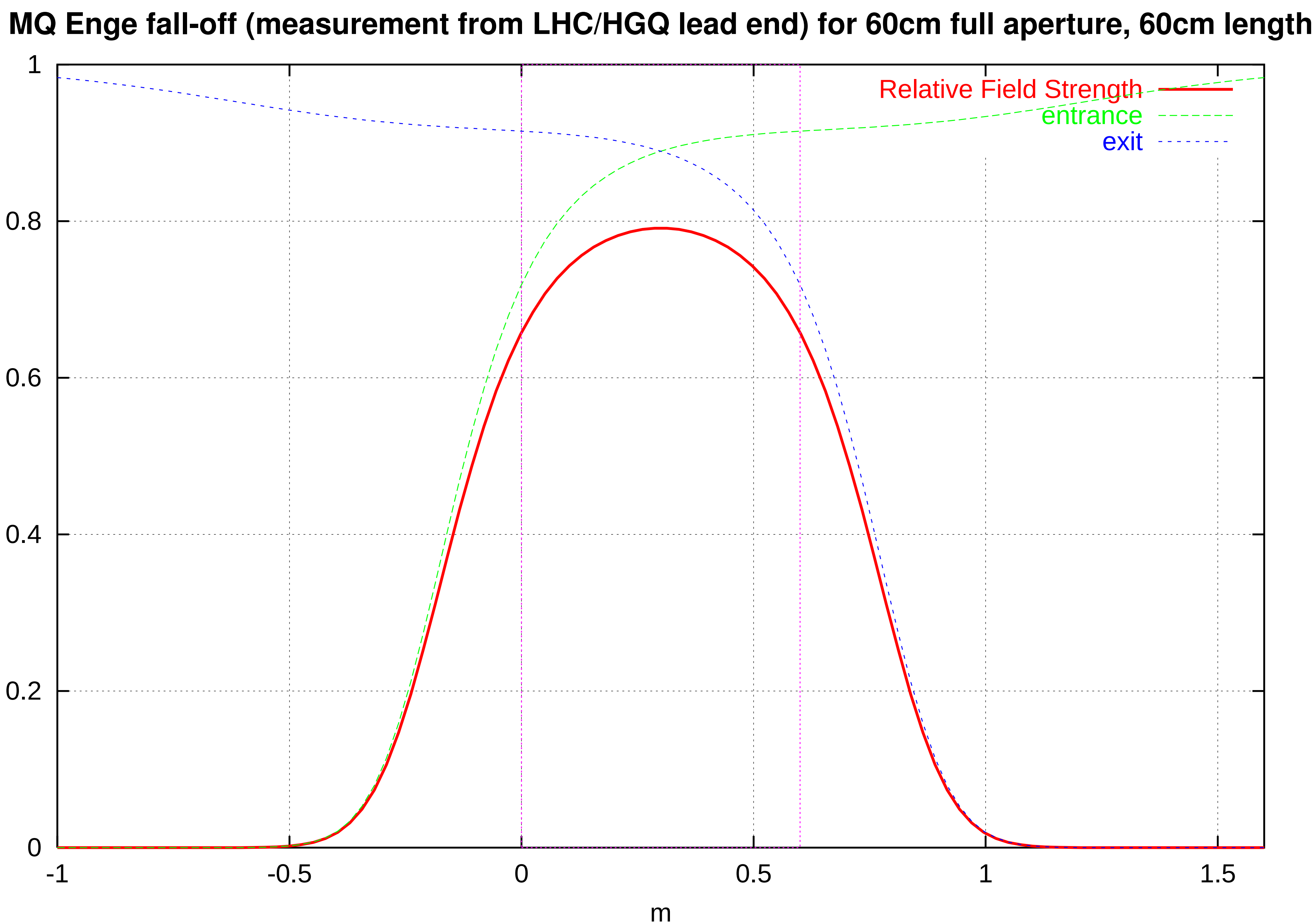




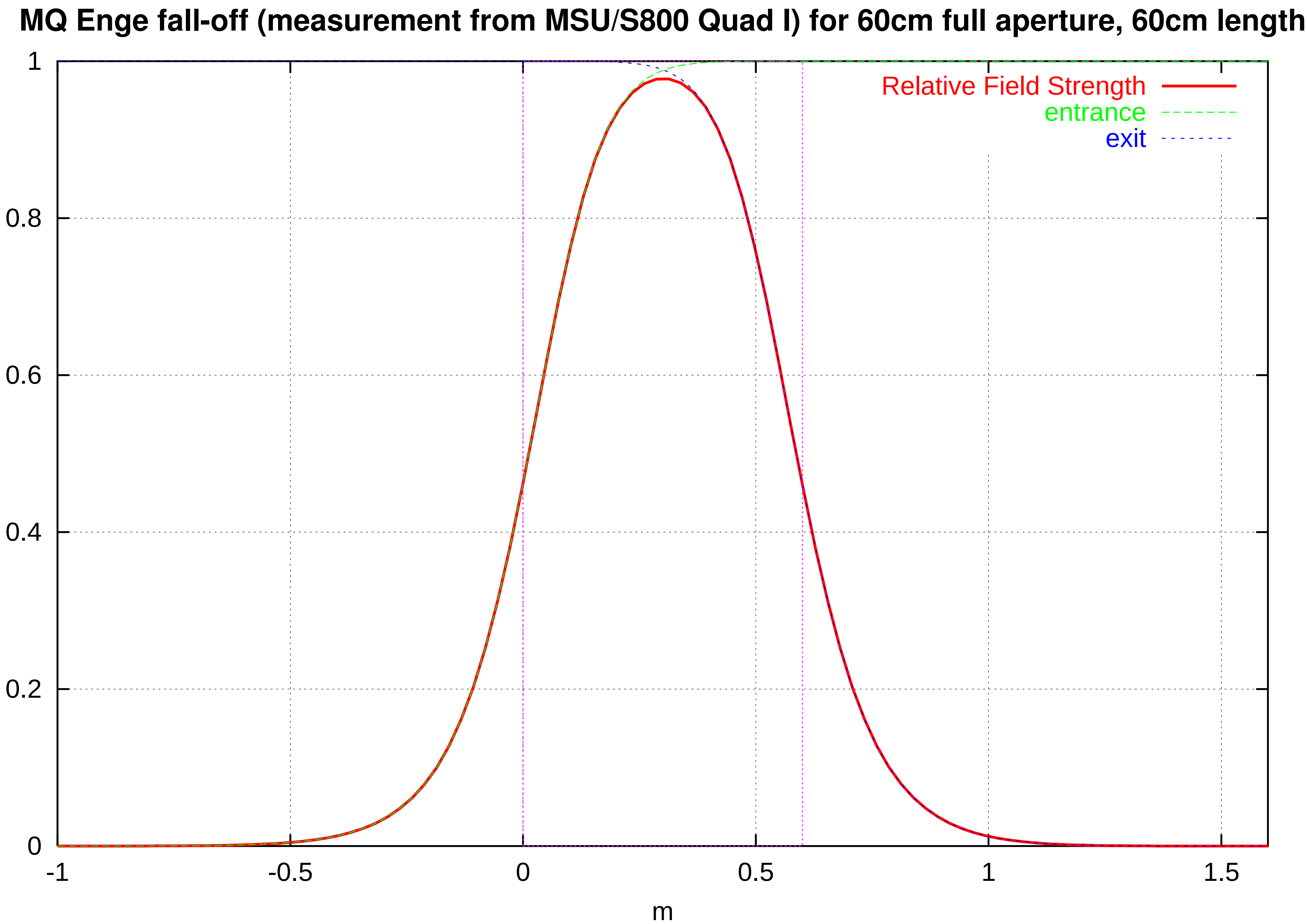



MQ Enge fall-off (measurement from PEP/SLAC) for 60cm full aperture, 60cm length
Relative Field Strength
entrance
exit
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
1.5
m


MQ Enge fall-off (measurement from PEP/SLAC) for 60cm full aperture, 30cm length
Relative Field Strength
entrance
exit
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
m



MQ Enge fall-off (measurement from LHC/HGQ lead end) for 60cm full aperture, 60cm length
Relative Field Strength
entrance
exit
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
1.5
m



MQ Enge fall-off (measurement from MSU/S800 Quad I) for 60cm full aperture, 60cm length
Relative Field Strength
entrance
exit
1
0.8
0.6
0.4
0.2
0
-1
-0.5
0
0.5
1
1.5
m

