# Beam and Spin Dynamics
# The Transfer Map Approach

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# Transfer Map Method and Differential Algebras

- The transfer map $\mathcal{M}$ is the flow of the system ODE.

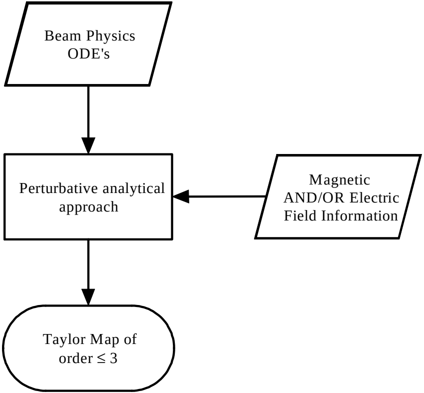
$$\vec{z}_f = \mathcal{M}(\vec{z}_i, \vec{\delta}),$$

  where $\vec{z}_i$ and $\vec{z}_f$ are the initial and the final condition, $\vec{\delta}$ is system parameters.
- For a repetitive system, only one cell transfer map has to be computed. Thus, it is much faster than ray tracing codes (i.e. tracing each individual particle through the system).
- The Differential Algebraic method allows a very efficient computation of high order Taylor transfer maps.
- The Normal Form method can be used for analysis of nonlinear behavior.

## Differential Algebras (DA)

- it works to arbitrary order, and can keep system parameters in maps.
- very transparent algorithms; effort independent of computation order.

The code **COSY Infinity** has many tools and algorithms necessary.




- Method can be used to compute transfer map of order $\leq 3$
- Analytic or local Taylor expansion (multipole decomposition) of the magnetic field should be specified

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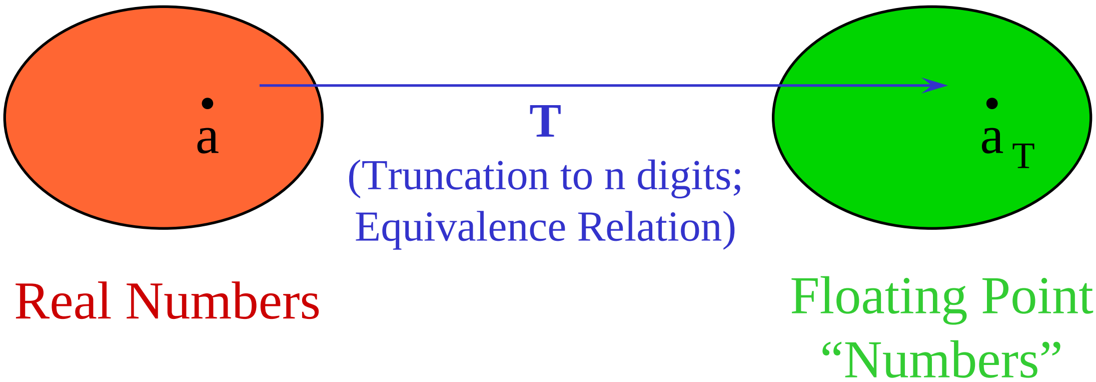
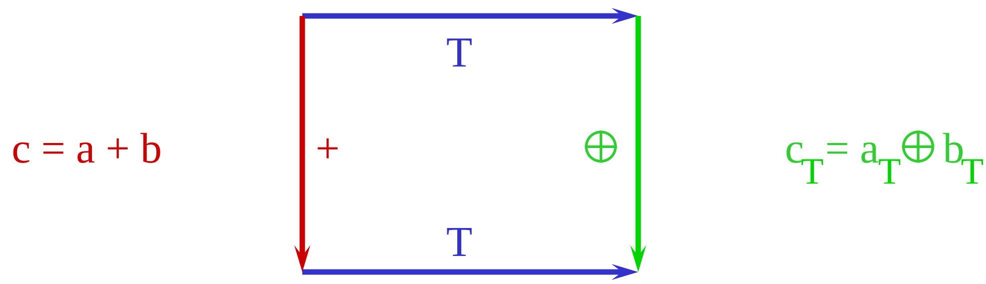
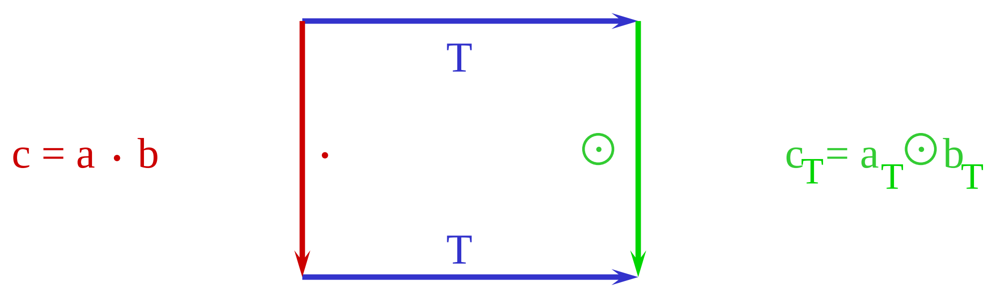
# Number Fields and Floating Point Numbers



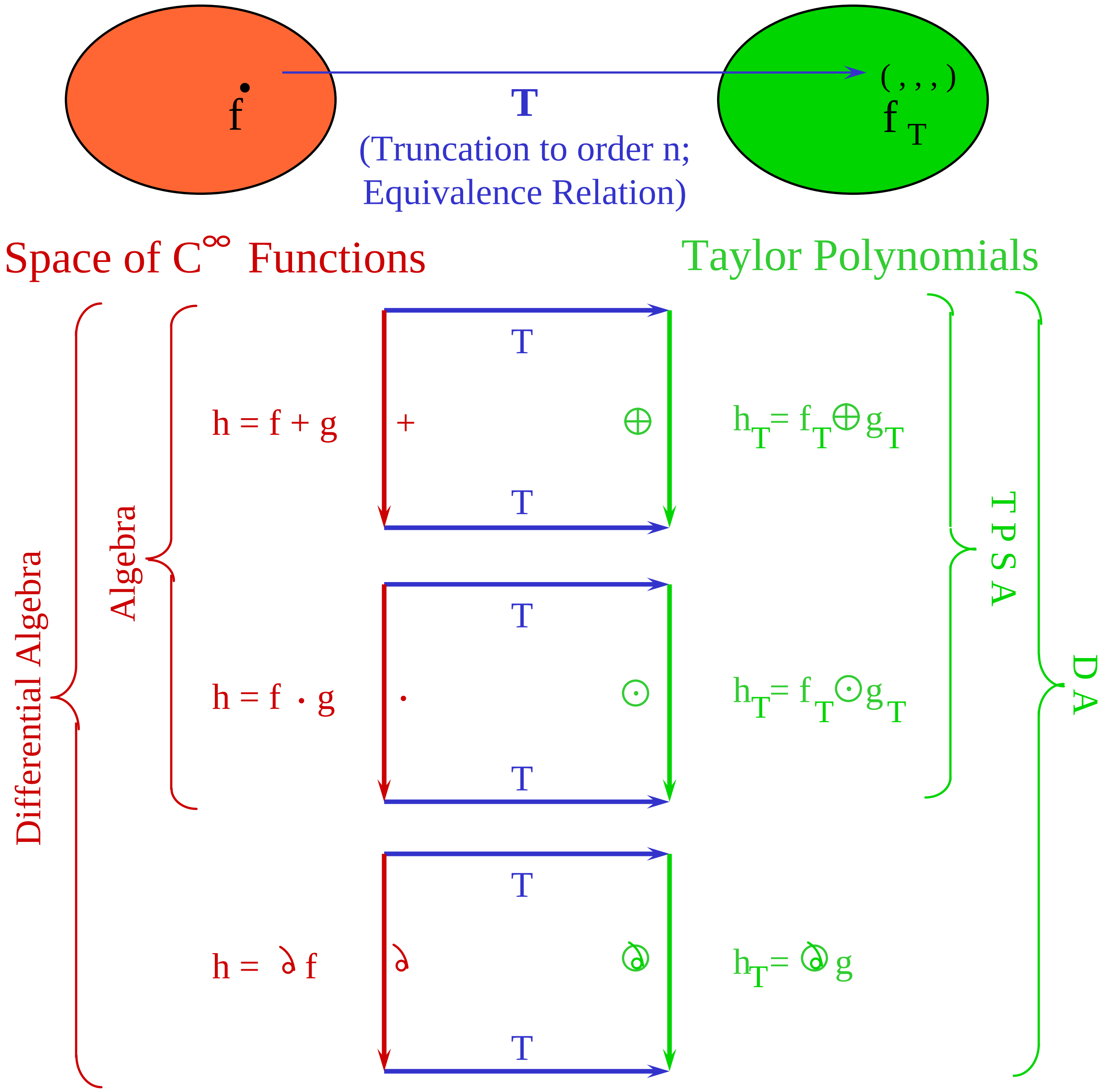

**Field**
(Also want “exp”, “sin”
etc: Banach Field)

Diagrams
commute
“approximately”

**Field**
(“approximately”)

T: Extracts Information
considered relevant

# Function Algebras




**Differential Algebra**
(also want "exp", "sin"
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Diagrams
commute
exactly

**Differential Algebra**
(even Banach DA)

T: Extracts Information
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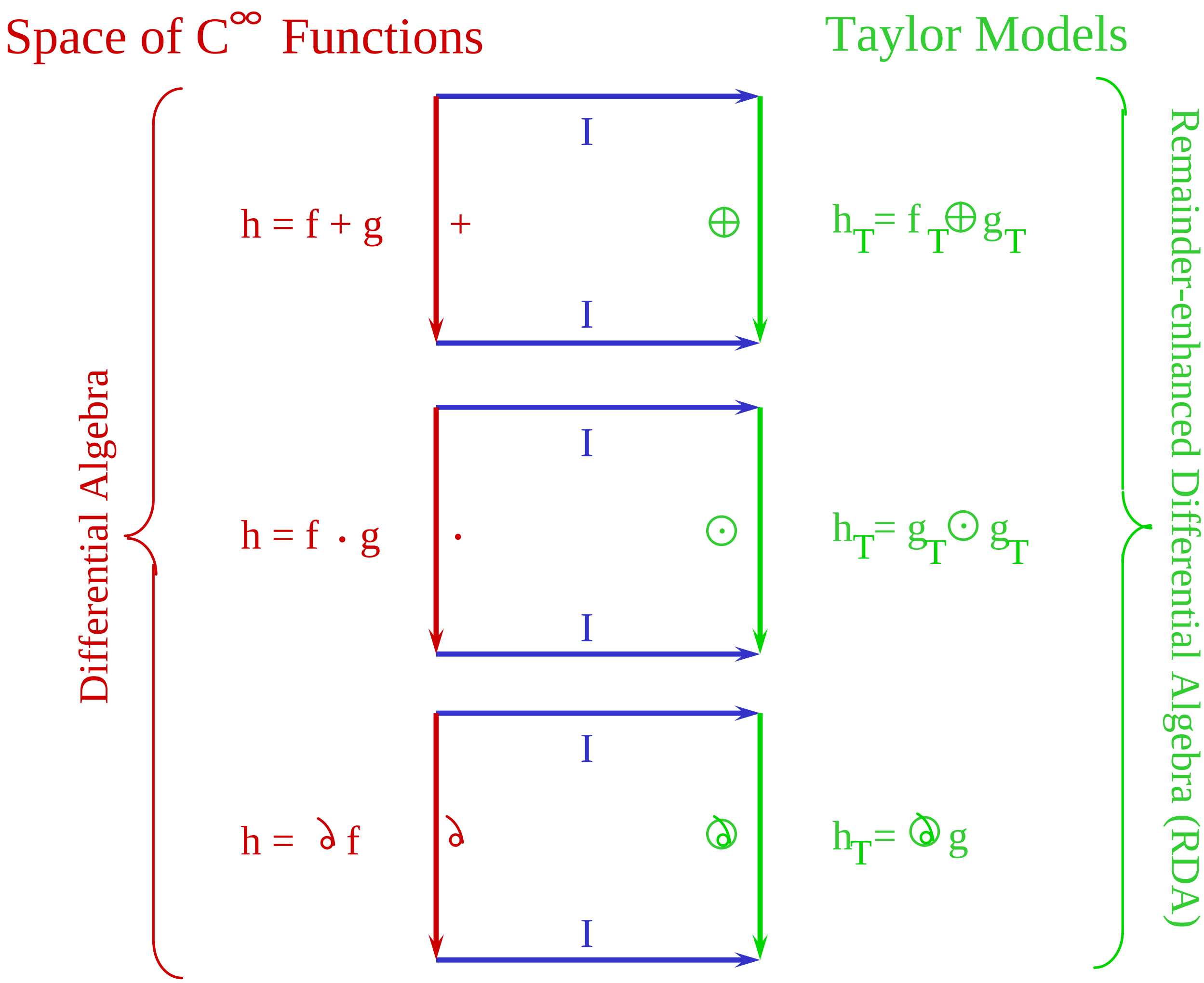
# Function Algebra Inclusions

