Leapfrog tracking of an electrostatic ring for the pEDM project. Code M3

Alfredo U. Luccio
Brookhaven National Laboratory, Upton, New York

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Foreword

we believe that simulation of an electrostatic storage ring for the pEDM is important and should be done by more than one method to compare and benchmark

There are codes using:

1. Integration of differential equations for orbit (Lorentz) and spin (Thomas-BMT) with Runge-Kutta type routines

2. Map description of machine elements or of the whole lattice

3. Symplectic finite kick propagation

Each has different characteristics of symplecticity, accuracy, flexibility to describe static or time-variable lattices, speed of execution.

In this contribution we will briefly describe a code M3 belonging to type 3

M3 orbit tracking is based on a leapfrog kick propagation.
It is a very simple algorithm.
e.g. see: Volker Springer in "Time Integration" -2006 Helmoltz School on Computational Astrophysics)

The algorithm can be found in some form in symplectic integrators *e.g.*by Ronald Ruth. It was also notably proposed by S. Mane. It seems to us to embody the fundamental requisites of symplecticity and speed of execution required by EDM tracking.

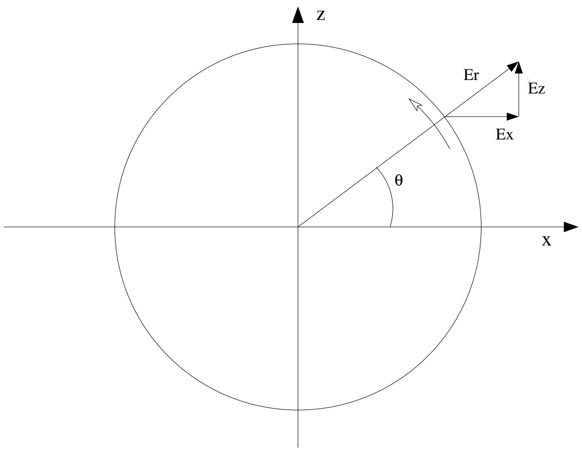
Here, we present M3 speoifically to track orbits and spin in a proposed electrostatic 0.7 GeV polarized proton ring for the pEDM experiment.

M3 spin tracking is based on the code SPINK by A.Luccio, that is also a kick propagator to match the leapfrog orbit

We will describe the formalism and present some results. Benchmarking is in progress.

Orbit coordinates

M3 uses Cartesian "laboratory" coordinates (x, z, y) -not the more common Frénet-Serret "accelerator" coordinates- with $\hat{y}$ vertical axis, and time as the independent variable. Vertical electric field component is calculated by a power expansion out of the "horizontal" x, z plane of the ring



The circular ring lattice shown is obtained by tracking a "reference particle" *i.e* at nominal energy injected tangentially.

Orbit Leapfrog formalism basics

Use a *ménagerie* of quantities for the game

$r_o[m]$	$=$	radius of curvature
$U_o[GeV]$	$=$	mc^2 rest – mass energy
a	$=$	magnetic anomaly
$pc[GeV]$	$=$	$U_o/\sqrt{a}$ moment
$U_T[GeV]$	$=$	$\sqrt{(pc)^2 + U_o^2}$, total energy
γ	$=$	$U_T/U_o, \quad \beta = \sqrt{1 - 1/\gamma^2}$
$B\rho[V \cdot s/m]$	$=$	$10^9 (pc)/c$, rigidity
$eE[eV/m]$	$=$	$(= pc/r_0)\beta c$ Electric bend field

Leapfrog formalism directly comes from the Hamilton equations

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}}{\partial p} \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial q} \tag{1}$$

Hamiltonian:

$$\mathcal{H} = \sqrt{(pc)^2 + (mc^2)^2} + e\phi. \tag{2}$$

On 3 examples (1) circular ring, (2) race-track structure and (3) 8-super-period structure with 8 bends, 8 drifts and 8 electrostatic quadrupoles, similar to what proposed by R.Talman.