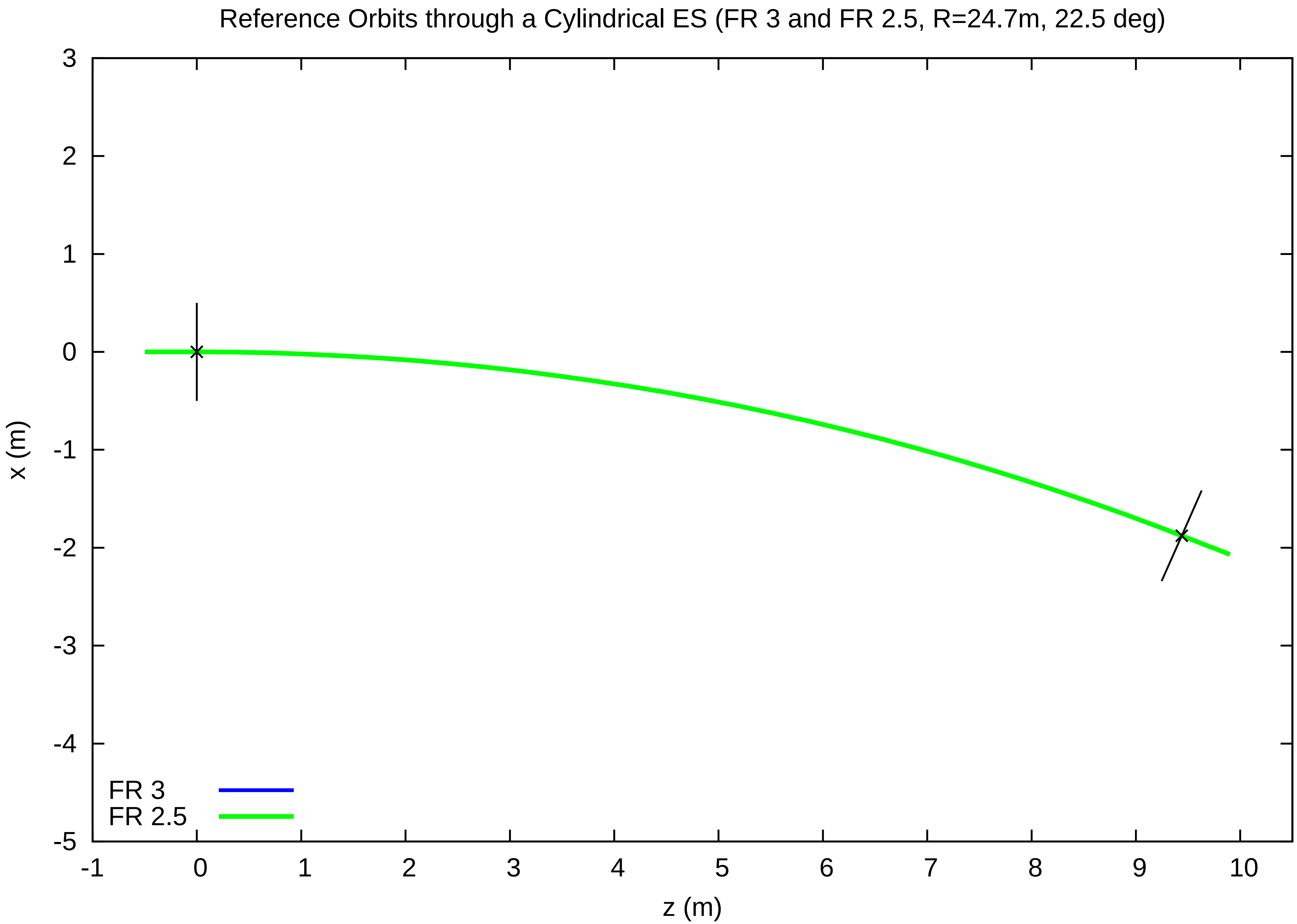
The Pain with Electrostatic Elements

- Unless one is very careful, there will be various undesirable effects:
  - The motion from before to after the element satisfies energy conservation, but the integrator does not know this
  - Repeated small violations of energy conservation can lead to either oscillations, or big long term effects
  - Particular problem: due to offset of reference orbit, it is very useful to re-align elements. This is normally done after each part:
  - After entrance fringe field, after main field, after exit fringe field
  - Each re-alignment causes small change in geometry, and hence small change in potential!

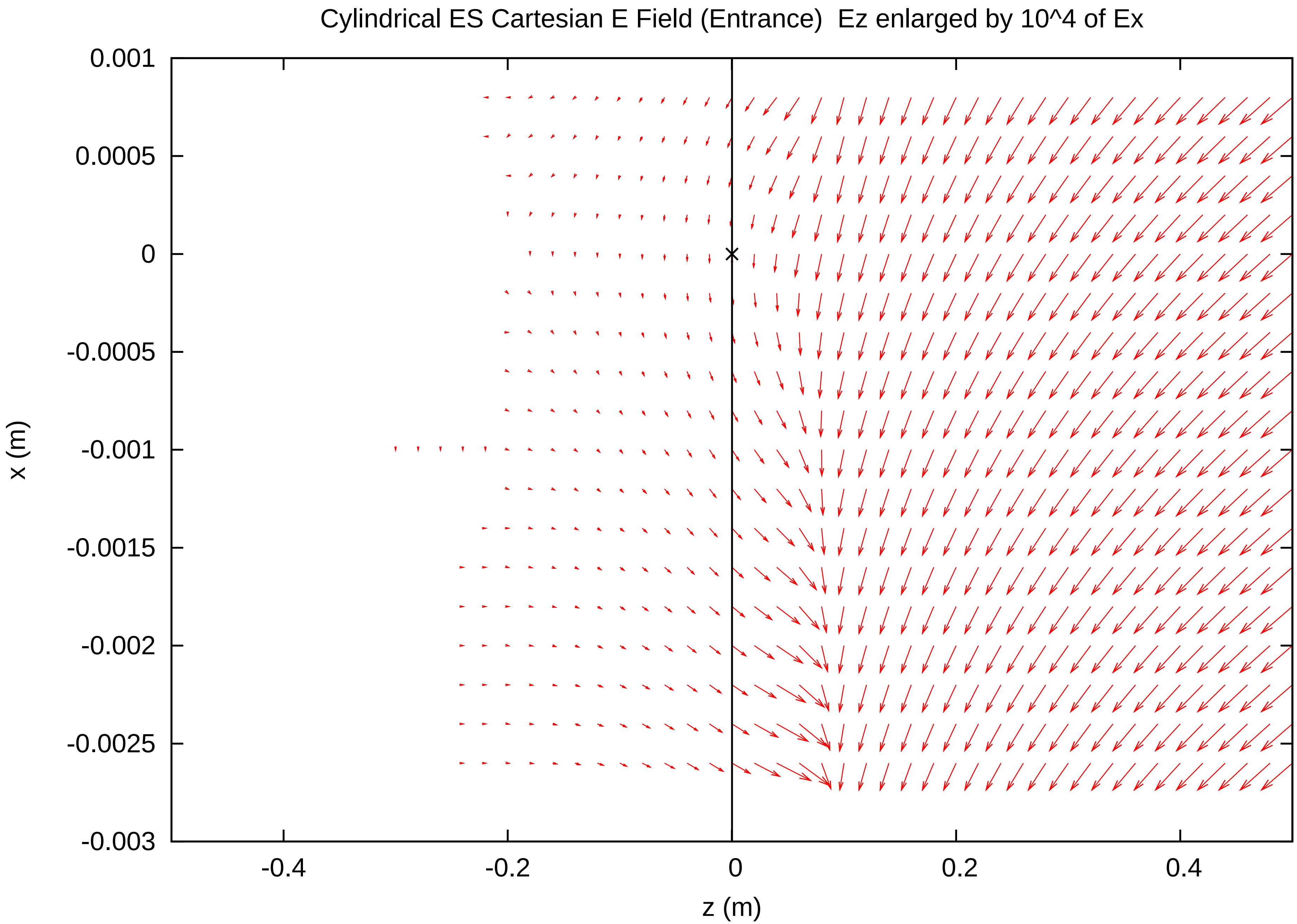
Electrostatic Elements - Study Case -

A cylindrical element with fringe fields (22.5°)

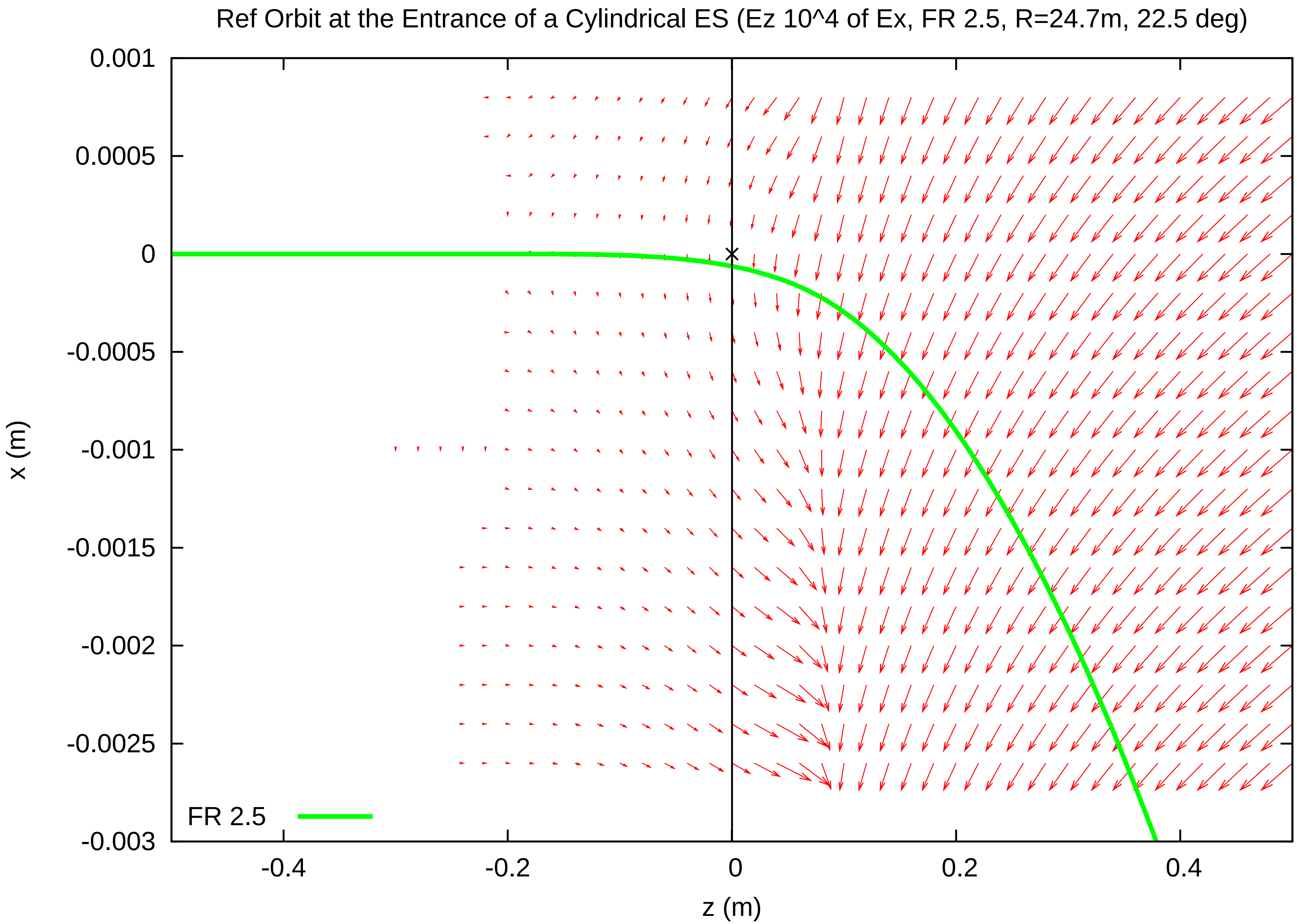
- Fields, especially fringe fields
- Reference orbits
- COSY fringe field computation modes FR 3 and FR 2.5
  - FR 2.5: Computes through the main and the fringe fields
  - FR 3: Further, preserves the mirror symmetry