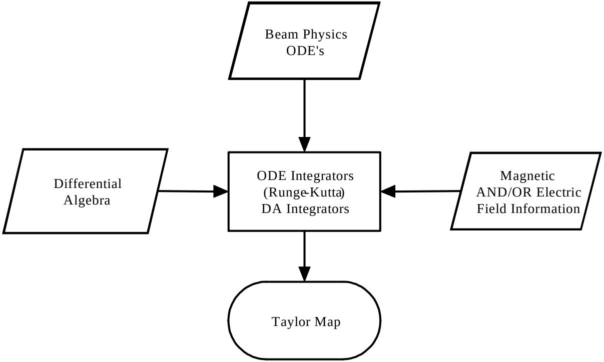




**Differential Algebra**
(also want “exp”, “sin” etc: Banach DA)

Diagrams commute exactly

**Differential Algebra with Remainder**

I: Extracts Information considered relevant




- DA methods were introduced in 1988 to compute maps to in principle arbitrary order
- Analytic formula or local expansion of the field should be specified

# COSY INFINITY

- Arbitrary order
- Maps depending on parameters
- No approximations in motion or field description
- Large library of elements
- Arbitrary Elements (you specify fields)
- Very flexible input language
- Powerful interactive graphics
- Errors: position, tilt, rotation
- Tracking through maps
- Normal Form Methods
- Spin dynamics
- Fast fringe field models using SYSCA approach
- Reference manual (80 pages) and Programming manual (90 pages)

## The Operator $\partial^{-1}$on Taylor Models

Let $(P_n, I_n)$ be an $n$-th order Taylor model of $f$. From this we can obtain a Taylor model for the indefinite integral $\partial_i^{-1} f = \int f \; dx_i'$ with respect to variable $x_i$.

Taylor polynomial part: $\int_0^{x_i} P_{n-1} dx_i'$,

Remainder Bound: $(B(P_n - P_{n-1}) + I_n) \cdot B(x_i)$, where $B(P)$ is a polynomial bound.

So define the operator $\partial_i^{-1}$on space of Taylor models as

$$\partial_i^{-1}(P_n, I_n) \\ = \left( \int_0^{x_i} P_{n-1} dx_i' \;\;, \quad (B(P_n - P_{n-1}) + I_n) \cdot B(x_i) \right)$$

## Taylor Models for the Flow

Goal: Determine a Taylor model, consisting of a Taylor Polynomial and an interval bound for the remainder, for the flow of the differential equation

$$\frac{d}{dt}\vec{r}(t) = \vec{F}(\vec{r}(t), t)$$

where $\vec{F}$ is sufficiently differentiable. The Remainder Bound should be fully rigorous for all initial conditions $\vec{r}_0$ and times $t$ that satisfy

$$\vec{r}_0 \in [\vec{r}_{01}, \vec{r}_{02}] = \vec{B}$$
$$t \in [t_0, t_1].$$

In particular, $\vec{r}_0$ itself may be a Taylor model, as long as its range is known to lie in $\vec{B}$.

## The Use of Schauder's Theorem

Re-write differential equation as integral equation

$$\vec{r}(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{F}(\vec{r}(t'), t')\, dt'.$$

Now introduce the operator

$$A : \vec{C}^0[t_0, t_1] \to \vec{C}^0[t_0, t_1]$$

on space of continuous functions via

$$A\left(\vec{f}\right)(t) = \vec{r}_0 + \int_{t_0}^{t} \vec{F}\left(\vec{f}(t'), t'\right)\, dt'.$$

Then the solution of ODE is transformed to a fixed-point problem on space of continuous functions

$$\vec{r} = A(\vec{r}).$$

**Theorem (Schauder):** *Let $A$ be a continuous operator on the Banach Space $X$. Let $M \subset X$ be compact and convex, and let $A(M) \subset M$. Then $A$ has a fixed point in $M$, i.e. there is an $\vec{r} \in M$ such that $A(\vec{r}) = \vec{r}$.*

## The Polynomial of the Self-Including Set

Attempt sets $M^*$ of the form

$$M^* = M_{\vec{P}^* + \vec{I}^*} \text{ where}$$
$$\vec{P}^* = \mathcal{M}_n(\vec{r}_0, t),$$

the $n$-th order Taylor expansion of the flow of the ODE. It is to be expected that $\vec{I}^*$ can be chosen smaller and smaller as order $n$ of $\vec{P}^*$ increases.

This requires knowledge of $n$th order flow $\mathcal{M}_n(\vec{r}_0, t)$, including time dependence. It can be obtained by iterating in polynomial arithmetic, or Taylor models without treatment of a remainder. To this end, one chooses an initial function $\mathcal{M}_n^{(0)}(\vec{r}, t) = \mathcal{I}$, where $\mathcal{I}$ is the identity function, and then iteratively determines

$$\mathcal{M}_n^{(k+1)} =_n A(\mathcal{M}_n^{(k)}).$$

This process converges to the exact result $\mathcal{M}_n$ in exactly $n$ steps.

# Field Description in Differential Algebra

There are various DA algorithms to treat the fields of beam optics efficiently. For example, **DA PDE Solver**

- requires to supply only
  - the midplane field for a midplane symmetric element.
  - the on-axis potential for straight elements like solenoids, quadrupoles, and higher multipoles.
- treats arbitrary fields straightforwardly.
  - Magnet (or, Electrostatic) fringe fields:
    The Enge function fall-off model
    $$F(s) = \frac{1}{1 + \exp(a_1 + a_2 \cdot (s/D) + ... + a_6 \cdot (s/D)^5)}$$
    where $D$ is the full aperture.
    Or, any arbitrary model including the measured data representation.
  - Solenoid fields including the fringe fields.
  - Measured fields: E.g. Use Gaussian wevelet representation.
  - Etc. etc.

# DA Fixed Point PDE Solvers

The **DA fixed point theorem** allows to **solve PDEs iteratively** in **finitely many steps** by rephrasing them in terms of a fixed point problem.

Consider the rather general PDE

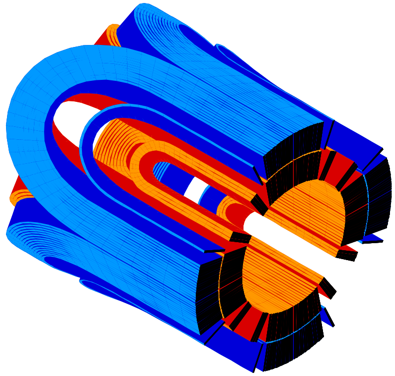
$$a_1 \frac{\partial}{\partial x}\left(a_2 \frac{\partial}{\partial x} V\right) + b_1 \frac{\partial}{\partial y}\left(b_2 \frac{\partial}{\partial y} V\right) + c_1 \frac{\partial}{\partial z}\left(c_2 \frac{\partial}{\partial z} V\right) = 0,$$

where $a_i$, $b_i$, $c_i$ are functions of $x$, $y$, $z$.

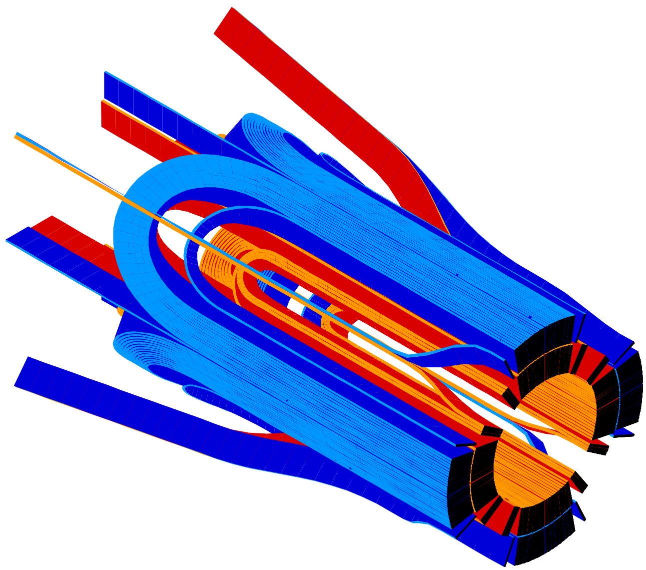
The PDE is re-written in **fixed point form** as

$$\begin{aligned} V = V|_{y=0} &+ \int_0^y \frac{1}{b_2}\left(b_2 \frac{\partial V}{\partial y}\right)\Bigg|_{y=0} \\ &- \int_0^y \frac{1}{b_2} \int_0^y \left(\frac{a_1}{b_1}\frac{\partial}{\partial x}\left(a_2 \frac{\partial V}{\partial x}\right) + \frac{c_1}{b_1}\frac{\partial}{\partial z}\left(c_2 \frac{\partial V}{\partial z}\right)\right) dy dy. \end{aligned}$$

Assume the derivatives of $V$ and $\partial V / \partial y$ with respect to $x$ and $z$ are **known in the plane** $y = 0$. Then the right hand side is **contracting** with respect to $y$ (which is necessary for the DA fixed point theorem), and the various orders in $y$ can be **iteratively** calculated by mere iteration.