Basic leapfrog cell is a sequence

$$\boxed{\text{drift} + \text{momentum kick} + \text{drift}}$$

Momentum kick follows Lorentz equation

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E}, \;\; \mathbf{E} = \nabla\phi \tag{3}$$

The potential, needed for the Hamiltonian, should obey the Laplace equation

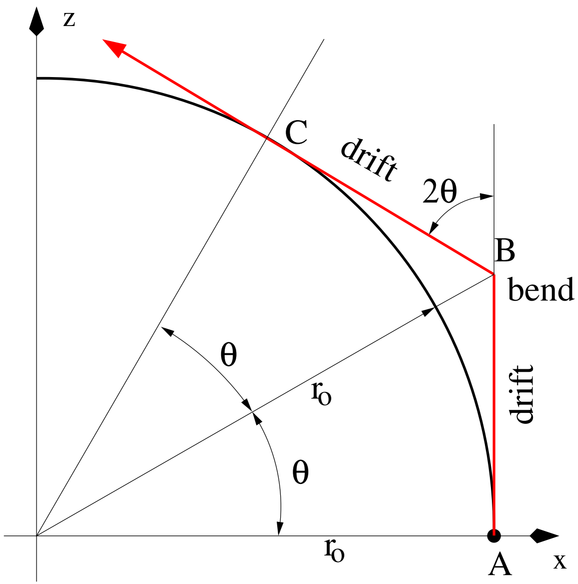
$$\nabla^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{4}$$

(an explicit solution is found by power expansion.)

The reference particle, around which the whole beam dances, is the magic particle whose spin would remain frozen in position during the propagation.

Leapfrog cell

Discuss what happens to a reference particle confined to the horizontal plane



A →B →C,
drift, kick-bend, drift

drift A-B

Start in **A** with Initial coordinates

$$(\mathbf{A}) \quad x = r_o,\ z = 0,\ (pc)_x = 0,\ (pc)_z = pc.$$

Eq's for the drift, with a time step dt for the drift **A**$\rightarrow$**B**:

$$\frac{dx}{dt} = \frac{(pc)_x}{U_o\gamma}c, \qquad \frac{dz}{dt} = \frac{(pc)_z}{U_o\gamma}c, \text{ or}$$
$$x := x + (pc)_x/(U_o\gamma)c\,dt, \qquad z := z + (pc)_z/(U_o\gamma)c\,dt \tag{5}$$

using the identity $pc = U_o\beta\gamma$, we obtain at the kick bend **B** the new position

$$(\mathbf{B}) \quad x = r_o,\ z = \beta c\,dt,\ (pc)_x = 0,\ (pc)_z = pc.$$

kick in B

In **B** a kick is imparted to the momentum **p**c, using the Lorentz Equation, with a time step δt, different from the dt of the drift.

$$(pc)_x := (pc)_x - eE_x c\,\delta t, \qquad (pc)_z := (pc)_z - eE_z c\,\delta t \tag{6}$$

For cylindrical bend the field E is purely radial, with components

$$eE_x = -eE\,r_o/r\cos\theta \qquad eE_y = eE\,r_o/r\sin\theta. \tag{7}$$

Now | find the relation between dt and δt for leapfrog | *i.e*:

1. Through the bend the value of the total momentum pc must be conserved

2. The trajectory in **C** should return tangent to the circle, as in the figure. Namely:

$$\arccos\left[(\mathbf{p}(A)\cdot\mathbf{p}(C))/p^2\right] = 2\theta \tag{8}$$

If both conditions hold, the basic trajectory will be a polygon circumscribed to the circle. Other particles in the beam will dance around it in betatron oscillations.

For condition (1): moment conservation, combining the preceding equations

$$(pc)_x = -pc/r \cos\theta\, \beta c\, \delta t, \;\; (pc)_z = pc\,(1 - (1/r)\sin\theta\, \beta c\, \delta t) \qquad (9)$$

then after kick (**C**):

$$(pc)_x^2 + (pc)_z^2 = (pc)^2 \left[1 + ((\beta c/r)\delta t)^2 - (2/r)\sin\theta\, \beta c\, \delta t\right]. \qquad (10)$$

Since: $\cos\theta = z/r$, $\sin\theta = x/r$, taking the value of x from Eq.(5), the term in [ ] in Eq.(10) above reduces to 1 when we set

$$\boxed{\delta t = 2\, dt}$$

For condition (2): new angle, it is calculated from the scalar product of the momentum before and after the kick