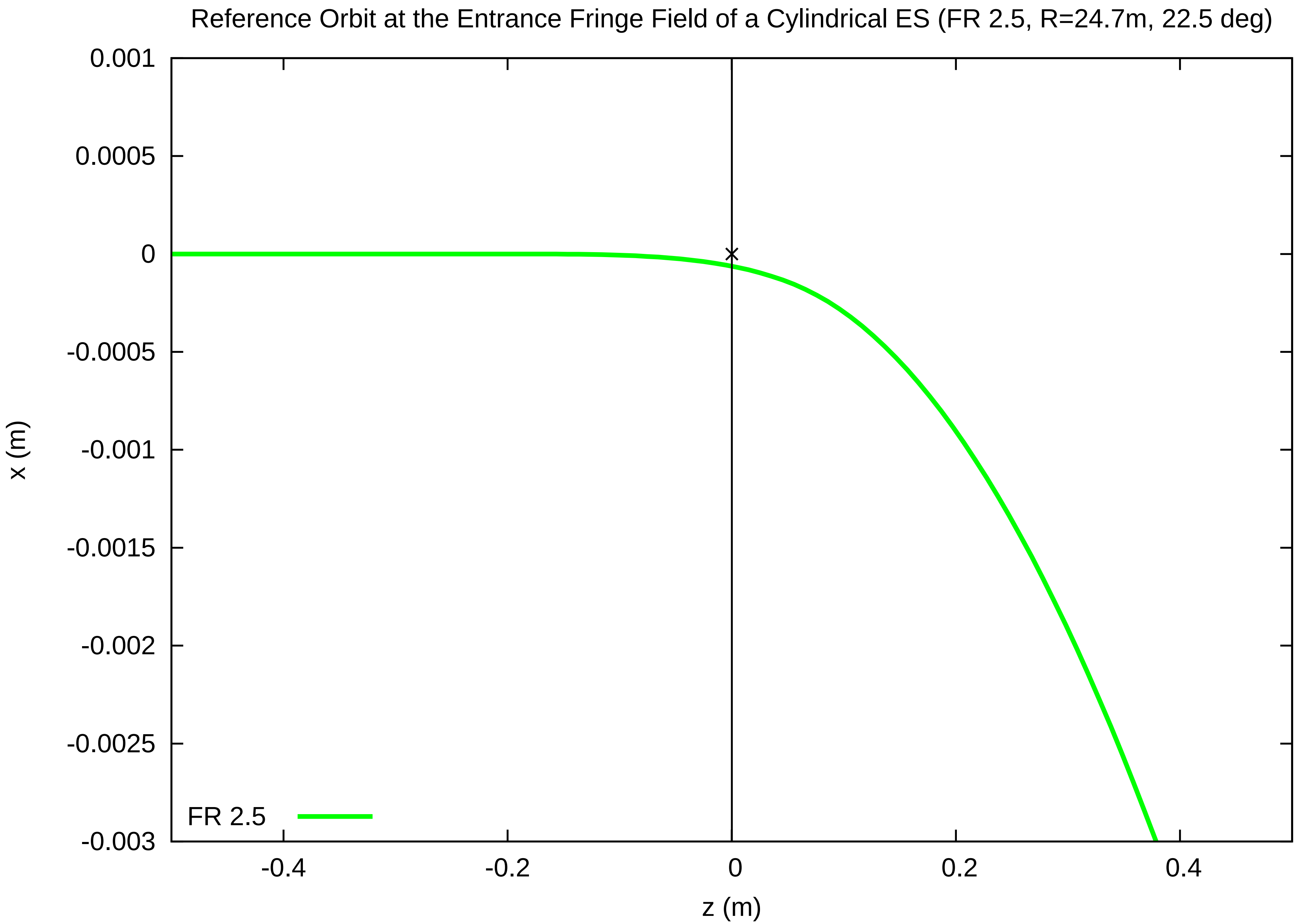


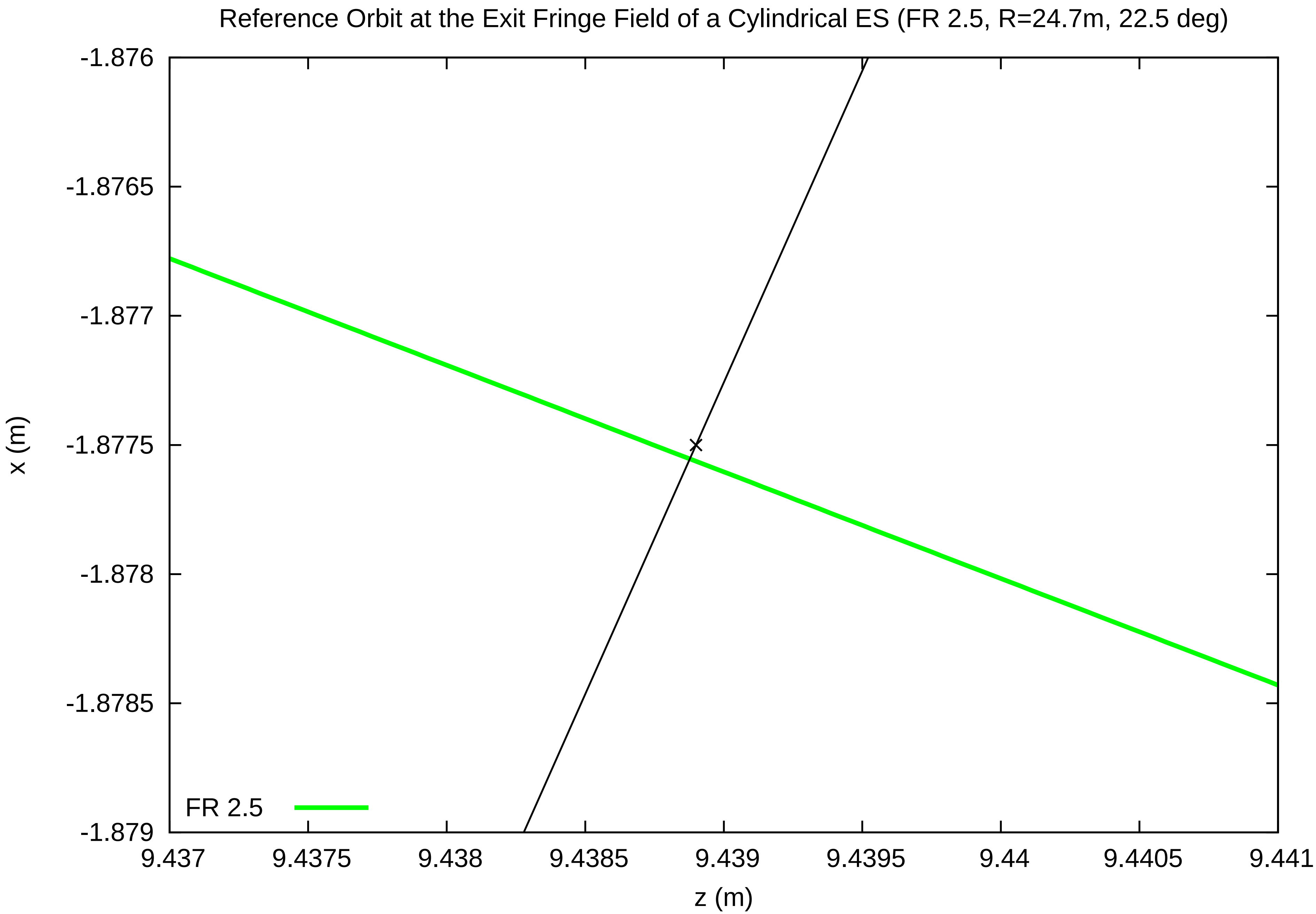
Reference Orbits through a Cylindrical ES (FR 3 and FR 2.5, R=24.7m, 22.5 deg)
3
2
1
0
-1
-2
-3
-4
-5
x (m)
-1
0
1
2
3
4
5
6
7
8
9
10
z (m)
FR 3
FR 2.5