The HGQ return end

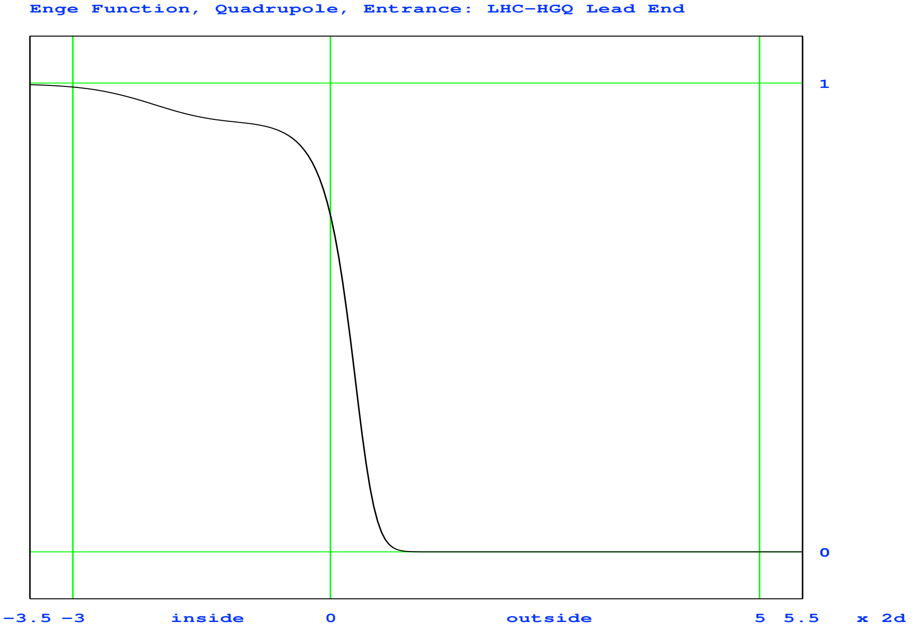


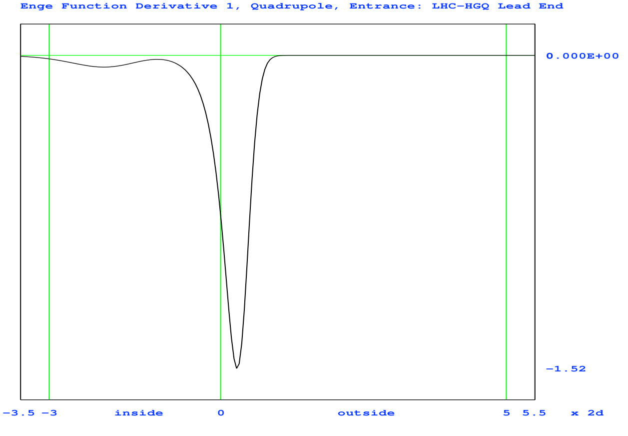
The HGQ lead end

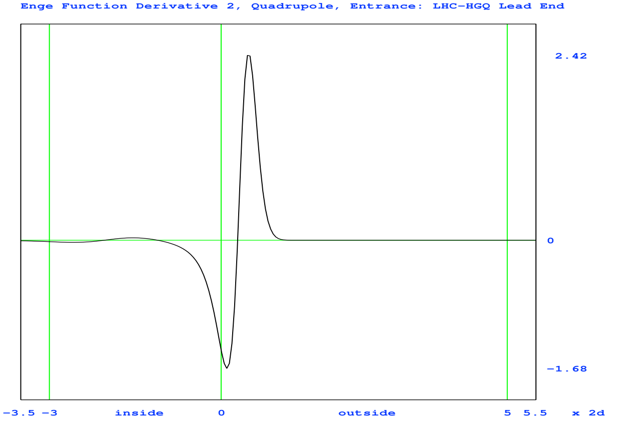
Pictures :

G. Sabbi, “Magnetic Field Analysis of HGQ Coil Ends”

# LHC-HGQ Lead End Enge Function



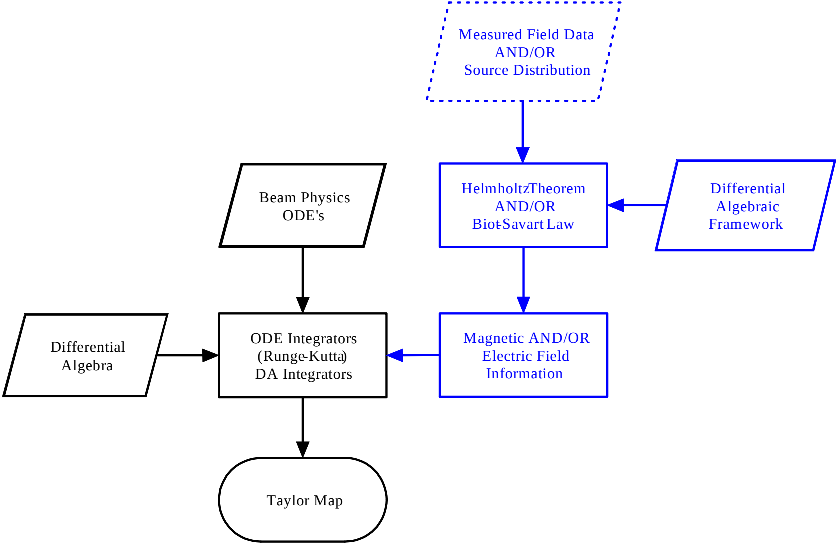



## Elements in COSY

- Magnetic and electric multipoles
- Superimposed multipoles
- Combined function bending magnets with curved edges
- Electrostatic deflectors
- Wien filters
- Wigglers
- Solenoids, various field configurations
- 3 tube electrostatic round lens, various configurations
- Exact fringe fields to all of the above
- Fast fringe fields (SYSCA)
- General electromagnetic element (measured data)
- Glass lenses, mirrors, prisms with arbitrary surfaces
- Misalignments: position, angle, rotation

All can be computed to arbitrary order, and the dependence on any of their parameters can be computed.


Measured Field Data
AND/OR
Source Distribution
Beam Physics
ODE's
HelmholtzTheorem
AND/OR
Biot-Savart Law
Differential
Algebraic
Framework
Differential
Algebra
ODE Integrators
(Runge-Kutta)
DA Integrators
Magnetic AND/OR
Electric Field
Information
Taylor Map

## Normal Form Theory

Goal: perform a nonlinear change of variables such that the motion in the new variable pairs is rotationally invariant:

$\mathcal{M} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{M}$

If the map is symplectic, this means circles. If the map is damped, we obtain logarithmic spirals.

Advantage: Tune with amplitude is trivial to compute, since each iteration of the map corresponds to the same angle advance.

Other Advantages: - Provides pseudo invariants the quality of which allows conclusions about the map; - sensitive to resonances, allows efficient study of resonances

# Decoupling of Planes in Linear Map

Goal: Coordinate Transformation such that linear Map that has only two-by-two blocks along diagonal.

If Linear Map is not in block form, it can be brought into block form if it has $n$ disctinct Eigenvalues. (Other cases are not of interest because they imply resonances)

Eigenvalues are either

- a complex conjugate pair $\rightarrow$ choose the real and imaginary parts of the eigenvector
- a real pair $\rightarrow$ choose the two eigenvectors

The resulting map is purely real. When it is fully diagonlized it becomes complex.

## The DA Normal Form Algorithm

Assume linear part of map has been diagonalized by a linear change of basis:

$$\mathcal{M} = \mathcal{R} + \mathcal{S}$$

where $\mathcal{R}$ has on its diagonal the values $r_j \cdot e^{\pm i\nu_j}$ (pair structure).

Now attempt to simplify the map by a nonlinear transformation. Choose transformation

$$\mathcal{A}_m = \mathcal{E} + \mathcal{T}_m$$

Up to order $m$, the inverse is $\mathcal{A}_m^{-1} =_m \mathcal{E} - \mathcal{T}_m$, and we obtain

$$\begin{aligned}
&\mathcal{A} \circ \mathcal{M} \circ \mathcal{A}^{-1} \\
&=_m (\mathcal{E} + \mathcal{T}_m) \circ (\mathcal{R} + \mathcal{S}_{m-1}) \circ (\mathcal{E} - \mathcal{T}_m) \\
&=_m (\mathcal{E} + \mathcal{T}_m) \circ (\mathcal{R} + \mathcal{S}_{m-1}) \circ (\mathcal{E} - \mathcal{T}_m) \\
&=_m \mathcal{R} + \mathcal{S}_{m-1} + (\mathcal{T}_m \circ \mathcal{R} - \mathcal{R} \circ \mathcal{T}_m)
\end{aligned}$$

## Removing Terms in the Normal Form Step

We can use the commutator $\mathcal{C}$ of $\mathcal{T}_m$ and $\mathcal{R}$ to remove terms from $\mathcal{S}_{m-1}$. We write

$$\mathcal{T}_{mj}^{\pm} = \sum (\mathcal{T}_{mj}^{\pm}|k_1^+, k_1^-, ..., k_n^+, k_n^-) \cdot (v_1^+)^{k_1^+}(v_1^-)^{k_1^-} \cdot ... \cdot (v_n^+)^{k_n^+}(v_n^-)^{k_n^-}$$

$$\mathcal{C}_{mj}^{\pm} = \sum (\mathcal{C}_{mj}^{\pm}|k_1^+, k_1^-, ..., k_n^+, k_n^-) \cdot (v_1^+)^{k_1^+}(v_1^-)^{k_1^-} \cdot ... \cdot (v_n^+)^{k_n^+}(v_n^-)^{k_n^-}$$

Because of the simple form of $\mathcal{R}$, we obtain

$$\begin{aligned}&(\mathcal{C}_j^{\pm}|k_1, k_1^-, ..., k_n^+, k_n^-)\\ &= -C_j^{\pm}(\vec{k}^+, \vec{k}^-) \ \cdot \ (\mathcal{T}_j^{\pm}|k_1^+, k_1^-, ..., k_n^+, k_n^-)\end{aligned}$$

where

$$C_j^{\pm}(\vec{k}^+, \vec{k}^-) = r_j \cdot e^{\pm i\nu_j} - \left( \prod_{j=1}^{n} (r_j)^{k_j^+ + k_j^-} \right) \cdot e^{i\vec{\nu}\cdot(\vec{k}^+ - \vec{k}^-)}$$

So we can remove every term for which $C_j^{\pm}(\vec{k}^+, \vec{k}^-)$ is nonzero!

## Removable Terms in the Symplectic Case

In the symplectic case, all $r_j$ are one (no damping). Then everything is removable except if

$$\vec{\nu} \cdot (\vec{k}^+ - \vec{k}^-) = l \cdot 2\pi \pm \nu_j \quad \forall l$$

This can occur in the following cases:

1. $\vec{n} \cdot \vec{\nu} = l \cdot 2\pi$ has nontrivial solutions (we are on a resonance; physics case)
2. $k_l^+ = k_l^- \ \ \forall\, l \neq j$, and $k_j^+ = k_j^- \pm 1$ (unavoidable; mathematics case)

# Resonance Correction and Tune Shifts

Amplitude dependent tune shifts obtained with DA normal form theory.Before and after resonance correction with sextupoles

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 0.4499999999999998 | 0 | 0 | 0 | 0 | 0 |
| 2 | 30.71450631162792 | 2 | 2 | 0 | 0 | 0 |
| 3 | -39734.01363530685 | 2 | 0 | 0 | 2 | 0 |
| 4 | 3077.595867175395 | 4 | 4 | 0 | 0 | 0 |
| 5 | -315212453990.7433 | 4 | 2 | 0 | 2 | 0 |
| 6 | -294628537556.6385 | 4 | 0 | 0 | 4 | 0 |

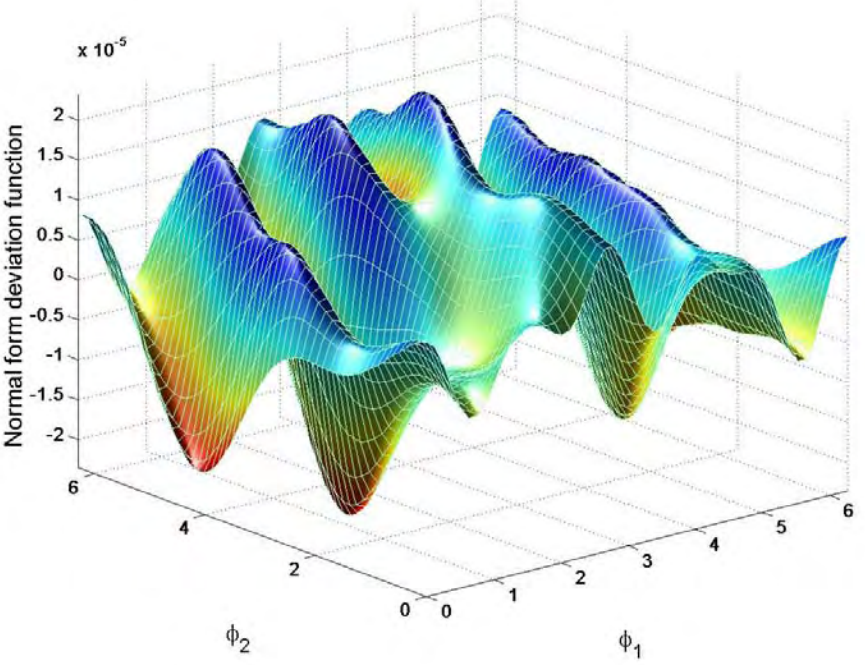
| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 0.4499999999999998 | 0 | 0 | 0 | 0 | 0 |
| 2 | 8.691395574893371 | 2 | 2 | 0 | 0 | 0 |
| 3 | 241.0982265780670 | 2 | 0 | 0 | 2 | 0 |
| 4 | 1247.729293865072 | 4 | 4 | 0 | 0 | 0 |
| 5 | 757332.8310454757 | 4 | 2 | 0 | 2 | 0 |
| 6 | 3727722.890469429 | 4 | 0 | 0 | 4 | 0 |