- (**A**) before kick: $(pc)_x = 0,\;\; (pc)_z = pc$
- (**C**) after kick: $(pc)_x = -(pc/r)\cos\theta\, \beta c\, \delta t,\;\; (pc)_z = pc\left(1 - 2\sin^2\theta\right)$

$$\text{angle} = \arccos\frac{pc(A)\cdot pc(B)}{(pc)^2} = \arccos\left(1 - 2\sin^2\theta\right) = \boxed{2\theta} \;\; q.e.d.$$

Reference Trajectory

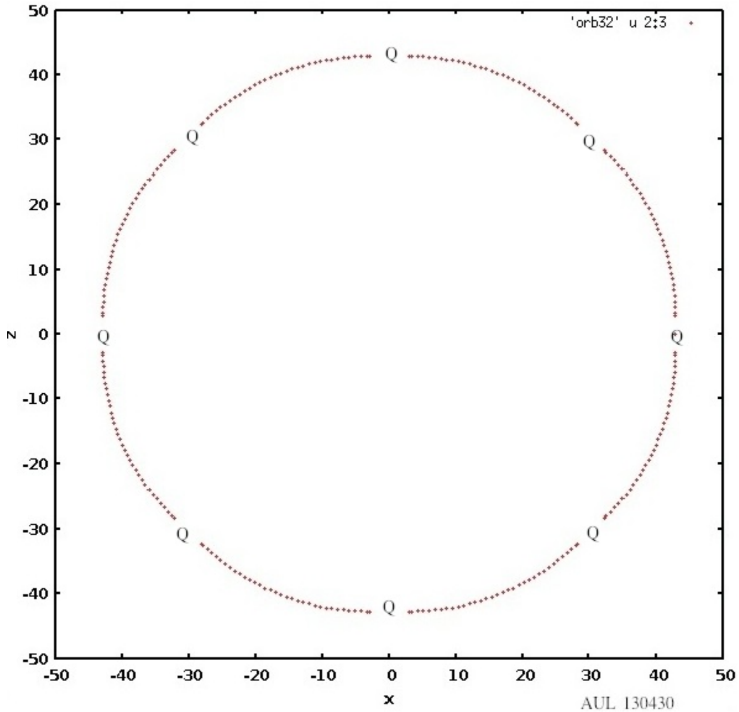
Let us produce a reference trajectory on the horizontal plane by Leapfrog tracking along a polygonal pattern tangent to a structure made of straights (drifts) and circular arcs (bends). So, The leap-frog orbit is slightly longer than the reference orbit. The more kicks we put in a bend the lesser this difference is.

In an example of a structure with 8 bends and 8 drifts of circa 270 m of total length, using 32 kicks in each bend of 36 m of radius, the difference in effective radius between the geometrical base line and the polygon is about 1 mm.

The step in M3 is much larger than the required step of a solution by integration for similar accuracy, with a very large gain in computing speed.

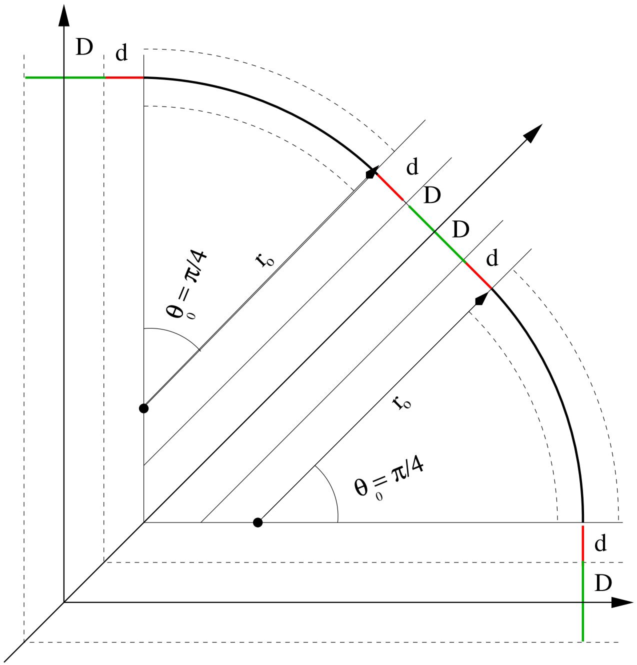
Tracking a reference particle will create a reference trajectory. An example is shown in the following picture.