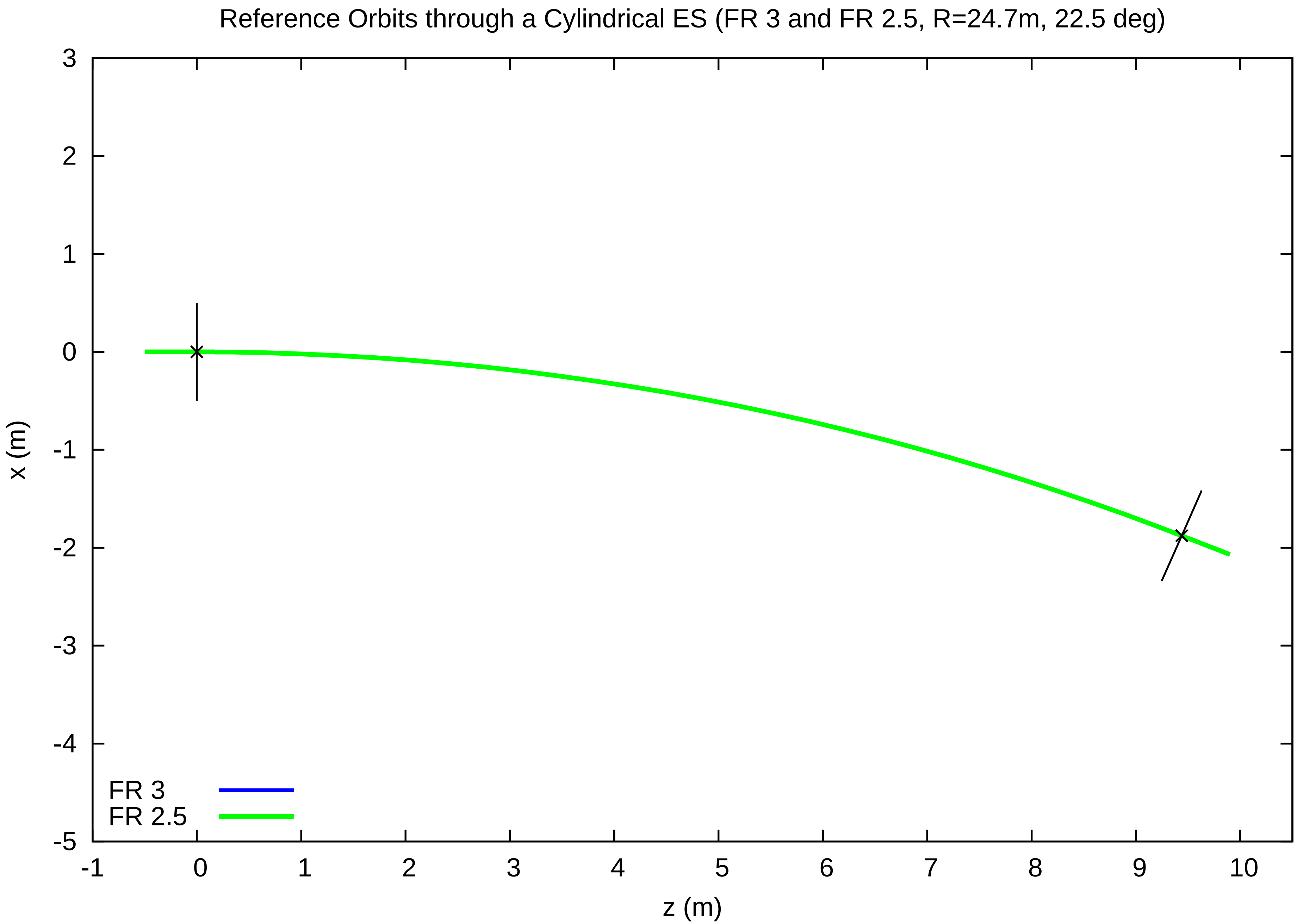


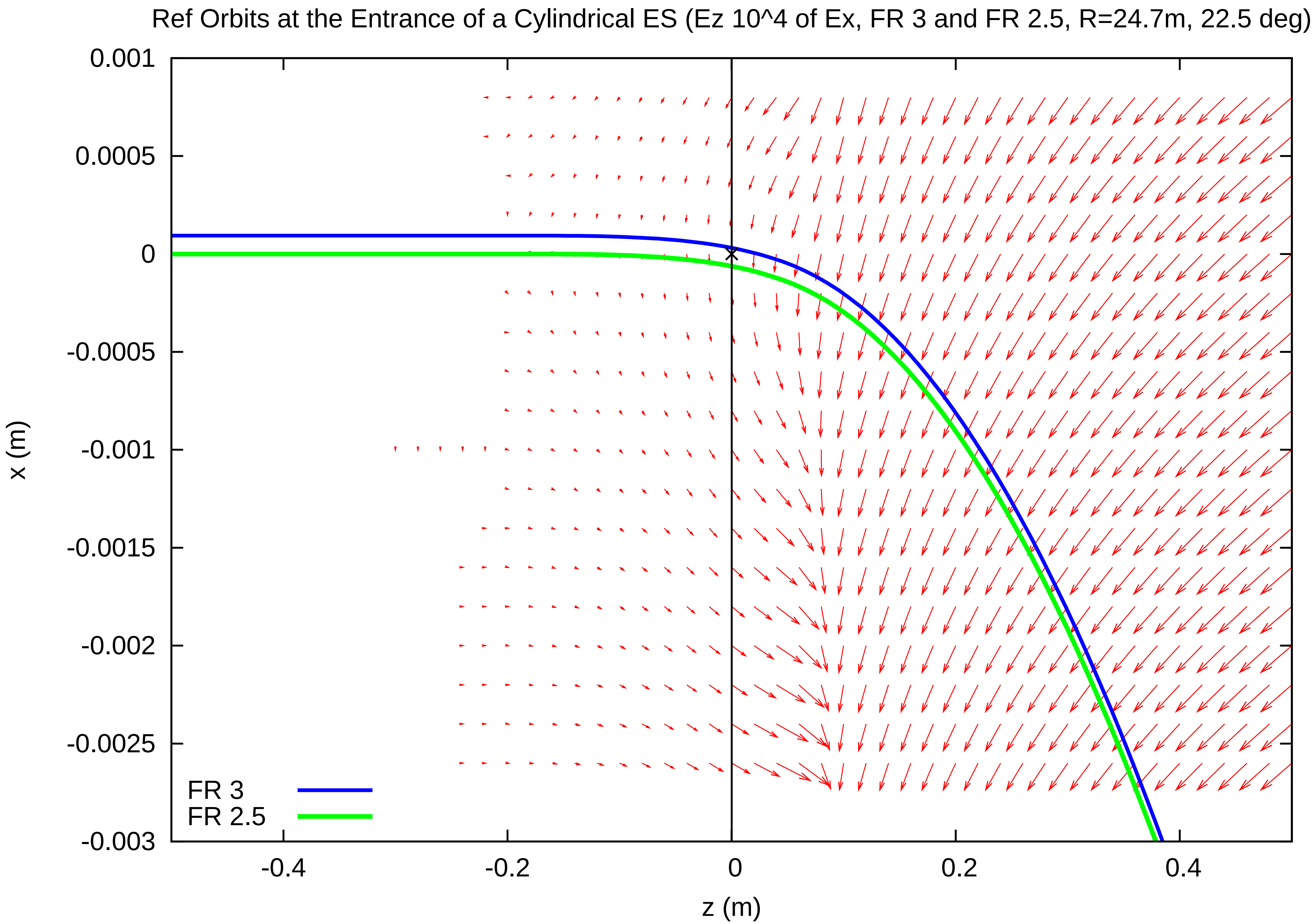
Cylindrical ES Cartesian E Field (Entrance) Ez enlarged by 10^4 of Ex
0.001
0.0005
0
-0.0005
-0.001
-0.0015
-0.002
-0.0025
-0.003
x (m)
-0.4
-0.2
0
0.2
0.4
z (m)

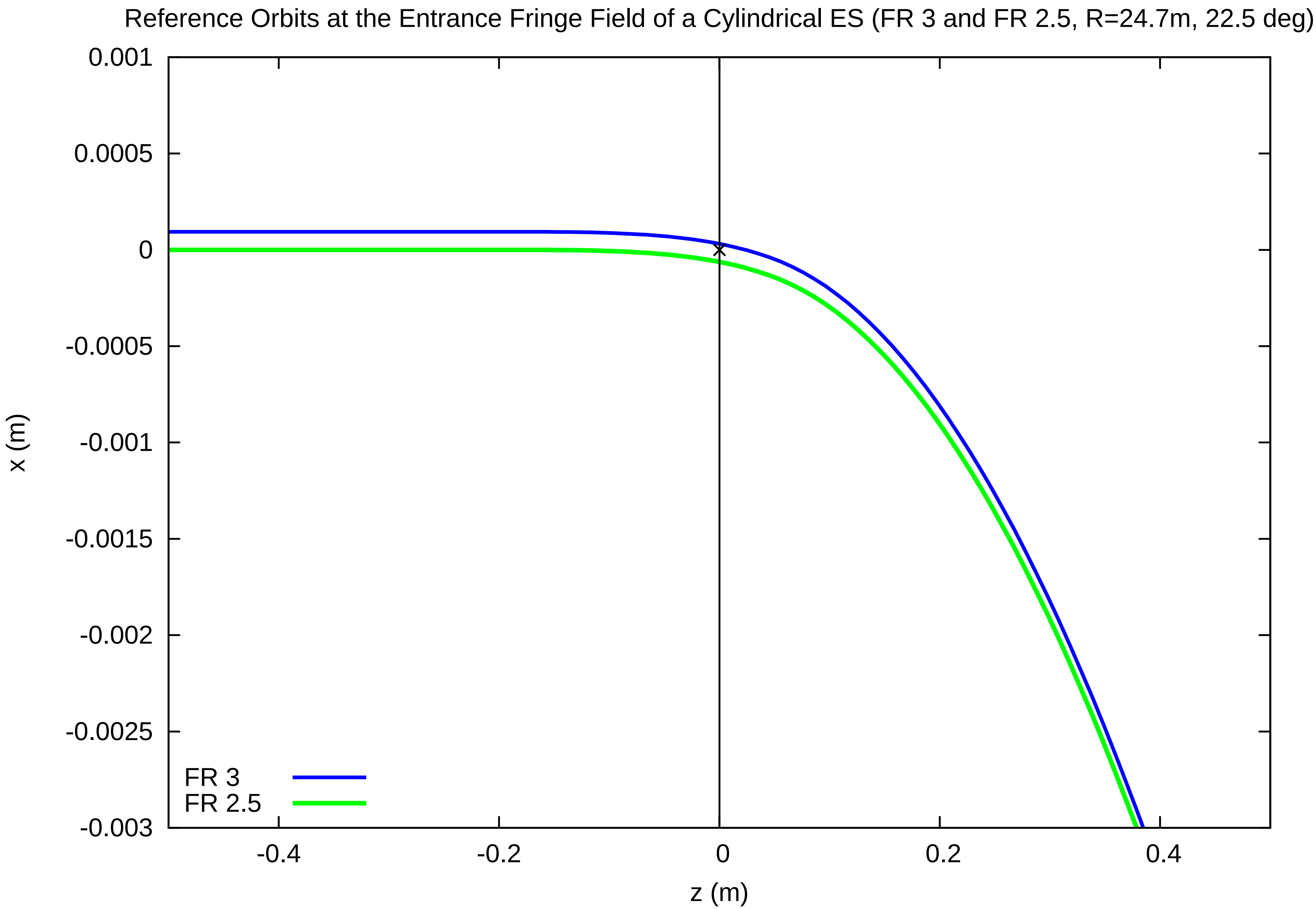

Ref Orbit at the Entrance of a Cylindrical ES (Ez 10^4 of Ex, FR 2.5, R=24.7m, 22.5 deg)
0.001
0.0005
0
-0.0005
-0.001
-0.0015
-0.002
-0.0025
-0.003
x (m)
-0.4
-0.2
0
0.2
0.4
z (m)
FR 2.5


Reference Orbit at the Entrance Fringe Field of a Cylindrical ES (FR 2.5, R=24.7m, 22.5 deg)
0.001
0.0005
0
-0.0005
-0.001
-0.0015
-0.002
-0.0025
-0.003
x (m)
-0.4
-0.2
0
0.2
0.4
z (m)
FR 2.5

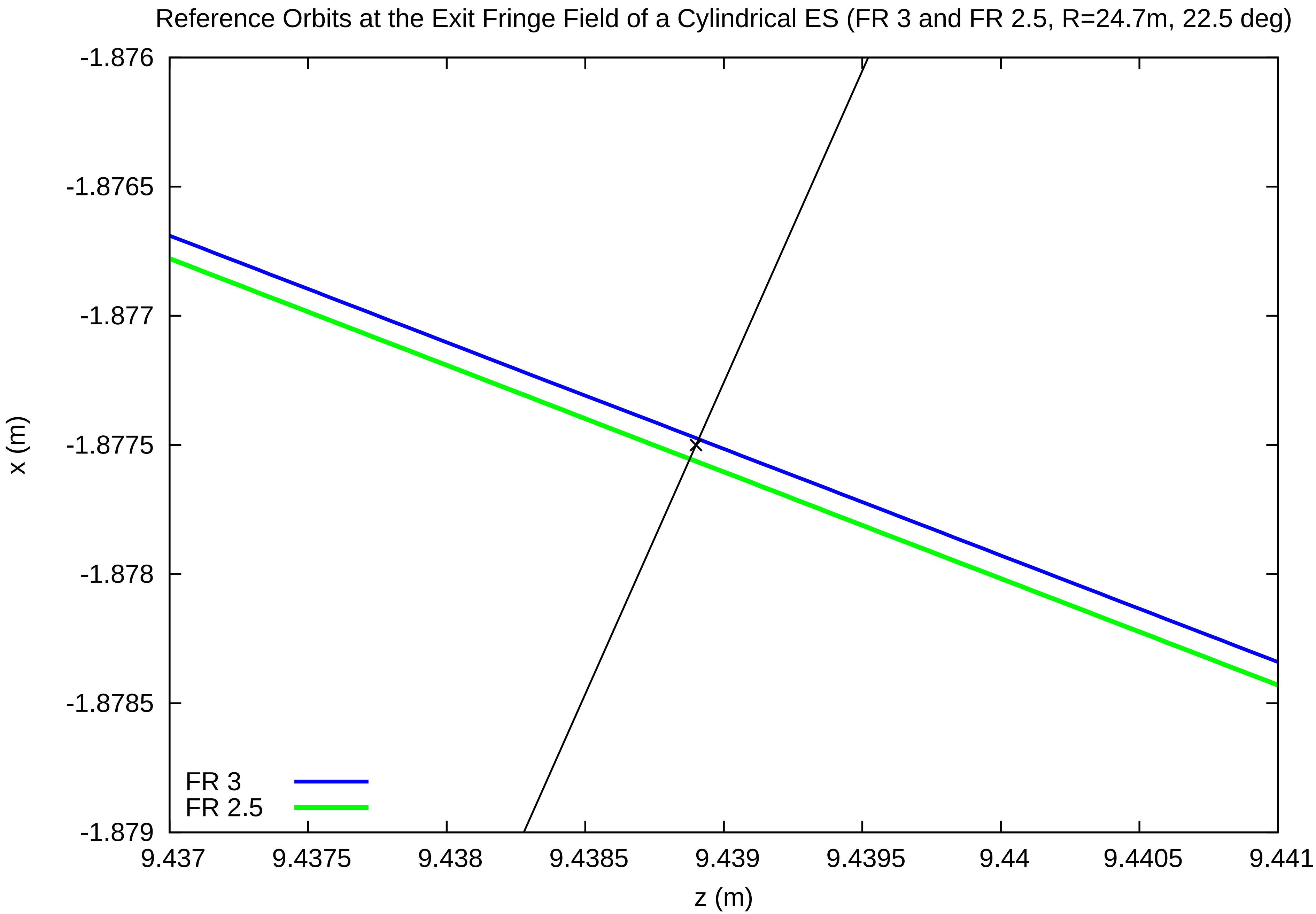

Reference Orbit at the Exit Fringe Field of a Cylindrical ES (FR 2.5, R=24.7m, 22.5 deg)
-1.876
-1.8765
-1.877
-1.8775
-1.878
-1.8785
-1.879
x (m)
9.437
9.4375
9.438
9.4385
9.439
9.4395
9.44
9.4405
9.441
z (m)
FR 2.5