# The Normal Form Defect Function

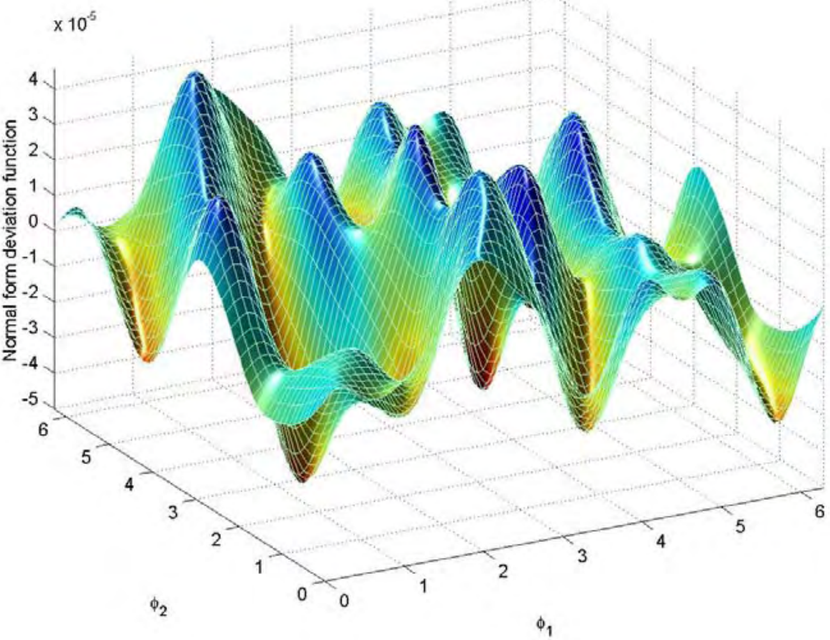
- **Extreme cancellation**; one of the reasons TM methods were invented
- Six-dimensional problem from dynamical systems theory
- Describes invariance defects of a particle accelerator
- Essentially composition of three tenth order polynomials
- The function vanishes identically to order ten
- Study for $a\cdot\ (1,1,1,1,1,1)$ for $a = .1$ and $a = .2$
- Interesting **Speed observation**: on same machine,
  * one CF in INTLAB takes 45 minutes
  * one TM of order 7 takes 10 seconds

$$f_4(x_1, .., x_6) = \sum_{i=1}^{3} \left( \sqrt{y_{2i-1}^2 + y_{2i}^2} - \sqrt{x_{2i-1}^2 + x_{2i}^2} \right)^2$$

$$\text{where } \vec{y} = \vec{P}_1 \left( \vec{P}_2 \left( \vec{P}_3(\vec{x}) \right) \right)$$



x 10^-5
Normal form deviation function
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
6
4
2
0
ϕ2
0
1
2
3
4
5
6
ϕ1


x 10^-5
Normal form deviation function
4
3
2
1
0
-1
-2
-3
-4
-5
φ2
0
1
2
3
4
5
6
φ1
0
1
2
3
4
5
6

## The BMT Equation

Classical equation of Motion for Spin:

$$\frac{d\vec{s}}{dt} = \vec{w} \times \bar{s}, \text{where}$$

$$\vec{w} = k \left( -(1+G\gamma)\vec{B} + \frac{G}{1+\gamma}(\vec{P} \cdot \vec{B})\ \vec{P} + (G + \frac{1}{1+\gamma})\ \vec{P} \times \frac{\vec{E}}{c} \right),$$

and $k = e/\gamma m_o c$, $G = (g-2)/g$, $\vec{P} = \vec{p}/m_0 c$.

In particle optical relative coordinates:

$$\frac{d\vec{s}}{ds} = t' \cdot \vec{w} \times \bar{s} + \vec{h} \times \vec{s}, \text{where } s\text{: arclength, } \vec{h}\text{: curvature}$$

Solution is a *linear orthogonal transformation* depending on orbital variables. Thus

$$\vec{s}_f = \hat{A}(\vec{z}) \cdot \vec{s}_i, \text{ where}$$

$$\hat{A}(\vec{z}) \in SO(3)$$

## Motion Of Spin Matrix

Nine dimensional motion of particle with spin, neglecting spin-orbit coupling

$$\begin{pmatrix} \vec{z} \\ \vec{s} \end{pmatrix}' = \vec{F}(\vec{z}, \vec{s}, s) = \begin{pmatrix} \vec{f}(\vec{z}, s) \\ \hat{W}(\vec{z}, s) \cdot \vec{s} \end{pmatrix}$$

$$\begin{pmatrix} \vec{z}_f \\ \vec{s}_f \end{pmatrix} = \vec{M}(\vec{z}_i, \vec{s}_i, s) = \begin{pmatrix} \mathcal{M}(\vec{z}_i, s) \\ \hat{A}(\vec{z}_i, s) \cdot \vec{s} \end{pmatrix}$$

To *reduce dimensionality* and *utilize linearity*, it is advantageous to set up EOM for $\hat{A}$. Insertion yields EOM for $3 \times 3$ spin matrix depending on only the 6 orbital variables:

$$\hat{A}'(\vec{z}, s) = \hat{W}(\vec{z}, s) \cdot \hat{A}(\vec{z}, s)$$

# DA Computation of Spin Motion

Instead of motion in nine variables, it is sufficient to use six dimensional DA to determine Taylor expansion of $\hat{A}(\vec{z})$:

*Autonomous Case* (Main Fields): Utilize existence of two invariant subspaces of mixed symplectic-orthogonal propagator $\exp(sL_{\vec{F}})$ with directional derivative

$$L_{\vec{F}} = \vec{f}^t \cdot \vec{\nabla}_{\vec{z}} + (\hat{W} \cdot \vec{s})^t \cdot \vec{\nabla}_{\vec{s}}$$

and exploit

$$\vec{M}(\vec{z}_i, \vec{s}_i, s) = \exp(sL_{\vec{F}})\ \vec{I}.$$

*Non-Autonomous Case* (Fringe Fields, Wigglers, Measured Fields etc.): Integrate equations of motion for orbit and spin matrix in DA.

*Composition of Maps:* Let $(\vec{M}_{1,2}, \hat{A}_{1,2})$ and $(\vec{M}_{2,3}, \hat{A}_{2,3})$ be given. Then we get

$$\vec{M}_{1,3} = \vec{M}_{2,3} \circ \vec{M}_{1,2}$$
$$\hat{A}_{1,3}(\vec{z}) = \hat{A}_{2,3}(\vec{M}_{1,2}) \cdot \hat{A}_{1,2}(\vec{z})$$

## Invariant Subspaces of $L_{\vec{F}}$

Define two spaces of functions $g(\vec{z}, \vec{s})$ on spin-orbit phase space as follows:

$Z$: space of functions depending on $\vec{z}$
$S$:space of linear forms in $\vec{s}$ with coefficients in $Z$

Then we have for $g \in Z$ :

$$L_{\vec{F}}\ g = (\vec{f}^t \cdot \vec{\nabla}_{\vec{z}} + (\hat{W} \cdot \vec{s})^t \cdot \vec{\nabla}_{\vec{s}})\ g = \vec{f}^t \cdot \vec{\nabla}_{\vec{z}}\ g = L_{\vec{f}}\ g$$

Similarly, we have for $g =< a_j >= \sum_j a_j \cdot s_j \in S$ :

$$L_{\vec{F}}\ < a_j >= (\vec{f}^t \cdot \vec{\nabla}_{\vec{z}} + (\hat{W} \cdot \vec{s})^t \cdot \vec{\nabla}_{\vec{s}})\ (\sum_j a_j \cdot s_j)$$
$$= \sum_j (\vec{f}^t \cdot \vec{\nabla}_{\vec{z}}) a_j \cdot s_j + \sum_{j,k} s_j W_{kj} a_k =< L_{\vec{f}} a_j + \sum_k W_{kj} a_k > .$$

Thus $Z$ and $S$ are invariant subspaces of $L_{\vec{F}}$. Further, the components of $\hat{I}$ are in $Z$ and $S$, and hence so is $\exp(sL_{\vec{F}})$.

Since elements in either space are charecterized by just six dimensional functions, $\exp(sL_{\vec{F}})$ can be computed in a six dimensional differential algebra.

## Spin Tracking

Spin tracking is performed iteratively with the following steps:

1. Evaluate orbit map on current orbit coordinates $\vec{z}_n$ to get new coordinates $\vec{z}_{n+1}$:
$$\vec{z}_{n+1} = \vec{M}(\vec{z}_n),$$
symplectify if needed or desired.

2. Insert new orbit coordinates into spin matrix to get
$$\hat{A}^* = \hat{A}(z_{n+1}),$$
orthogonalize $\hat{A}^*$ if needed or desired; simply requires renormalization of $\vec{s}_{n+1}$.

3. Multiply current spin matrix $\hat{A}^*$ with current spin coordianates $\vec{s}_n$ to get new spin coordinates $\vec{s}_{n+1}$

4. Display coordinates of $\vec{s}_{n+1}$

## The Spin Tune Shifts

The determination of spin amplitude tune shifts requires the following steps:

1. Bring orbital motion to normal form $N$ describing amplitude dependent rotations.
2. Express spin matrix $\hat{A}$ in terms of orbital normal form
3. Solve $3 \times 3$ eigenvalue problem for $\hat{A}$
4. Linear part of conjugate non-unity eigenvalues give spin tune
5. Nonlinear part of non-unity eigenvalues give spin tune dependence on orbital amplitudes

## The Invariant Axis $\bar{n}$

One of the quantities of prime interest: Invariant polarization axis $\bar{n}(\vec{z})$. It depends on orbit quantities and satisfies

$$\hat{A}(\vec{z}) \cdot \bar{n}(\vec{z}) = \bar{n}(\vec{\mathcal{M}}(\vec{z}))$$

We develop a method to obtain $\bar{n}(\vec{z})$ with coupled $SO(3)$ and symplectic normal form methods.

To begin, assume orbital map is already in normal form and linear spin map is diagonalized.

For zeroth order, observe that $\vec{\mathcal{M}}(\vec{z}) =_0 0$. Denote $\bar{n}_0 = \bar{n}(0)$.The equation reduces to

$$\hat{A}_0 \cdot \bar{n}_0 = \bar{n}_0$$

Thus, $\bar{n}_0$ is the eigenvector to unit eigenvalue of the constant part of $\hat{A}$.