Reference Trajectory by tracking



32 kicks per bend
bend length=28.276 m
drift length 2×2.83 m
intra bend drift length = 0.44 m
nominal curvature radius = 36 m
$E_{cyl} = -1.1647455 10^7 V/m$

Evaluation of the electric field



In a general lattice the center of curvature for the calculation of the electric field continuously changes and has to be re-evaluated every time

The sketch (for the preceding lattice) suggests how

'D' is any added drift space
'd' is a leapfrog inner-bend drift

General tracking

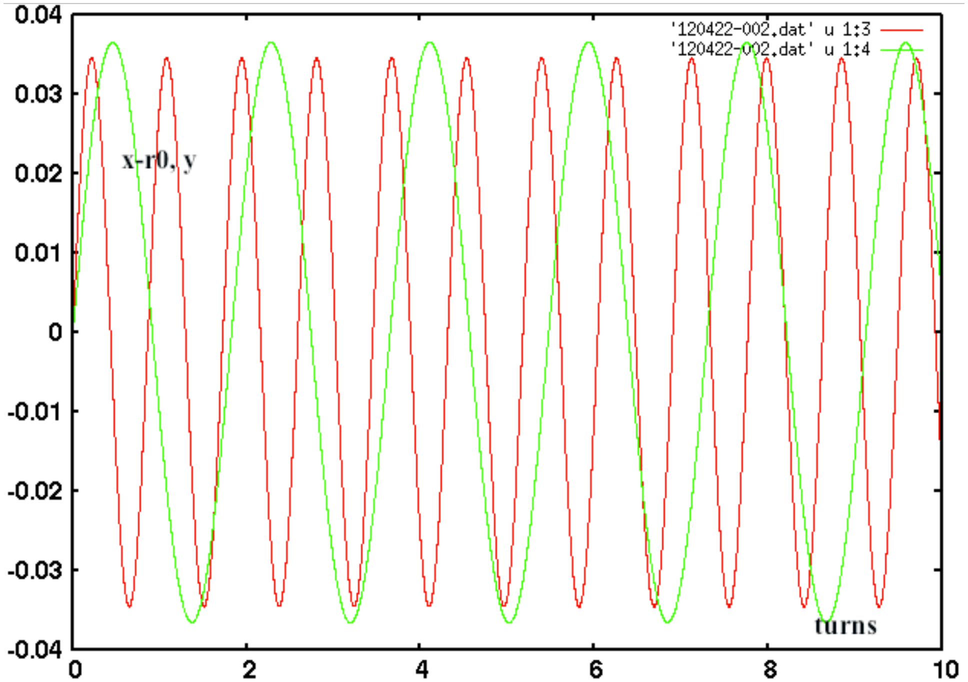
The Leapfrog formalism extends to 3 dimensions and applies unchanged to particles that don't have a magic energy or are injected in the lattice on a finite transverse emittance.

Eqs.(5) and .(6) in 3 dimensions are

$$\left\{ \begin{array}{l} x := x + (pc)_x/(U_o\gamma)c\,dt \\ y := y + (pc)_y/(U_o\gamma)c\,dt \\ z := z + (pc)_z/(U_o\gamma)c\,dt \end{array} \right. , \quad \begin{array}{l} (pc)_x := (pc)_x - eE_x 2c\,dt \\ (pc)_y := (pc)_y - eE_y 2c\,dt \\ (pc)_z := (pc)_z - eE_z 2c\,dt \end{array} . \qquad (11)$$

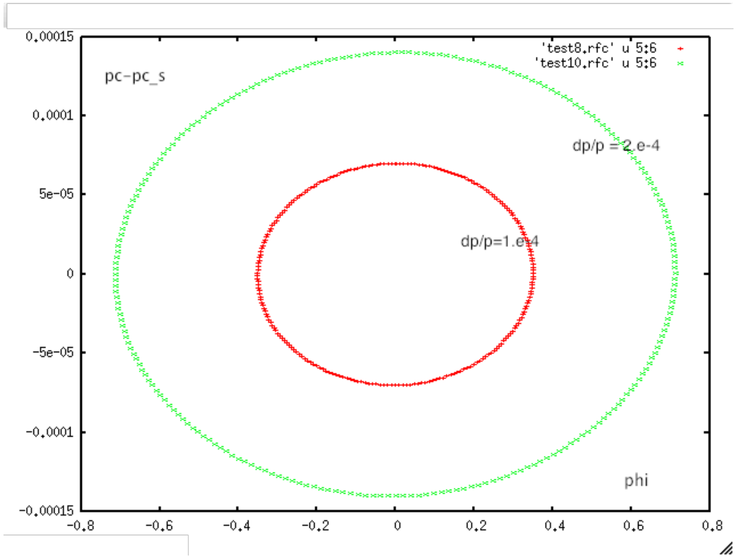
However, In a general case the leapfrog conditions (1) for momentum and angle are not fully satisfied in a bend because, due to transverse oscillations, the particle sees a tangential component of the electric field that modulates the energy.