Reference Orbits through a Cylindrical ES (FR 3 and FR 2.5, R=24.7m, 22.5 deg)
3
2
1
0
-1
-2
-3
-4
-5
x (m)
-1
0
1
2
3
4
5
6
7
8
9
10
z (m)
FR 3
FR 2.5

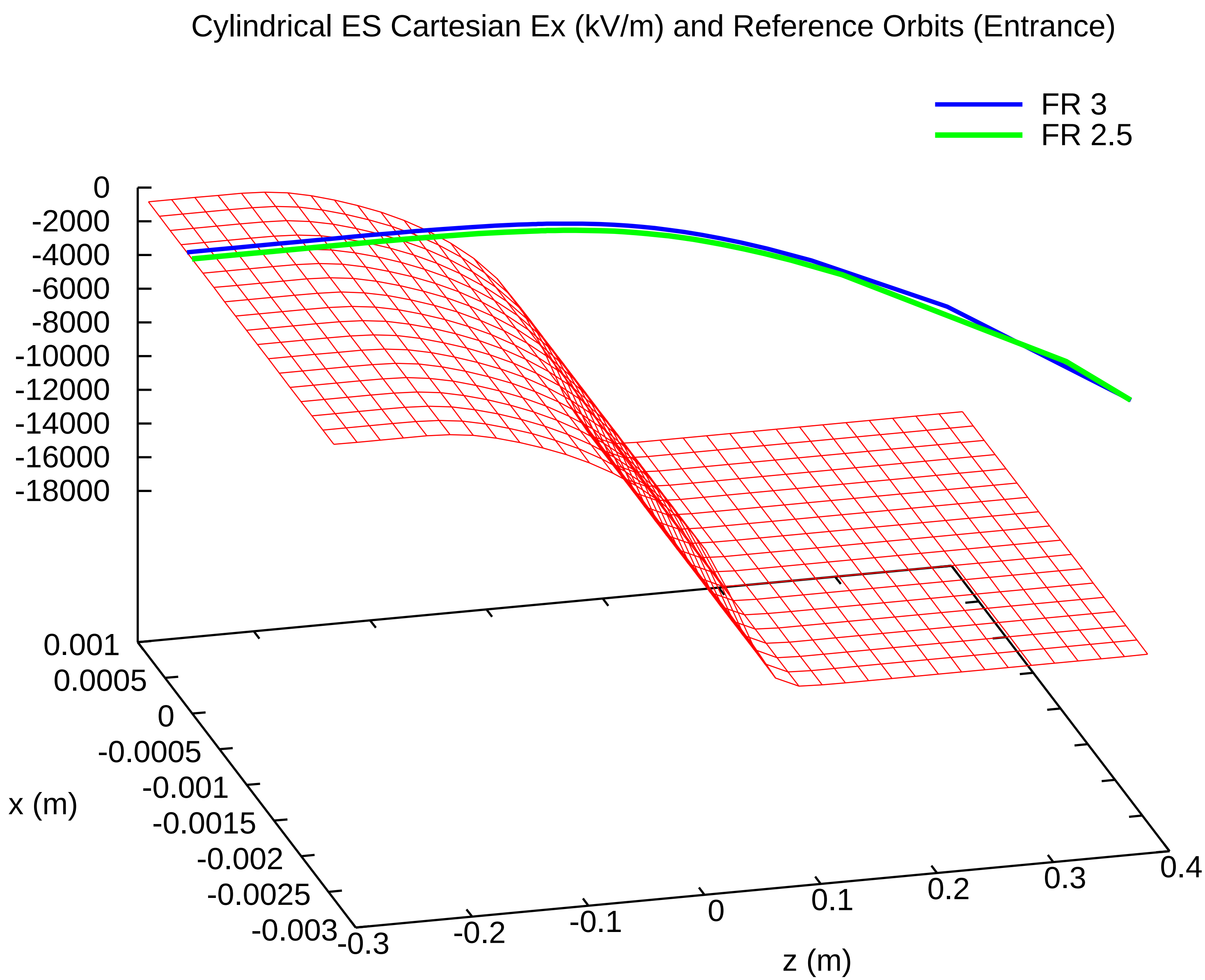

Ref Orbits at the Entrance of a Cylindrical ES (Ez 10^4 of Ex, FR 3 and FR 2.5, R=24.7m, 22.5 deg)
0.001
0.0005
0
-0.0005
-0.001
-0.0015
-0.002
-0.0025
-0.003
x (m)
-0.4
-0.2
0
0.2
0.4
z (m)
FR 3
FR 2.5



Reference Orbits at the Entrance Fringe Field of a Cylindrical ES (FR 3 and FR 2.5, R=24.7m, 22.5 deg)
0.001
0.0005
0
-0.0005
-0.001
-0.0015
-0.002
-0.0025
-0.003
x (m)
-0.4
-0.2
0
0.2
0.4
z (m)
FR 3
FR 2.5

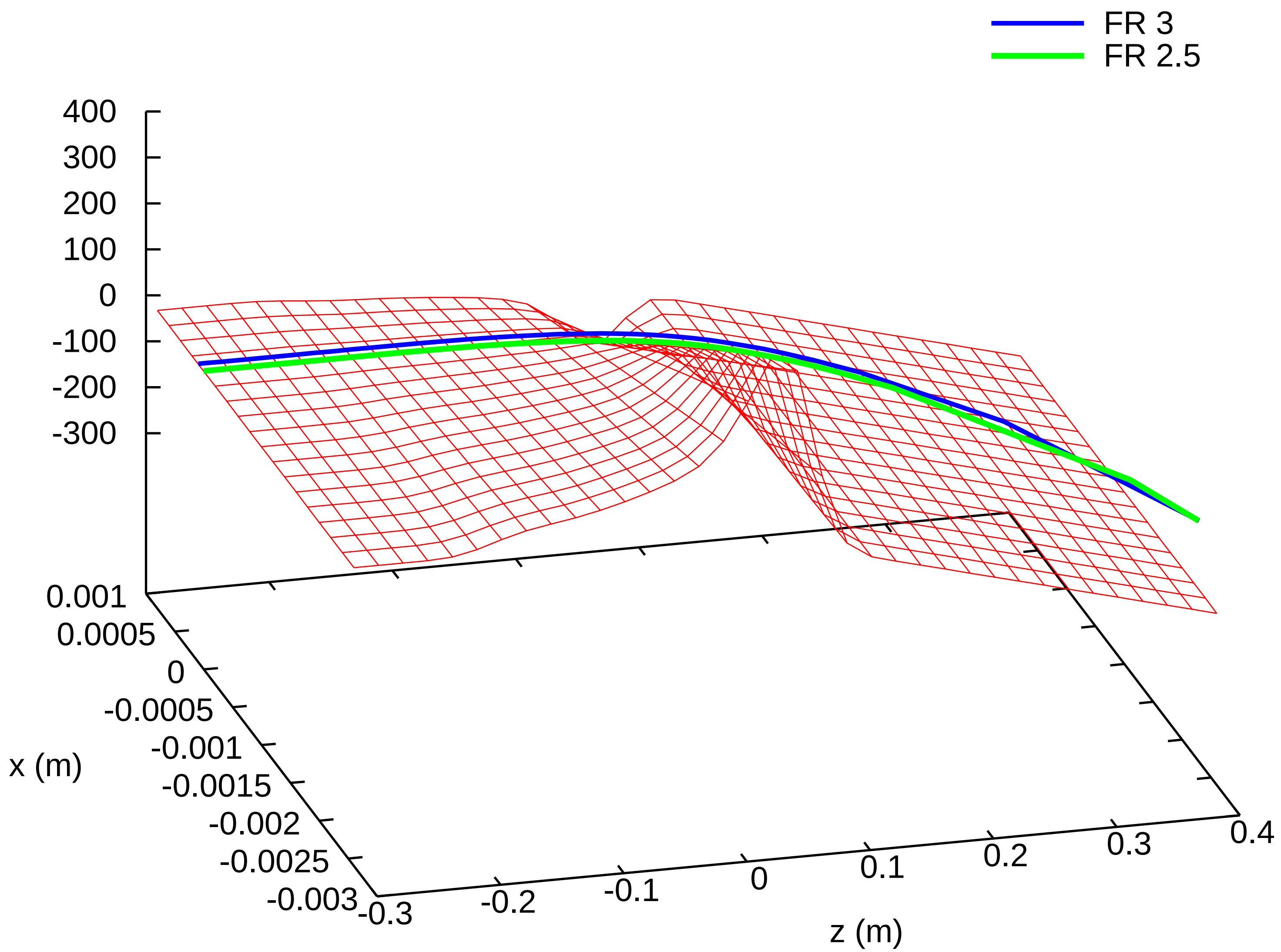


Reference Orbits at the Exit Fringe Field of a Cylindrical ES (FR 3 and FR 2.5, R=24.7m, 22.5 deg)
-1.876
-1.8765
-1.877
-1.8775
-1.878
-1.8785
-1.879
x (m)
9.437
9.4375
9.438
9.4385
9.439
9.4395
9.44
9.4405
9.441
z (m)
FR 3
FR 2.5