## The Invariant Axis $\bar{n}$ - Higher Orders

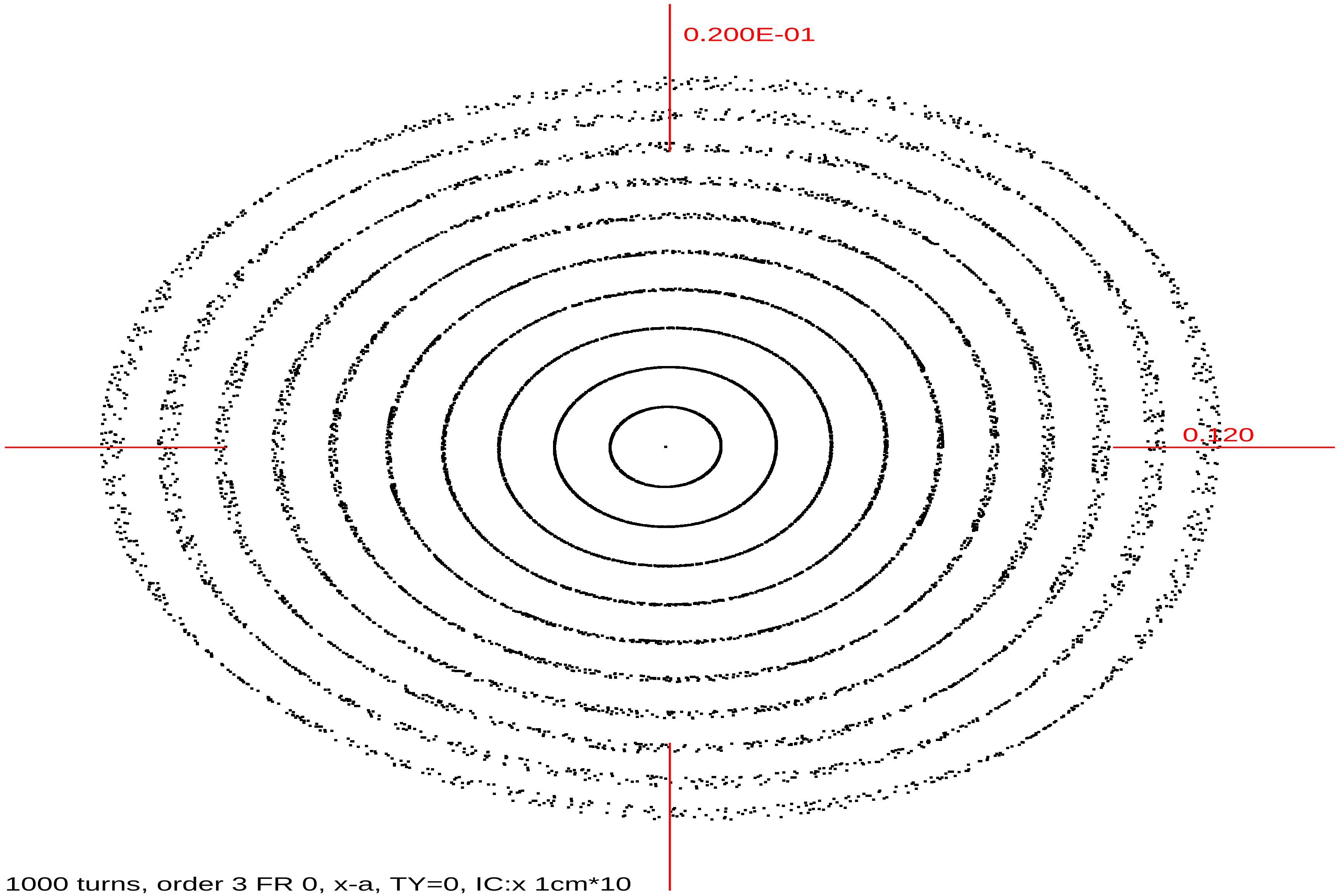
Now proceed iteratively; assume we know $\bar{n}$ to order $m-1$ and want to determine it to order $m$. Assume $\vec{\mathcal{M}}(\vec{z})$ is in normal form, i.e. $\vec{\mathcal{M}}(\vec{z}) = \mathcal{R} + \mathcal{N}$. Write $\hat{A} = \hat{A}_0 + \hat{A}_{\geq 1}$, $\bar{n} = \bar{n}_{<m} + \bar{n}_m$, and obtain

$$(\hat{A}_0 + \hat{A}_{\geq 1}) \cdot (\bar{n}_{<m} + \bar{n}_m) = (\bar{n}_{<m} + \bar{n}_m) \circ (\mathcal{R} + \mathcal{N})$$

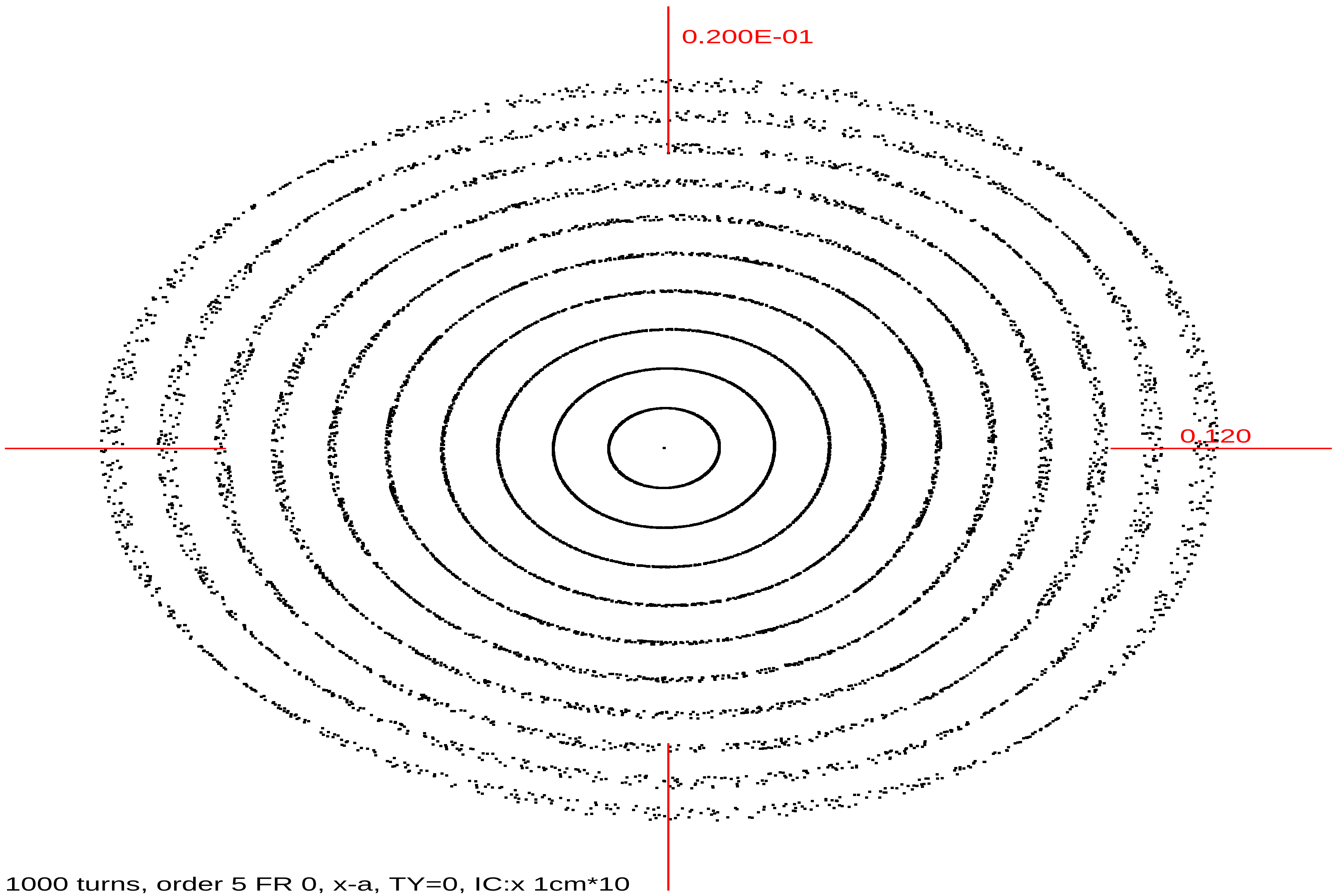
To order $m$, this can be rewritten

$$\begin{gathered} \hat{A} \cdot \bar{n}_{<m} + \hat{A}_0 \cdot \vec{n}_m =_m \bar{n}_{<m} \circ (\mathcal{R} + \mathcal{N}) + \bar{n}_m \circ R, \text{ or} \\ \hat{A}_0 \bar{n}_m - \bar{n}_m \circ \mathcal{R} =_m \bar{n}_{<m} \circ (\mathcal{R} + \mathcal{N}) - \hat{A} \cdot \vec{n}_{<m}. \end{gathered}$$

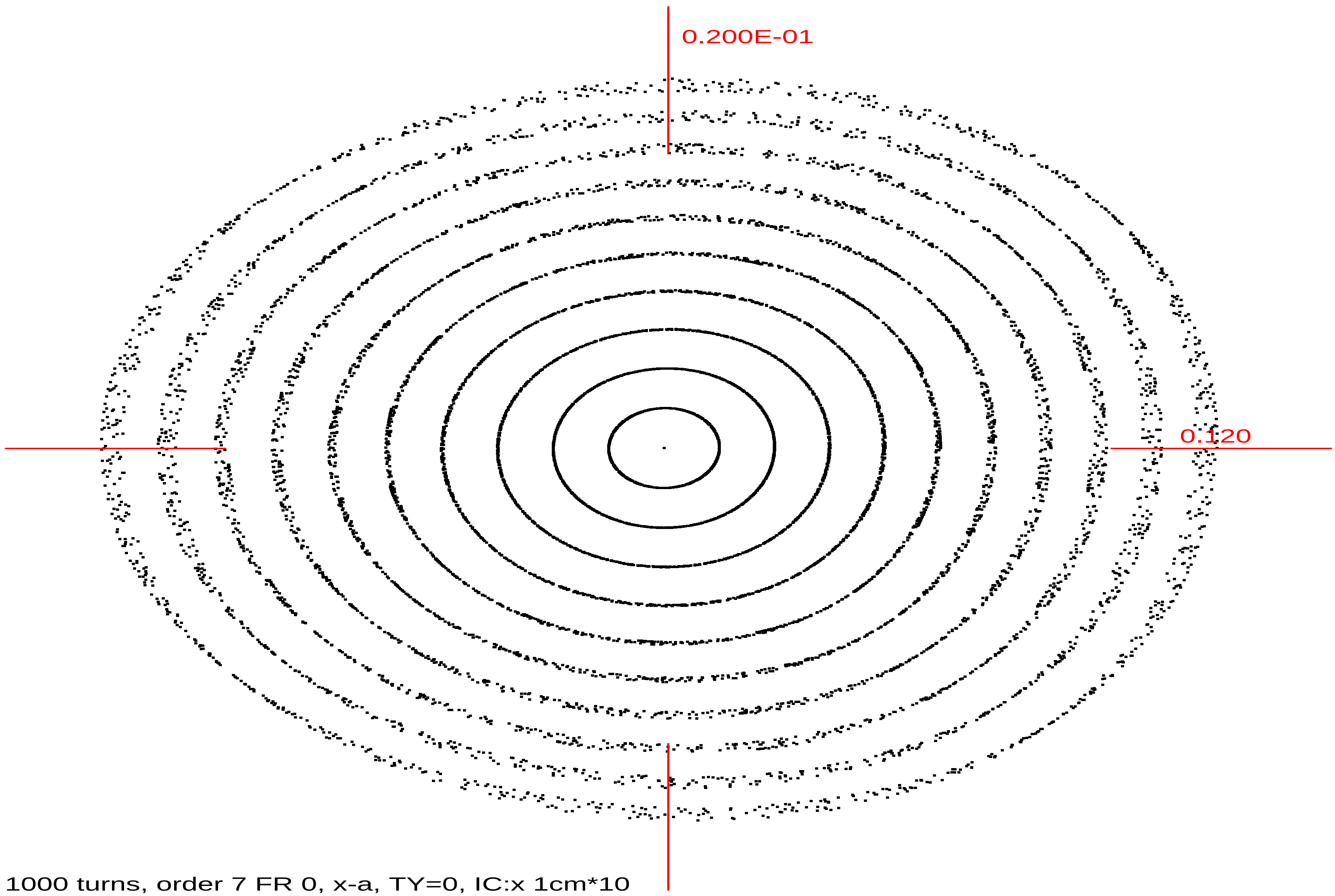
The right hand side has to be balanced by choosing $\bar{n}_m$ appropriately. But like in orbit case, the coefficients of $\hat{A}_0 \vec{n}_m - \vec{n}_m \circ \mathcal{R}$ differ from those of $\bar{n}_m$ only by resonance denominators.