During tracking the Hamiltonian is continuously calculated. It conserves its initial valiue.



x,y betatron oscillations vs. turn #

Add a RF - Example of RF bucket



Phase space of $\Delta \times pc$ for two particles, with $dp/p = 1.e^{-4}$ and $2.e^{-4}$, respectively, with $V_{RF} = 1000V/m$ and $h = 24$. Number of turns for a complete oscillations is 335, corresponding synchrotron frequency $\nu_s = 0.002985$ oscillations per turn

Briefly on Spin Dynamics: BMT equation

The code SPINK uses the T-BMT equation for the evolution of the spin **S** of the proton

$$\frac{d\mathbf{S}}{dt} = -\frac{q}{m\gamma}\mathbf{f} \times \mathbf{S} \tag{12}$$

For a 1/2-spin particle, $\mathbf{S}$ is treated as a real 3-dimensional spin vector. $\mathbf{f}$ is a function of the position, the momentum and of the electric field encountered by the proton. In a pure electrostatic ring **f** reduces to

$$\mathbf{f} = \left(a\gamma - \frac{\gamma}{\gamma^2 - 1}\right)\frac{\mathbf{E} \times \mathbf{v}}{c^2}. \tag{13}$$

M3-SPINK calculates the kick matrix $\mathcal{M}$ for kick propagation of the spin vector

$$\mathbf{S} := \mathcal{M}\mathbf{S} \tag{14}$$

Matrix elements are function of field and dynamics variables.

Briefly on Spin Dynamics: EDM

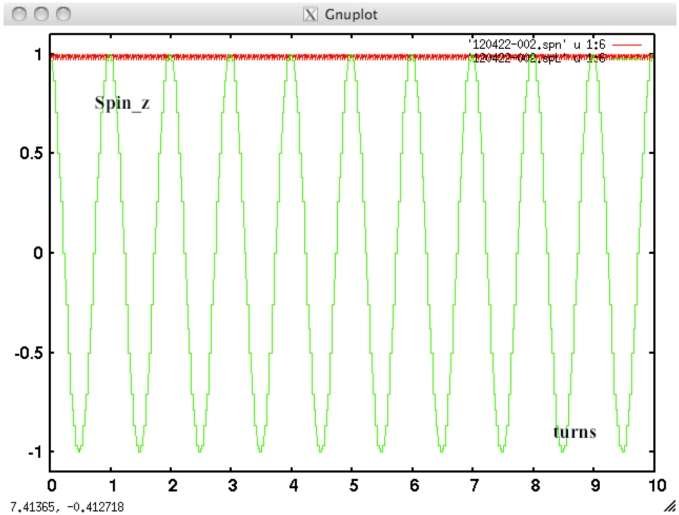
Spin kicks, applied at each Bend and Quad, follow the leapfrog pattern of the orbit.

At the magic energy it is $\mathbf{F} = \mathbf{0}$ and the spin remains frozen
If the proton has an EDM, the spin is Not completely frozen: in the rest frame of the particle, the electric field appears as a magnetic field **B'** $\perp$ to **E** and another small term is added to **f** in Eq.(13)

$$\mathbf{B}' = -\gamma\vec{\beta} \times \mathbf{E}. \quad \mathbf{f} := \mathbf{f} + \eta\mathbf{B}' \times \mathbf{v}. \tag{15}$$

The spin will make a precession around this magnetic field and a spin vertical component will appear, that can be measured. For a magic proton this is the only non vanishing additional spin component.

Spin dynamics of a frozen spin



Longit. component of the frozen spin: red line in accelerator coordinates, green line, in laboratory coordinates. The red line shows little wiggles because the responsible proton is on purpose not perfectly magic and there are betatron oscillations.

Thank You !