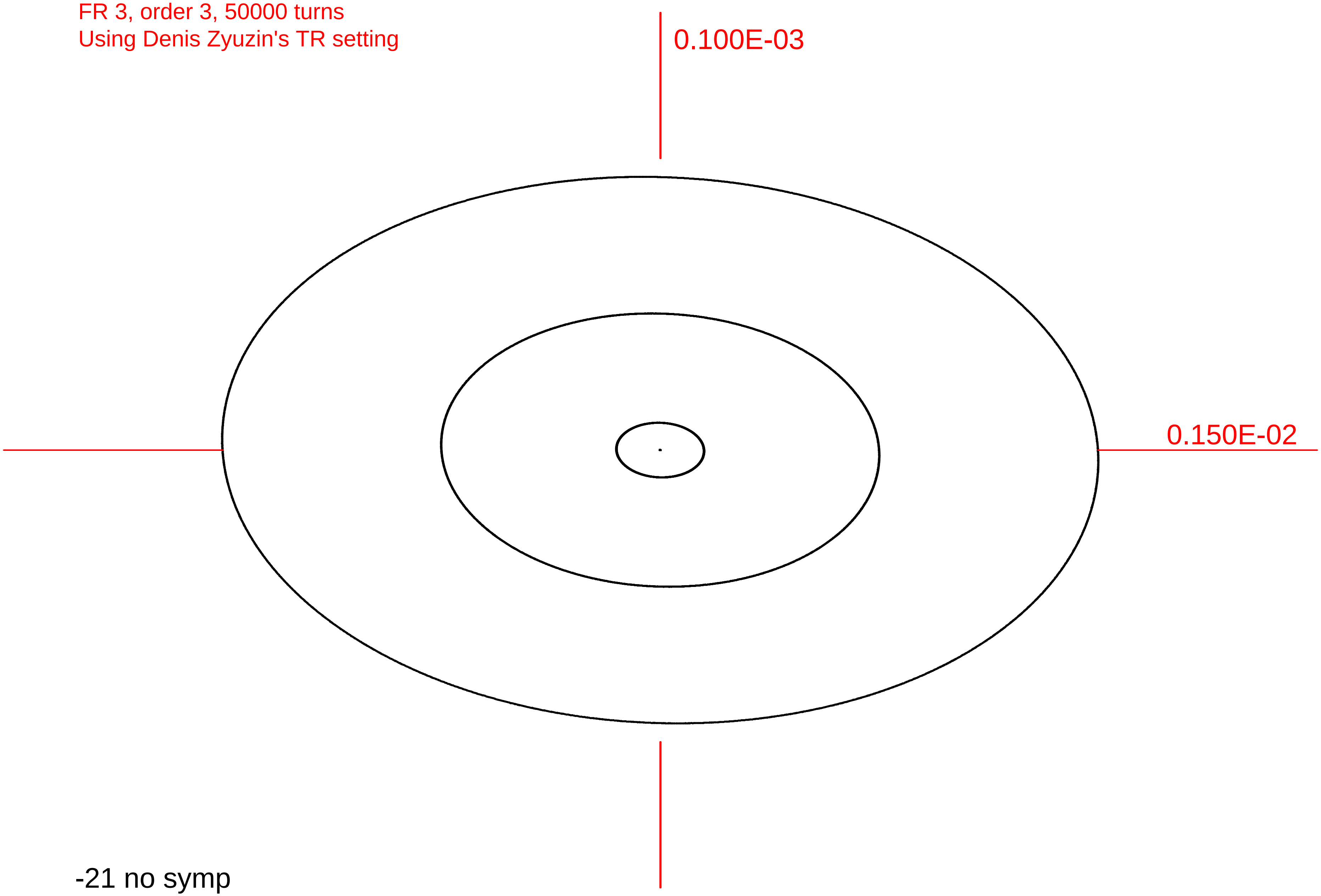
Cylindrical ES Cartesian Ex (kV/m) and Reference Orbits (Entrance)
FR 3
FR 2.5
0
-2000
-4000
-6000
-8000
-10000
-12000
-14000
-16000
-18000
0.001
0.0005
0
-0.0005
-0.001
-0.0015
-0.002
-0.0025
-0.003
x (m)
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
z (m)

Cylindrical ES Cartesian Ez (kV/m) and Reference Orbits (Entrance)



```
***** ES cylindrical FR 3 *****
0.8764065   -0.2498426E-1  0.000000     0.000000   -0.3431232   100000
 9.282307    0.8764065     0.000000     0.000000    -1.697380   010000
 0.000000     0.000000     1.000588   0.1214327E-3  0.000000   001000
 0.000000     0.000000     9.686087     1.000588    0.000000   000100
 0.000000     0.000000     0.000000     0.000000    1.000000   000010
 1.697380    0.3431232     0.000000     0.000000    1.712180   000001
Reversed Map
0.8764065   -0.2498426E-1  0.000000     0.000000   -0.3431232   100000
 9.282307    0.8764065     0.000000     0.000000    -1.697380   010000
 0.000000     0.000000     1.000588   0.1214327E-3  0.000000   001000
 0.000000     0.000000     9.686087     1.000588    0.000000   000100
 0.000000     0.000000     0.000000     0.000000    1.000000   000010
 1.697380    0.3431232     0.000000     0.000000    1.712180   000001
***** ES cylindrical FR 2.5 *****
0.8764077   -0.2498421E-1  0.000000     0.000000   -0.3431202   100000
 9.282277    0.8764068     0.000000     0.000000    -1.697352   010000
 0.000000     0.000000     1.000588   0.1216104E-3  0.000000   001000
 0.000000     0.000000     9.686121     1.000590    0.000000   000100
 0.000000     0.000000     0.000000     0.000000    1.000000   000010
 1.697364    0.3431199     0.000000     0.000000    1.712164   000001
Reversed Map
0.8764068   -0.2498421E-1  0.000000     0.000000   -0.3431199   100000
 9.282277    0.8764077     0.000000     0.000000    -1.697364   010000
 0.000000     0.000000     1.000590   0.1216104E-3  0.000000   001000
 0.000000     0.000000     9.686121     1.000588    0.000000   000100
 0.000000     0.000000     0.000000     0.000000    1.000000   000010
 1.697352    0.3431202     0.000000     0.000000    1.712164   000001
```

FR 3, order 3, 50000 turns
Using Denis Zyuzin's TR setting



-21 no symp

Testing Electrostatic Elements

- The Spherical Capacitor is an excellent test case for studying the performance of codes:
  - Observe that just in the gravitational case, the motion follows Kepler orbits
  - Set up a "lattice" consisting of n spherical capacitors of angle $2\pi/n$
  - Compute dynamics for one revolution, it should be the identity
  - In particular, in map picture, this gives a nice effect because ALL transfer matrix elements have to vanish
  - Very important: need non-relativistic motion for testing! (Remember precession of perihelion of Mercury)

MS <radius> <angle> <aperture> $< n_1 >< n_2 >< n_3 >< n_4 >< n_5 >$;

ES <radius> <angle> <aperture> $< n_1 >< n_2 >< n_3 >< n_4 >< n_5 >$;

The radius is measured in meters, the angle in degrees, and the aperture is in meters and corresponds to half of the gap width. The indices n_i describe the midplane radial field dependence which is given by

$$F(x) = F_0 \cdot \left[1 - \sum_{i=1}^{5} n_i \cdot (\frac{x}{r_0})^i \right]$$

where r_0 is the bending radius. Note that an electric cylindrical condenser has $n_1 = 1$, $n_2 = -1$, $n_3 = 1$, $n_4 = -1$, $n_5 = 1$, etc, and an electric spherical condenser has $n_1 = 2$, $n_2 = -3$, $n_3 = 4$, $n_4 = -5$, $n_5 = 6$, etc. Homogeneous dipole magnets have $n_i = 0$.

Since an electric cylindrical condenser is invariant under translation along the y axis, the y motion is like a drift. An offset in the y direction does not alter the motion, so a map produced by **ES**, $\mathcal{M}_{ES}$, and a y offset map $\mathcal{M}_{\triangle y}$, produced by "**SA** 0 DA(5) ;" for example, commute. A similar consistency test can be performed for an electric spherical condenser. Consider an offset $\triangle\theta$ along the radius R sphere toward the positive y direction, and call the map $\mathcal{M}_{\triangle}$. For example, such a map $\mathcal{M}_{\triangle}$ can be produced by "**SA** -R 0 ; **RA** DA(5) ; **SA** R 0 ;". The map of a spherical condenser for a 180° travel, represented by $\mathcal{M}_{ES180}$, agrees with the map of a $\triangle$ offset, a 180° travel in the condenser and a $\triangle$ offset again. Another test is based on the observation that the motion is that of a Kepler problem. The motion should return to the original state after one cycle travel, thus $\mathcal{M}_{ES360}$ should become an identity map.