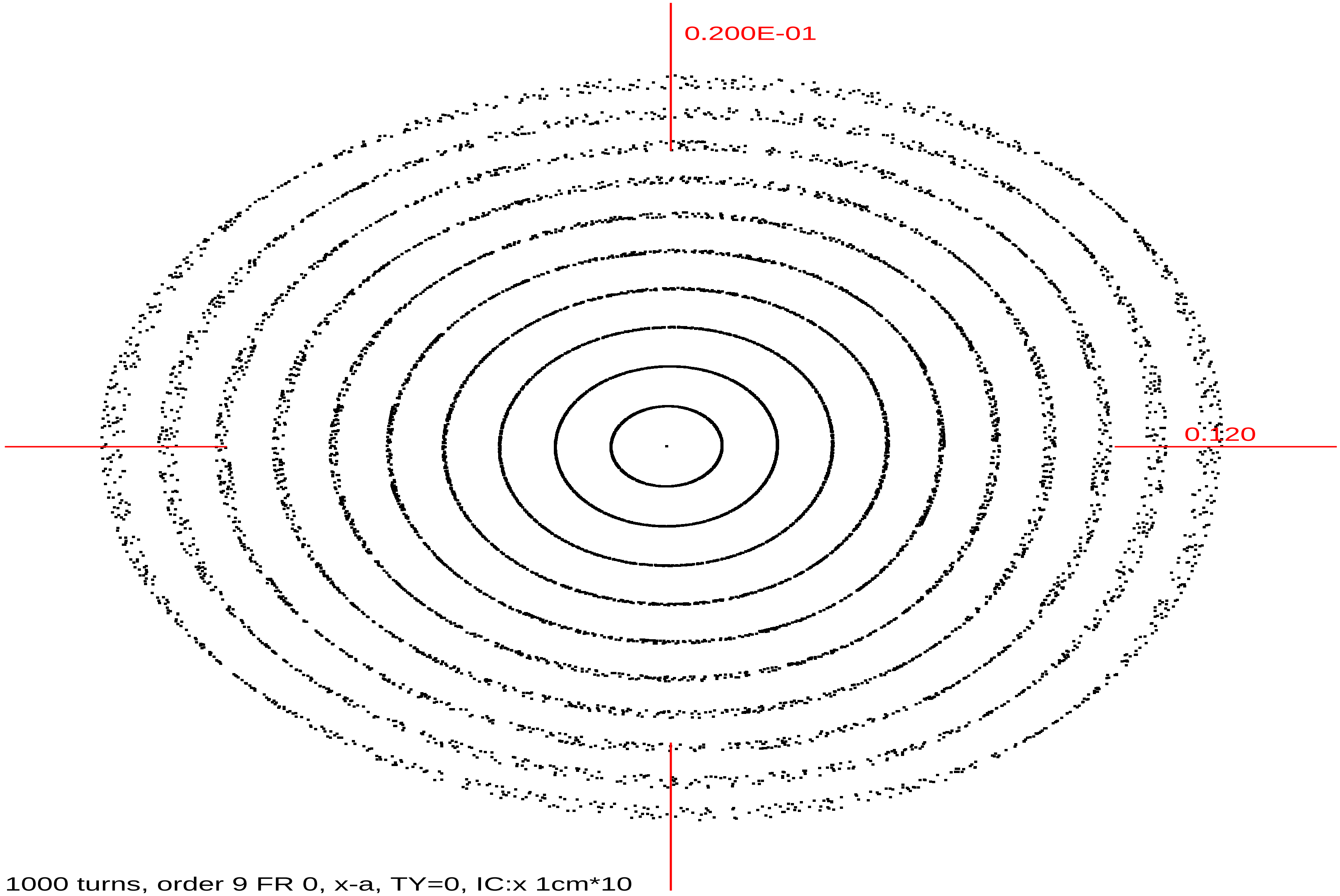

1000 turns, order 3 FR 0, x-a, TY=0, IC:x 1cm*10



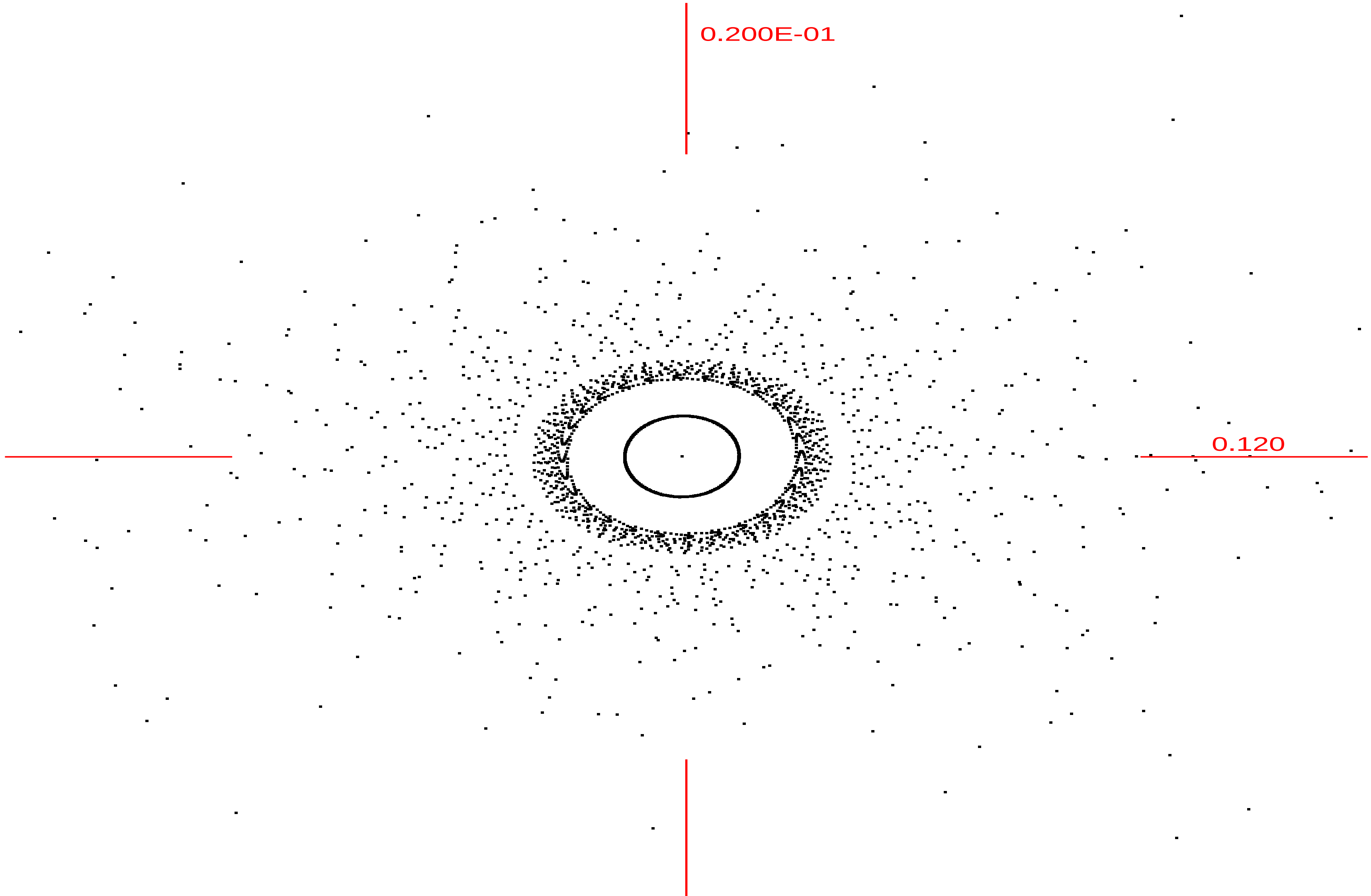

1000 turns, order 5 FR 0, x-a, TY=0, IC:x 1cm*10



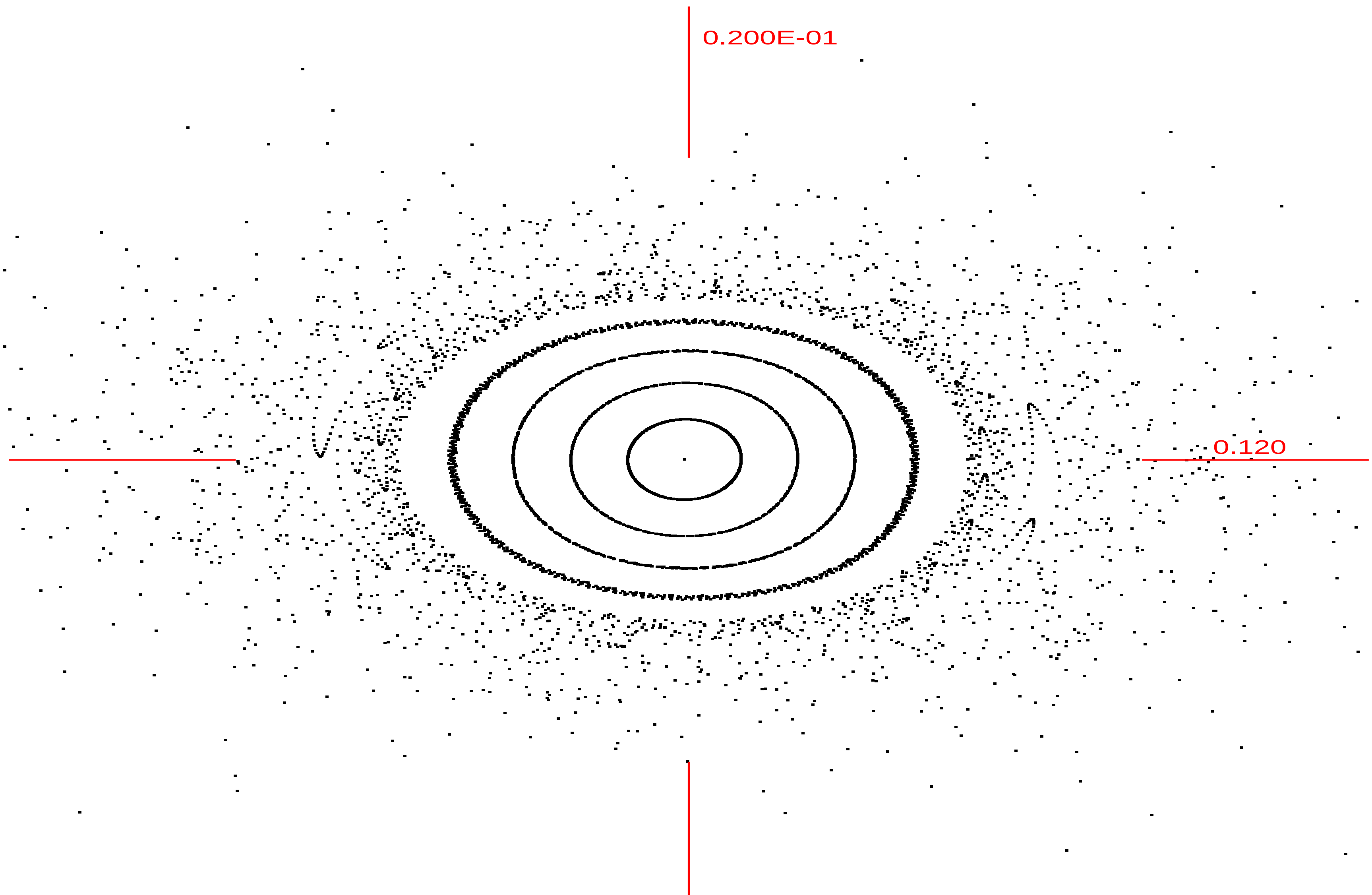

1000 turns, order 7 FR 0, x-a, TY=0, IC:x 1cm*10



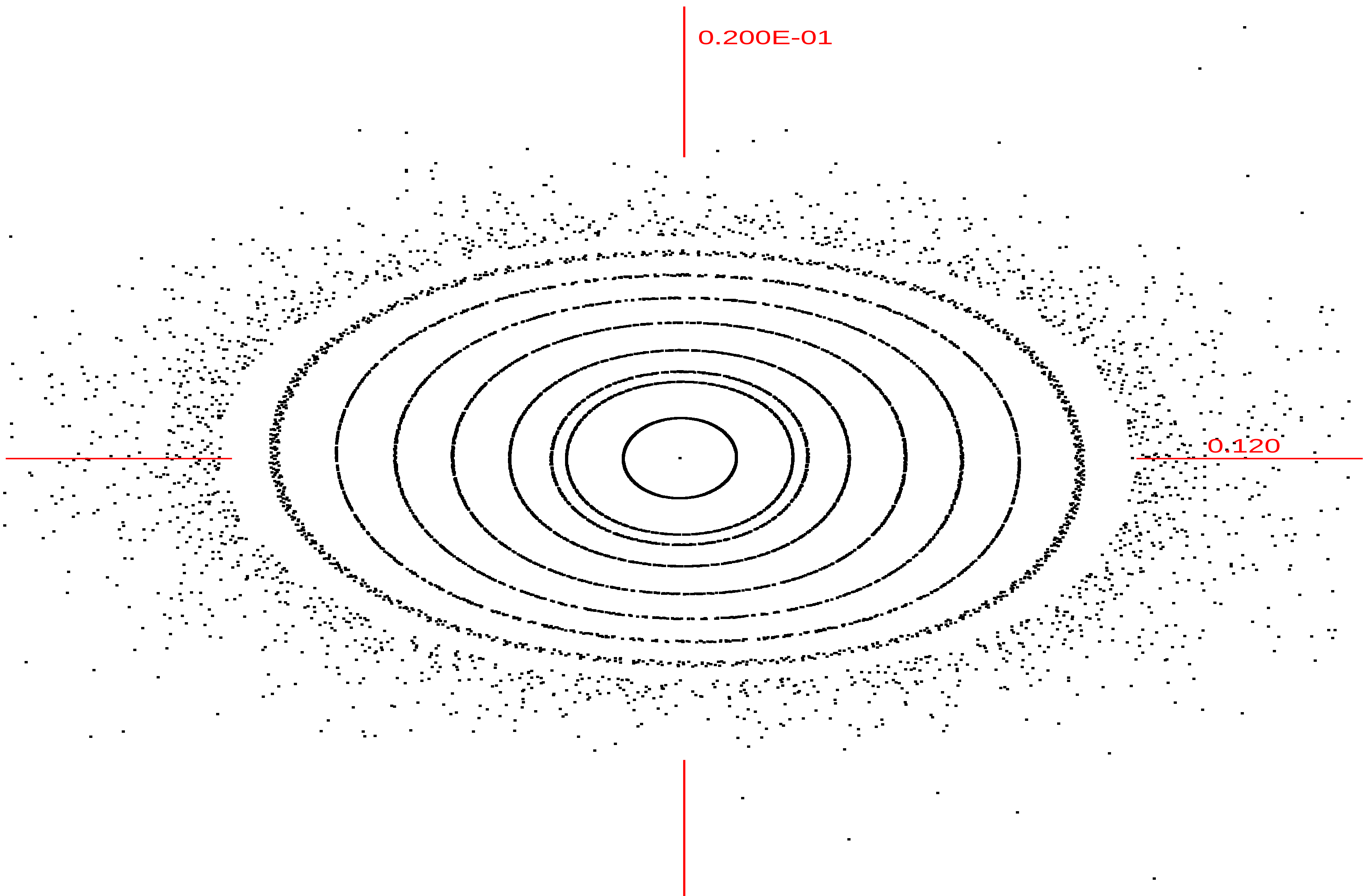

