Academic Lecture: Lensing of the Cosmic Microwave Background

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I. DEFLECTION OF LIGHT

- Point mass

$$\delta\theta = \frac{4GM}{Rc^2} \tag{1}$$

- More generally define

$$\Phi(\vec{\theta}) \equiv \frac{2}{D_s}\int_0^{D_s} dD\, \phi(x^i = D\theta^i, D; t(D))\, \frac{D_s - D}{D} \tag{2}$$

and then

$$\delta\theta_i \equiv \alpha_i = \frac{\partial\Phi}{\partial\theta^i}. \tag{3}$$

- Deflection angle for light traveling from very far away is of order a few arc minutes, but induced by coherent structures of order a degree

- We cannot observe $\delta\theta$, but are sensitive to its derivatives

$$\Psi_{ij} \equiv \frac{\partial^2\Phi}{\partial\theta^i\partial\theta^j} = \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix}. \tag{4}$$

- Measure the components of the distortion tensor and infer the projected gravitational potential

- Power spectra

$$\langle \tilde{\Phi}(\vec{l})\tilde{\Phi}^*(\vec{l})\rangle = (2\pi)^3\, \delta^3(\vec{l} = \vec{l}')P_\Phi(l) \tag{5}$$

II. GRAVITATIONAL POTENTIALS IN THE UNIVERSE

- Balance between gravitational accretion and expansion

- Broken if some of energy does not cluster: e.g. dark energy

- Neutrinos

- Production Rate $\Gamma \sim n_e \alpha^2 T^2/m_W^4 \sim \alpha^2 T^5/m_W^4$
- Compare to expansion rate $H = \sqrt{8\pi G\rho/3} \propto T^2/m_{\rm Pl}$
- At high T, neutrinos in equilibrium, so they are produced even if not present initially. They drop out of equilibrium at $T \sim (m_W^4/\alpha^2 m_{\rm Pl})^{1/3} \sim 5\,MeV$
- But they maintain a distribution associated to massless fermion; just that it gets shifted (cooled) as the universe expands
- Calibrate the number of neutrinos off the number of photons: $n_\nu = 112$ cm^{-3} per generation
- Their contribution to the energy density is just the sum of the rest masses times this number density, so

$$f_\nu \equiv \frac{\rho_\nu}{\rho_m} \simeq 10^{-2} \frac{\sum m_\nu}{0.1\,{\rm eV}} \tag{6}$$

- Since this fraction doesn't clump for most of the history of the universe on small scales, the spectrum is suppressed by more than a percent (about 5% for the projected potential spectrum)

III. CMB REVIEW: TEMPERATURE ANISOTROPY

- Acoustic Peaks, mode by mode

$$\ddot{T} + k^2 c_s^2 T = F[\phi] \tag{7}$$

- Projected on to the sky in C_l's
- Gaussian, so two-point function captures (almost) everything

IV. CMB REVIEW: POLARIZATION

- Generation from Quadrupole
- Out of Phase with Monopole
- E-B Decomposition: E-modes only

V. SMOOTHING

$$
\begin{aligned}
T'(\vec{\theta}) &= T(\vec{\theta} - \vec{\alpha}) \\
&\simeq T(\vec{\theta}) - \frac{\partial T}{\partial \theta_i}\,\alpha_i + \frac{1}{2}\,\frac{\partial^2 T}{\partial\theta_i \partial\theta_j}\,\alpha_i\alpha_j
\end{aligned} \tag{8}
$$

Fourier transform:

$$
\begin{aligned}
\tilde{T}'(\vec{l}) &= \tilde{T}(\vec{l}) + \int \frac{d^2 l_1}{(2\pi)^2}\,\vec{l}_1 \cdot (\vec{l} - \vec{l}_1)\tilde{T}(\vec{l}_1)\,\tilde{\Phi}(\vec{l} - \vec{l}_1) \\
&+ \frac{1}{2}\int \frac{d^2 l_1}{(2\pi)^2}\,\tilde{T}(\vec{l}_1)\int \frac{d^2 l_2}{(2\pi)^2}\,(\vec{l}_1 \cdot \vec{l}_2)\,\vec{l}_1 \cdot (\vec{l} - \vec{l}_1 - \vec{l}_2)\tilde{\Phi}(\vec{l}_2)\tilde{\Phi}(\vec{l} - \vec{l}_1 - \vec{l}_2) \\
&.
\end{aligned} \tag{9}
$$

$$
C_l' = C_l + l^2 R\left(\bar{C}_l - C_l\right) \tag{10}
$$

with

$$
R \equiv \int \frac{d^2 l_1}{(2\pi)^2}\,(\vec{l} \cdot \vec{l}_1)^2 C^{\Phi}(l_1). \tag{11}
$$

Physically, photons from a hot spot are spread out so the hot spot appears less hot, and similarly the cold spot appears less cold.

VI. QUADRATIC ESTIMATOR FOR κ

- Non-identical Fourier modes are now coupled

$$
\begin{aligned}
\langle \tilde{T}(\vec{l})\tilde{T}(\vec{l}')\rangle &= 2\int \frac{d^2 l_1}{(2\pi)^2}\,\vec{l}_1 \cdot (\vec{l} - \vec{l}_1)\tilde{\Phi}(\vec{l} - \vec{l}_1)\langle \tilde{T}(\vec{l}')\tilde{T}(\vec{l}_1)\,\rangle \\
&= 2C_{l'}\vec{l}' \cdot (\vec{l} + \vec{l}')\tilde{\Phi}(\vec{l} + \vec{l}').
\end{aligned} \tag{12}
$$

  There is an "optimal" way to weight all these modes to get an estimator for $\tilde{\Phi}$.

- Another way to think of this is the power spectrum in a small patch is larger than normal if κ is positive. The patch gets projected to a larger angular size $\theta \to \theta(1+\kappa)$. Therefore, its conjugate $l \to l(1-\kappa)$. So

$$
C_l \to C_{l[1-\kappa]} \simeq C_l - \kappa \frac{\partial C_l}{\partial \ln l}. \tag{13}
$$

  So, κ is larger in regions with large C_l.

VII. LENSING B-MODES

Appendix A: Some cosmological facts

- Mpc $= 3 \times 10^{24}$ cm
- FRW metric: $ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2$
- Hubble expansion rate: $H = \dot{a}/a$
- Hubble rate today is called the *Hubble constant* and is measured to be about 70 km/sec/Mpc, sometimes written as $H_0 = 100h$ km/sec/Mpc
- Friedmann equation: $H^2 = 8\pi G\rho/3$ where ρ counts all the contributions to the energy density
- Energy census in units of critical density: Species i today contributes a fraction $\Omega_i \equiv \rho_i/\rho_{\rm cr}$, with $\rho_{\rm cr} \equiv 3H_0^2/(8\pi G) = 4 \times 10^{-11}\,{\rm eV}^4$
- Photon density: $\Omega_\gamma = 1.2 \times 10^{-5}$
- Baryon density: $\Omega_b = 0.05$
- Matter density: $\Omega_m = 0.26$
- Total density: $\Omega = 1$
- $a_{\rm EQ} = 4.15 \times 10^{-5}/(\Omega_m h^2) = 2.91 \times 10^{-4}$