The element

unity map test (for ES Spherical)

```
  1.000000     0.6745274E-08  0.000000      0.000000      0.000000     1000000
-0.6745265E-08  1.000000      0.000000      0.000000      0.000000     0100000
  0.000000      0.000000      1.000000     0.7076424E-14  0.000000     0010000
  0.000000      0.000000     0.1078906E-14  1.000000      0.000000     0001000
 0.1960073E-13-0.6745267E-08  0.000000      0.000000      0.000000     0000001
 0.1250518E-11 0.4073090E-14  0.000000      0.000000      0.000000     2000000
-0.1349320E-07-0.5151423E-14  0.000000      0.000000      0.000000     1100000
-0.2007760E-11 0.6745271E-08  0.000000      0.000000      0.000000     0200000
  0.000000      0.000000    -0.3080058E-12 0.6745268E-08  0.000000     1010000
  0.000000      0.000000    -0.6748311E-08-0.1367794E-13  0.000000     0110000
 0.5851285E-12 0.3371156E-08  0.000000      0.000000      0.000000     0020000
  0.000000      0.000000    -0.7270661E-12 0.2327415E-13  0.000000     1001000
  0.000000      0.000000    -0.9280394E-12 0.6745264E-08  0.000000     0101000
-0.6744571E-08-0.3180281E-11  0.000000      0.000000      0.000000     0011000
-0.1771120E-11-0.1011791E-07  0.000000      0.000000      0.000000     1000001
 0.3368683E-08-0.9681441E-14  0.000000      0.000000      0.000000     0100001
  0.000000      0.000000    -0.3136933E-11-0.6745272E-08  0.000000     0010001
-0.6305797E-12 0.2567664E-12  0.000000      0.000000      0.000000     0002000
  0.000000      0.000000     0.3741119E-11-0.4766519E-14  0.000000     0001001
 0.1817235E-12 0.3372632E-08  0.000000      0.000000      0.000000     0000002
------------------------------------------------------------------------
```

```
 0.3661618E-03-0.1707484E-08  0.000000      0.000000      0.000000     0202001
  0.000000      0.000000      0.7662359E-03-0.2194845E-07  0.000000     1012001
  0.000000      0.000000     -0.6488619E-03-0.3581858E-08  0.000000     0112001
-0.4835207E-03 0.1994027E-07  0.000000      0.000000      0.000000     0022001
  0.000000      0.000000      0.1494336E-03 0.1015586E-12  0.000000     2001002
  0.000000      0.000000      0.2281940E-03 0.5143277E-07  0.000000     1101002
  0.000000      0.000000     -0.5831523E-03 0.4309075E-12  0.000000     0201002
-0.3915750E-03 0.2464082E-10  0.000000      0.000000      0.000000     1011002
-0.9667518E-03 0.4642991E-07  0.000000      0.000000      0.000000     0111002
  0.000000      0.000000      0.1705244E-03 0.2448350E-09  0.000000     0021002
-0.1624803E-03-0.2086817E-06  0.000000      0.000000      0.000000     2000003
-0.1060660E-02-0.6621711E-12  0.000000      0.000000      0.000000     1100003
-0.2591765E-03-0.4300154E-07  0.000000      0.000000      0.000000     0200003
  0.000000      0.000000      0.2825920E-03-0.4426537E-07  0.000000     1010003
  0.000000      0.000000     -0.2309635E-03-0.3811017E-12  0.000000     0110003
-0.1588611E-03-0.2128057E-07  0.000000      0.000000      0.000000     0020003
-0.9345362E-04-0.1703433E-06  0.000000      0.000000      0.000000     1004000
-0.1763581E-04-0.3705832E-07  0.000000      0.000000      0.000000     0104000
  0.000000      0.000000     -0.1721734E-04-0.5052216E-04  0.000000     0014000
  0.000000      0.000000      0.4354669E-03 0.1546786E-08  0.000000     1003001
  0.000000      0.000000     -0.6749205E-05-0.5807740E-09  0.000000     0103001
-0.3855160E-03 0.5475406E-06  0.000000      0.000000      0.000000     0013001
-0.2215067E-04-0.3428119E-08  0.000000      0.000000      0.000000     1002002
-0.1719209E-04-0.3570590E-10  0.000000      0.000000      0.000000     0102002
  0.000000      0.000000     -0.3401219E-03 0.9564552E-08  0.000000     0012002
  0.000000      0.000000      0.2936376E-04-0.4636029E-13  0.000000     1001003
  0.000000      0.000000     -0.4288197E-03-0.8853119E-08  0.000000     0101003
-0.1870915E-03 0.1902511E-10  0.000000      0.000000      0.000000     0011003
-0.4417931E-03 0.8995486E-07  0.000000      0.000000      0.000000     1000004
-0.3436738E-04-0.8123444E-12  0.000000      0.000000      0.000000     0100004
  0.000000      0.000000     -0.1481154E-03 0.8009784E-08  0.000000     0010004
  0.000000      0.000000      0.5465326E-05-0.2912335E-04  0.000000     0005000
-0.1852932E-04-0.7348978E-08  0.000000      0.000000      0.000000     0004001
  0.000000      0.000000      0.2584482E-03 0.9979594E-09  0.000000     0003002
-0.1560408E-03 0.1685662E-08  0.000000      0.000000      0.000000     0002003
  0.000000      0.000000     -0.1663985E-03-0.2632633E-13  0.000000     0001004
-0.2456936E-04-0.1280640E-07  0.000000      0.000000      0.000000     0000005
------------------------------------------------------------------------------
```

360 degree test (for ES Cylindrical)

```
-0.8582162    -0.7258994      0.000000      0.000000      0.000000     1000000
 0.3629497    -0.8582162      0.000000      0.000000      0.000000     0100000
  0.000000      0.000000      1.000000      0.000000      0.000000     0010000
  0.000000      0.000000      6.283185      1.000000      0.000000     0001000
 0.9291081      0.3629497     0.000000      0.000000      0.000000     0000001
-0.6170785     0.8667631E-01  0.000000      0.000000      0.000000     2000000
-0.8477916     0.4437621      0.000000      0.000000      0.000000     1100000
 -1.085123    -0.5877627      0.000000      0.000000      0.000000     0200000
  0.000000      0.000000     0.7258994      0.000000      0.000000     1001000
  0.000000      0.000000      1.858216      0.000000      0.000000     0101000
 0.6170785      0.2762734     0.000000      0.000000      0.000000     1000001
 0.6053707    -0.2218810      0.000000      0.000000      0.000000     0100001
-0.9291081    -0.3629497      0.000000      0.000000      0.000000     0002000
  0.000000      0.000000      2.778643      0.000000      0.000000     0001001
 0.7800739E-01-0.6906835E-01  0.000000      0.000000      0.000000     0000002
```

4.4 Repetitive Tracking

COSY allows efficient repetitive tracking of particles through maps. The command

TR <N> <NP> <ID1> <ID2> <D1> <D2> <TY> <NF> <IU>;

tracks the momentary particles selected with **SR** or **ER** through the momentary map for the required number of iterations N. After each NP iterations the position of the phase space projection ID1-ID2 is drawn to unit IU. The phase space numbers 1 through 6 correspond to x, a, y, b, d, t, and the numbers -1, -2, -3 correspond to the x, y and z components of the spin. If any of these components get larger than D1, D2, they will not be drawn.

TY specifies the mode of symplectification, and NF turns on and off the display mode in normal form variables.

TY=0

Symplectic tracking using the EXPO generating function is performed [39] [40] [38] [81].

$1 \leq |\mathbf{TY}| \leq 4$

Symplectic tracking using the generating function of type |TY| (see [8] [14]) is performed. For TY$>$ 0, a fixed-point iteration is used to determine the symplectified map. For TY$<$ 0, a Newton iteration is used to determine the symplectification. While the Newton method is more robust, the fixed point iteration tends to be faster if it works.

TY=-12 or **TY=-13**

Symplectic tracking is performed by symplectifying the linear map $\mathcal{M}_L$ and representing the map $\mathcal{N} = \mathcal{M} \circ \mathcal{M}_L^{-1}$ by the generating function of type |TY+10|. Because the linear part of $\mathcal{N}$ is the unity map, only the generating functions of type 2 (for TY$= -12$) and 3 (for TY$=-13$) can be used for that purpose.

TY=-21

The tracking is performed without symplectification.

Inversion of Maps

Suppose we want to obtain the inverse of A. We write

$$A = A_1 + A_{\geq 2}$$

$$A^{-1} = \sum_{\nu=1}^{\infty} M_\nu$$

Composing, we obtain

$$\begin{aligned}(A_1 + A_{\geq 2}) \circ \sum_{\nu=1}^{\infty} M_\nu = E \Rightarrow \\ A_1 \circ \sum_{\nu=1}^{\infty} M_\nu = E - A_{\geq 2} \circ \sum_{\nu=1}^{\infty} M_\nu \Rightarrow \\ M_{\leq i} = A_1^{-1} \circ (E - A_{\geq 2} \circ M_{\leq i-1})\end{aligned}$$

In the last step use has been made of the fact that to i-th order $A_{\geq 2} \circ M_{\leq i-1} = A_{\geq 2} \circ M_{\leq i}$.

Computation of A_1^{-1} is a linear matrix inversion.

The Symplectic Condition

As a solution of a Hamiltonian system, a Transfer Map $\mathcal{M}$ has to satisfy the symplectic condition

$$M \cdot J \cdot M^t = J, \text{ or alternatively } M \cdot J = (M \cdot J)^t$$

where M is the Jacobian Matrix of partial derivatives of $\mathcal{M}$, and J has the form

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

Generating Functions, Symplectic Tracking

We are interested in one of four "mixed" relationships, for example

$$(\vec{q_f}, \vec{p_i}) = \pm \frac{\partial \mathcal{F}(\vec{q_i}, \vec{p_f})}{\partial \vec{q_i}, \vec{p_f}}$$

While these mixed relationships are awkward, they have one remarkable property:

Theorem: The transformation defined by the above implicit condition is symplectic

Algorithm: First, Determine a generating function representing the transfer map. Then in each step:

1. Find approximately symplectic coordinates from transfer map
2. Correct coordinates by solving implicit equation via Newton etc.

Result: A method that leads to fully symplectic maps, at a cost only slightly higher than map evaluation

Obtaining Generating Functions

Obtaining $\mathcal{F}$ (if possible) from $\mathcal{M}$ is achieved through "partial inversion". We write $\mathcal{M} = (\mathcal{M}_1, \mathcal{M}_2)$, $\mathcal{E} = (\mathcal{E}_1, \mathcal{E}_2)$. Set

$$\mathcal{N} = (\mathcal{E}_1, \mathcal{M}_2)$$

Then we obtain:

$$(\vec{q_i}, \vec{p_f}) = \mathcal{N}(\vec{q_i}, \vec{p_i})$$

If $\mathcal{N}$ is invertible, we get

$$(\vec{q_i}, \vec{p_i}) = \mathcal{N}^{-1}(\vec{q_i}, \vec{p_f})$$

Composing $(\mathcal{M}_1, \mathcal{E}_2)$ and $\mathcal{N}^{-1}$, we finally obtain the "mixed" relationship:

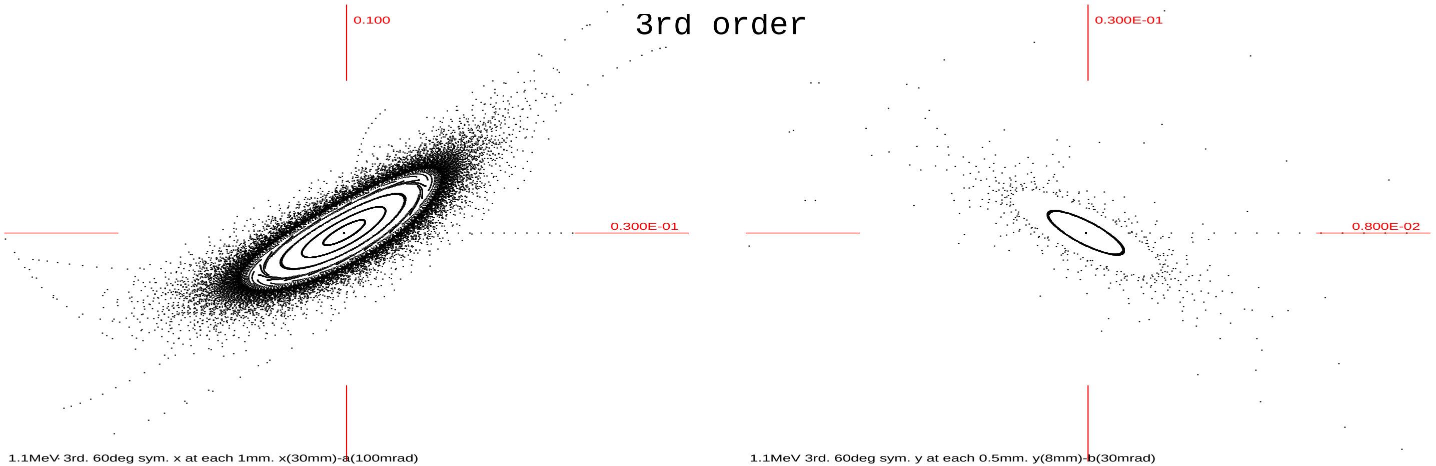
$$(\vec{q_f}, \vec{p_i}) = \left((\mathcal{M}_1, \mathcal{E}_2) \circ \mathcal{N}^{-1}\right)(\vec{q_i}, \vec{p_f}) = \mathcal{F}(\vec{q_i}, \vec{p_f})$$

The generating function $\mathcal{F}$ can be obtained from here by path integration.