1000 turns, order 9 FR 0, x-a, TY=0, IC:x 1cm*10



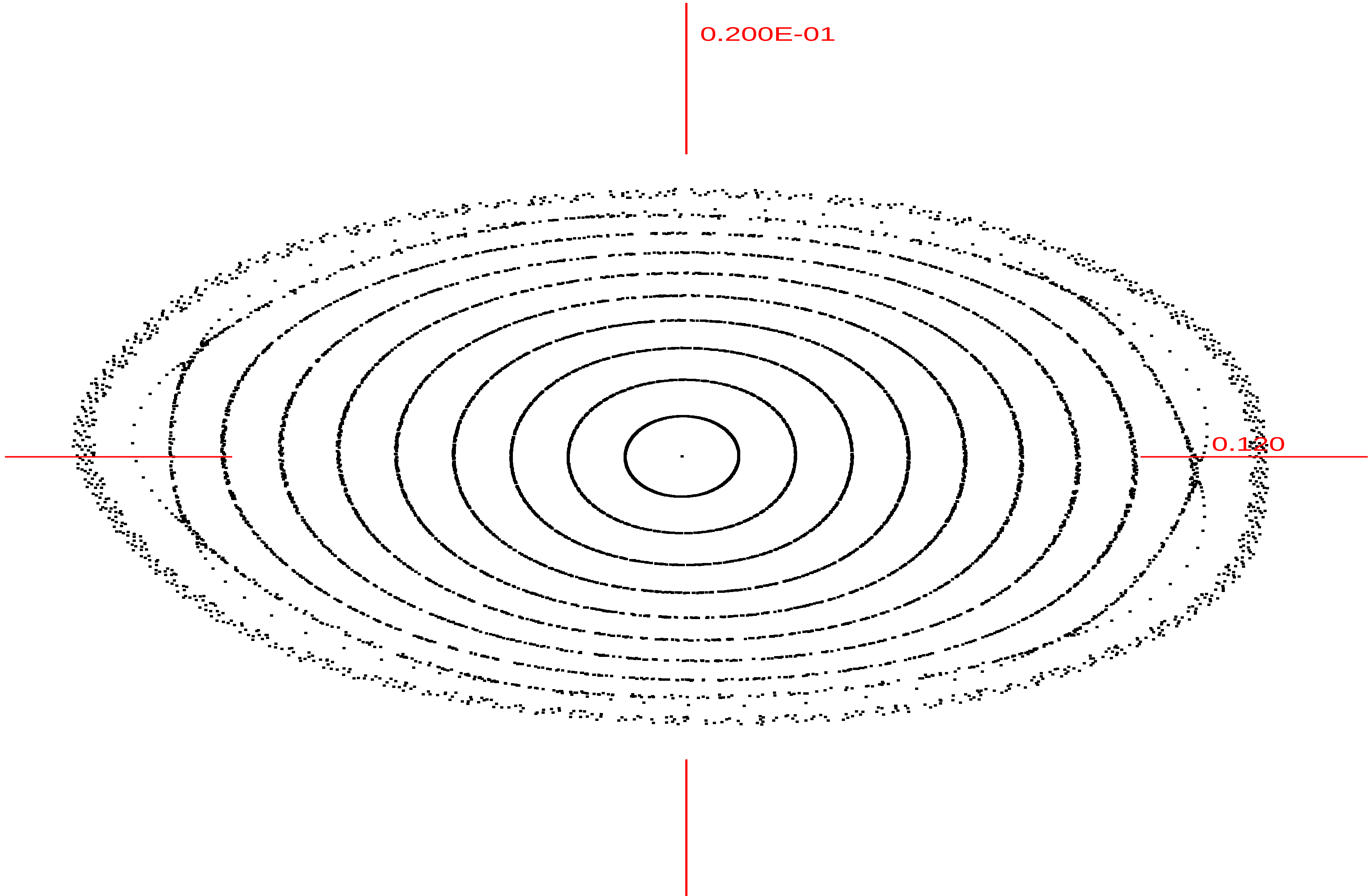

1000 turns, order 3 FR 1.9, x-a, TY=-21, IC:x 1cm*10



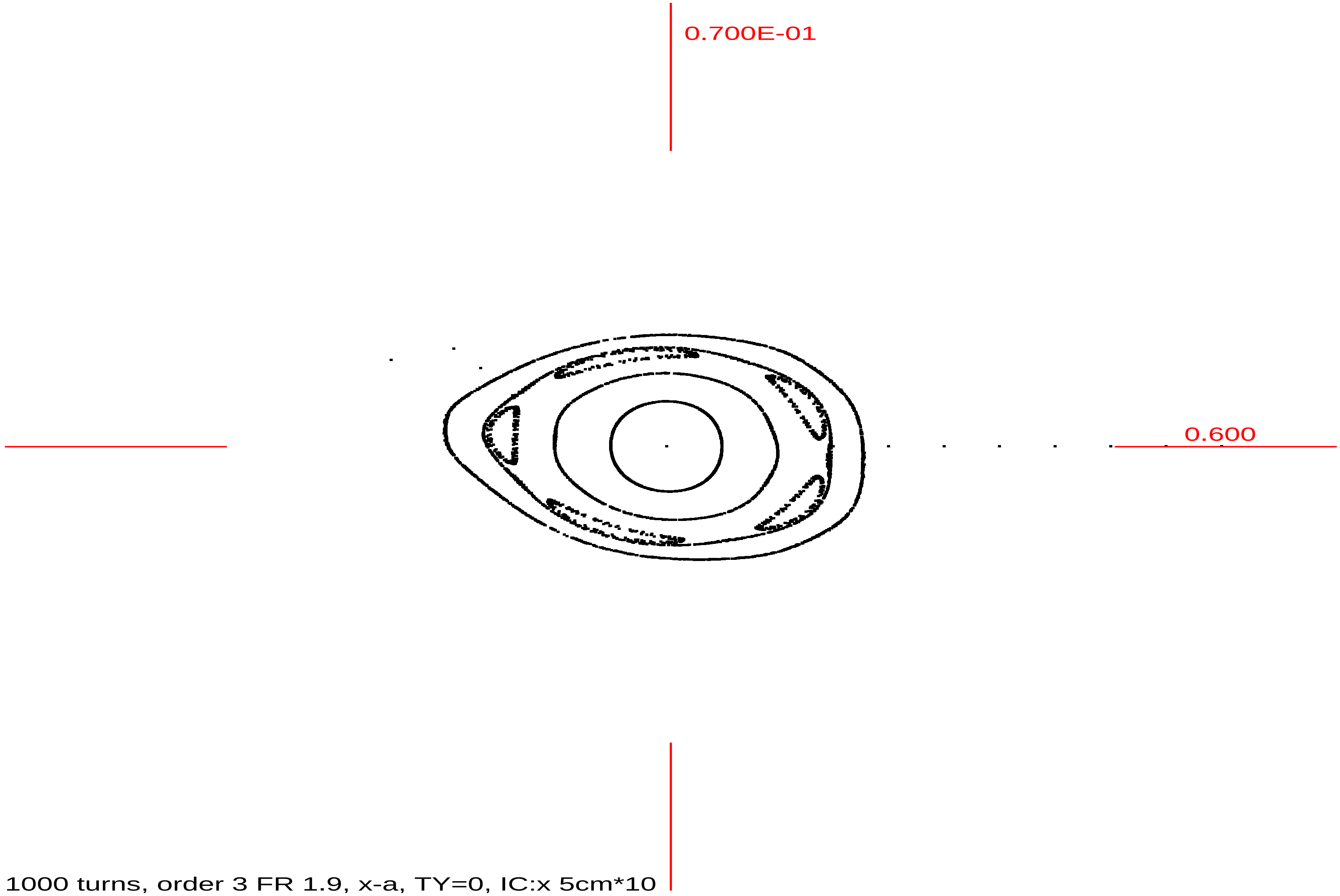

1000 turns, order 5 FR 1.9, x-a, TY=-21, IC:x 1cm*10




1000 turns, order 7 FR 1.9, x-a, TY=-21, IC:x 1cm*10




1000 turns, order 9 FR 1.9, x-a, TY=-21, IC:x 1cm*10

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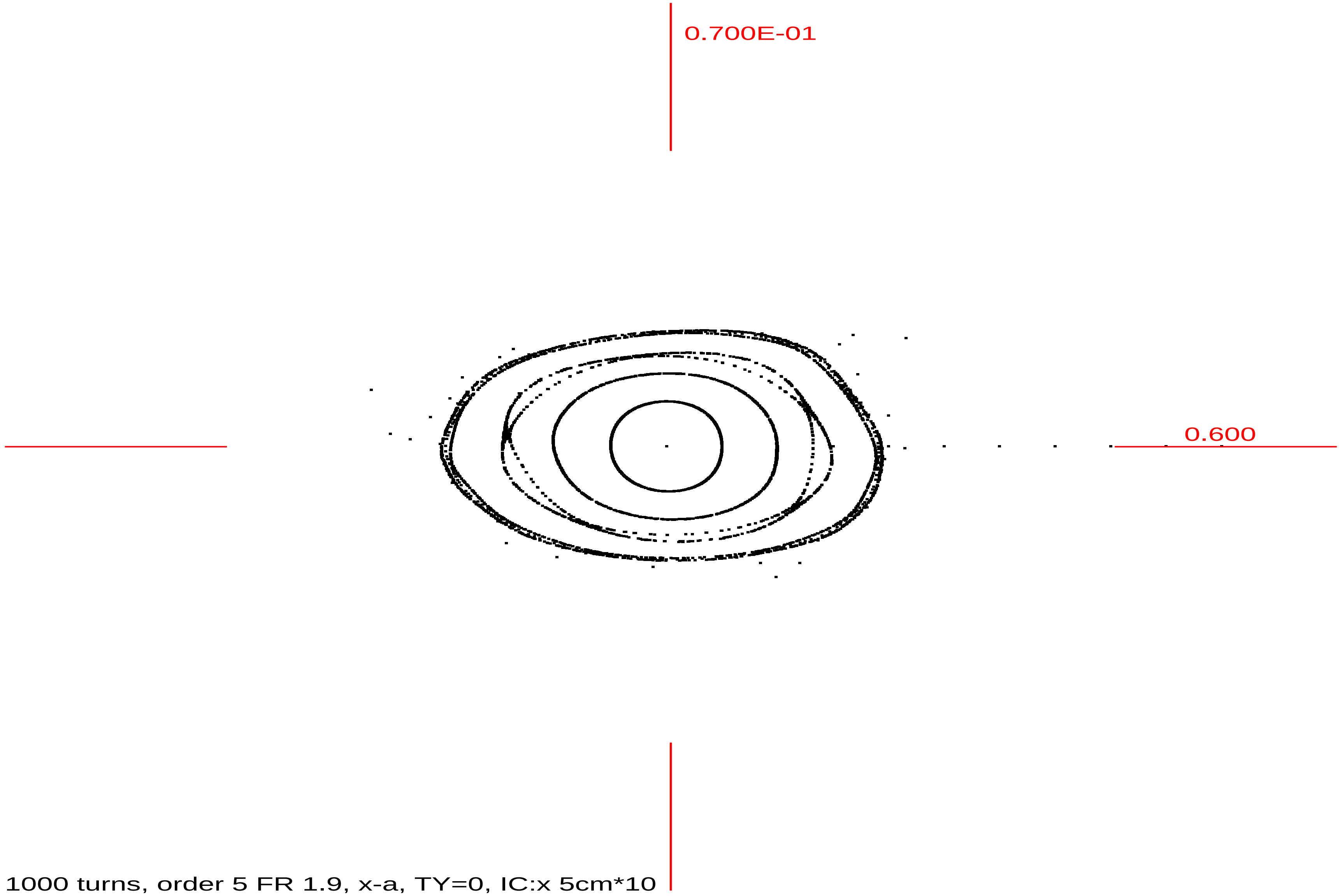
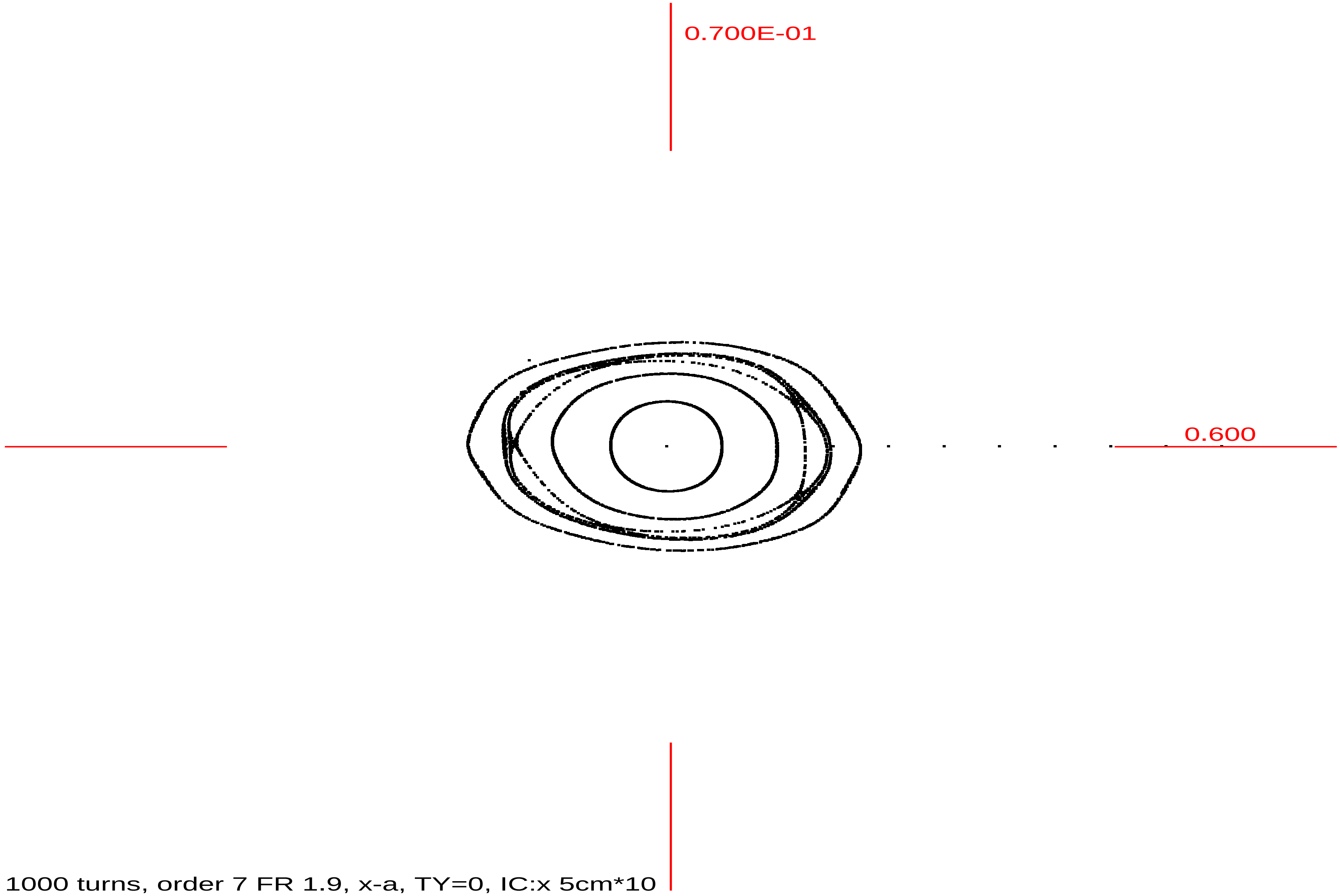
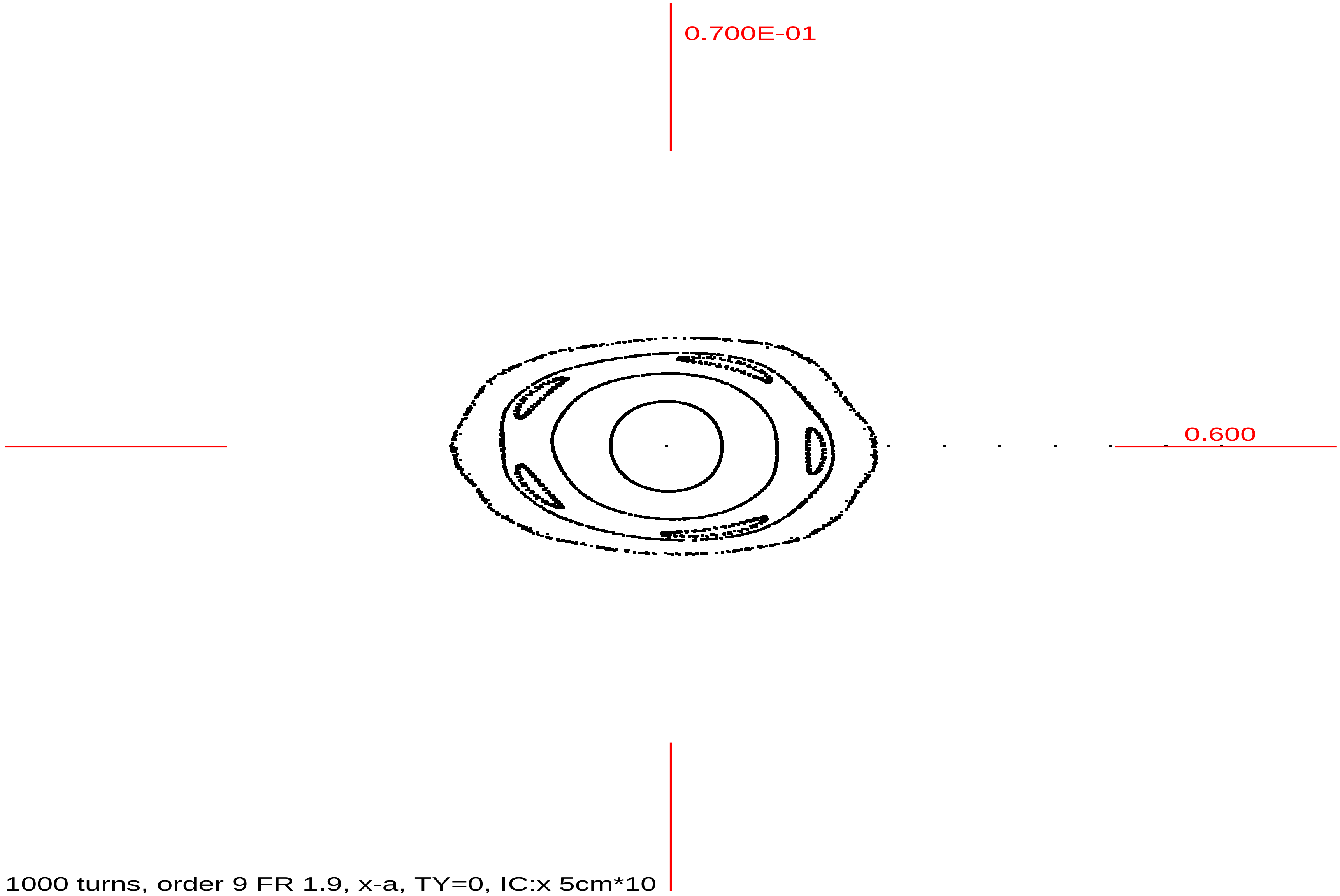



1000 turns, order 5 FR 1.9, x-a, TY=0, IC:x 5cm*10




1000 turns, order 7 FR 1.9, x-a, TY=0, IC:x 5cm*10




1000 turns, order 9 FR 1.9, x-a, TY=0, IC:x 5cm*10