Tracking with/without Symplectification

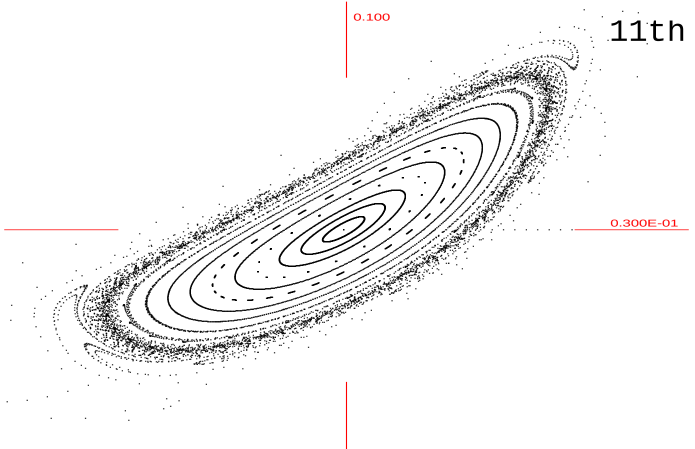
- 3rd order computations without symplectification
- 11th order computations without symplectification
- 11th order computations with EXPO

3rd order

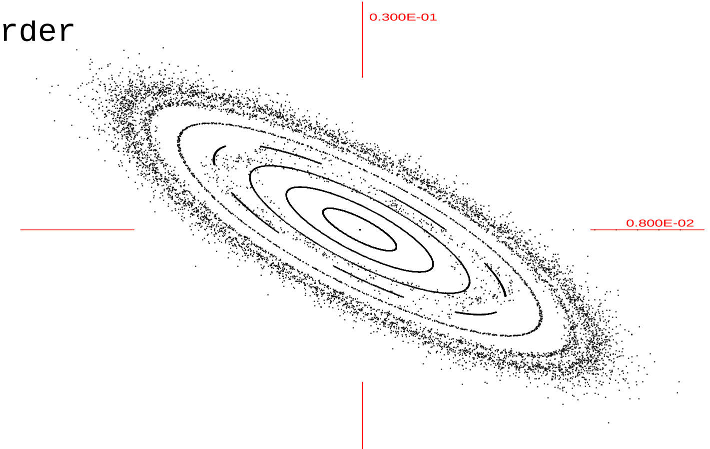


0.100
0.300E-01
1.1MeV 3rd. 60deg sym. x at each 1mm. x(30mm)-a(100mrad)
0.300E-01
0.800E-02
1.1MeV 3rd. 60deg sym. y at each 0.5mm. y(8mm)-b(30mrad)

11th order



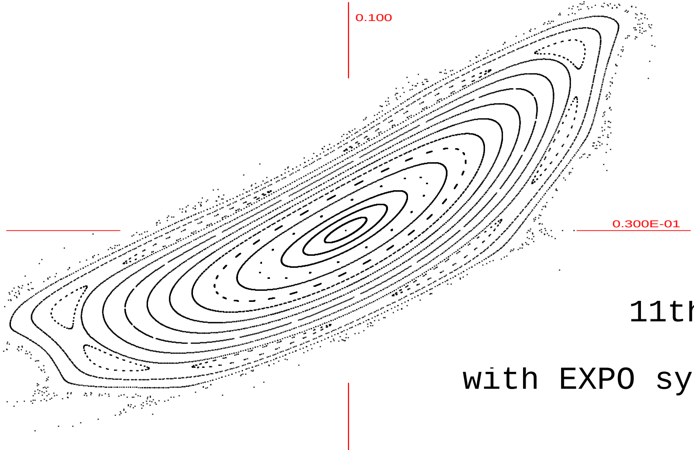
1.1MeV 11th. 60deg sym. x at each 1mm. x(30mm)-a(100mrad)



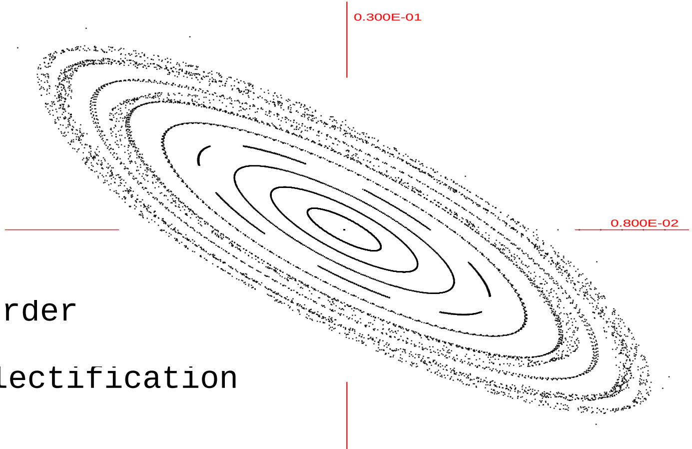
1.1MeV 11th. 60deg sym. y at each 0.5mm. y(8mm)-b(30mrad)

11th order

with EXPO symplectification



1.1MeV Symplectic 11th. 60deg sym. x at each 1mm. x(30mm)-a(100mrad)



1.1MeV Symplectic 11th. 60deg sym. y at each 0.5mm. y(8mm)-b(30mrad)

A Purely Electrostatic Ring (Courtesy Senichev, Zyuzin)

The question is: how the fringe fields influence spin orbit motion?

Using COSY Infinity we found that:

- No changes in orbit motion.
- Spin is very sensitive to sextupole component of fringe fields.
- Spin decoherence due to fringe fields can be suppressed using sextupoles.

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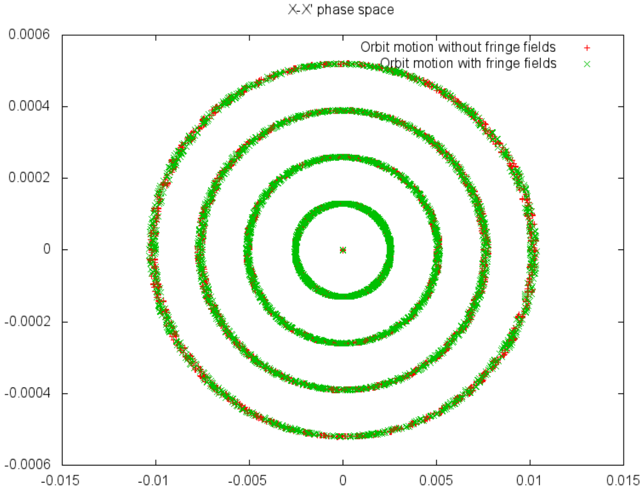
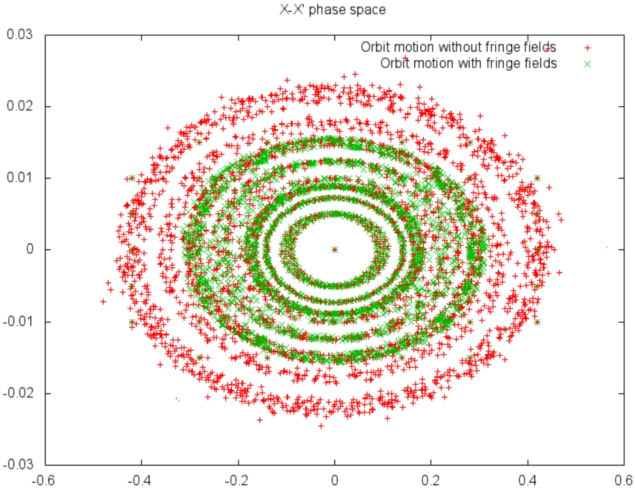
A Purely Electrostatic Ring (Courtesy Senichev, Zyuzin)

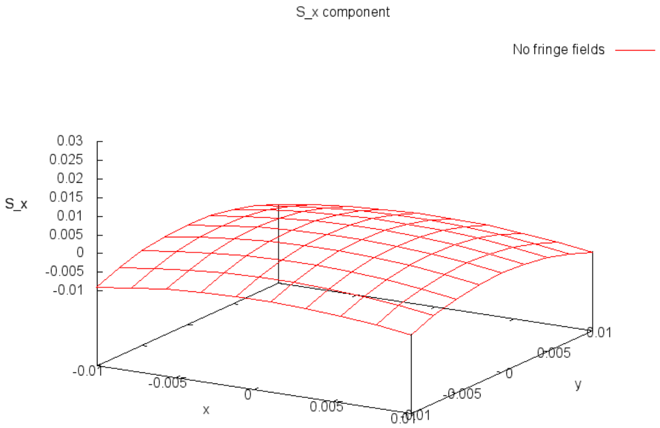
Unfortunately vertical spin component grows very slow and if we assume $\eta = 10^{-15}$, machine precision is not enough to accumulate S_y component. In COSY Infinity we can simulate S_y growth with $\eta > 10^{-11}$.


X-X' phase space
Orbit motion without fringe fields
Orbit motion with fringe fields
0.0006
0.0004
0.0002
0
-0.0002
-0.0004
-0.0006
-0.015
-0.01
-0.005
0
0.005
0.01
0.015

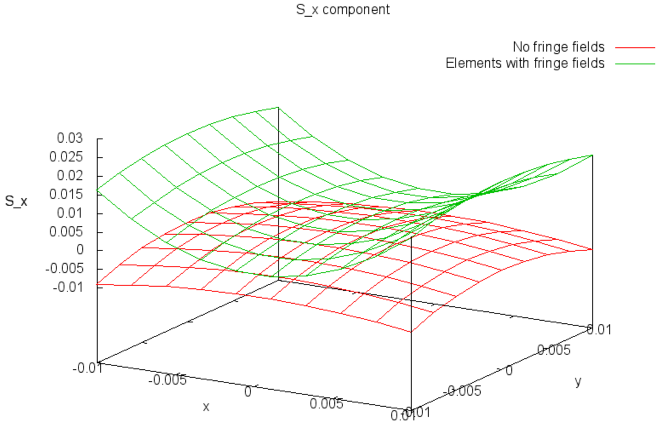


X-X' phase space
Orbit motion without fringe fields
Orbit motion with fringe fields
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.6
-0.4
-0.2
0
0.2
0.4
0.6

S_x component after 100 thousand turns, different x, y, $\Delta p/p = 0$.



S_x component after 100 thousand turns, different x, y, $\Delta p/p = 0$.



We need to make this surface as flat as possible. This could be done using sextupoles, like we did before, and get SCT several thousand seconds.

Summary

- Transfer map method
- Differential Algebras (DA)
- DA field computations
  - Fringe fields
  - Electrostatic elements – Fringe fields, Benchmark tests
- Repetitive tracking
  - Symplectic tracking
  - Examples