Muon $(g-2)$: State of the Theoretical Art

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Fermilab Academic Lecture Series

Feel Like a Number? $a = \mu\frac{2m}{e\hbar} - 1 = \frac{1}{2}(g-2)$

$$10^{14} a_e = \begin{array}{ll} 115\,965\,218\,073(28) & \text{expt} \\ 115\,965\,218\,178(76) & \text{SM with } \alpha \text{ from } {}^{87}\mathrm{Rb} \end{array}$$

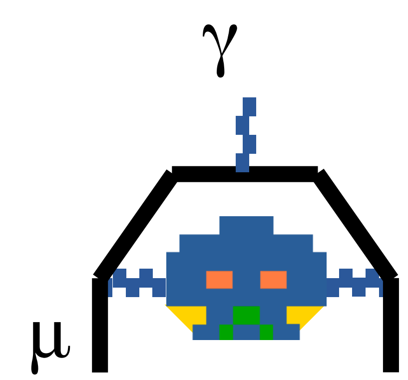
$\Rightarrow$ better α

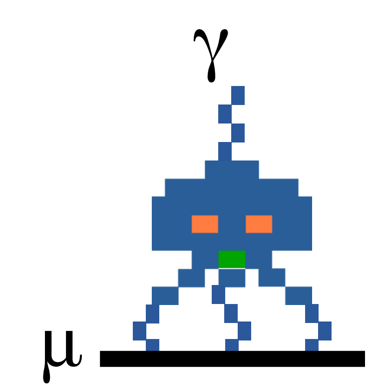
$$10^{11} a_\mu = \begin{array}{ll} 116\,59\boxed{2\,089(63)} & \text{expt} \\ 116\,59\boxed{1\,802(49)} & \text{SM with HVP from } e^+e- \end{array}$$

$$10^8 a_\tau = \begin{array}{ll} \text{between } -5\,200\,000 \text{ and } 1\,300\,000 & \text{expt} \\ 117\,721(5) & \text{SM} \end{array}$$

Outline

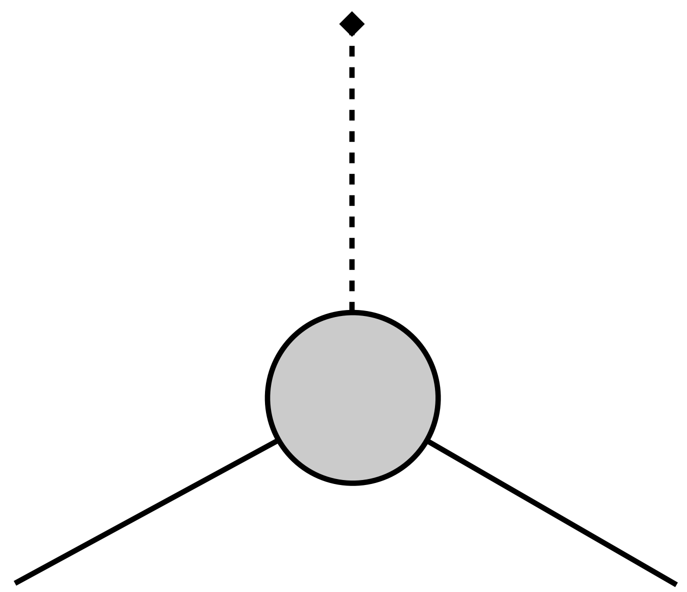
- Recap from last week: $(g-2)/2$ in quantum field theory.
- QED-EW & QED-BSM contributions to $(g-2)/2$:
  - on the one hand, the discrepancy is evidence for susy; yet, on the other, ...
  - … the agreement provides a strong constraint on susy [Bechtle *et al.*, arXiv:0907.2589].
- QED-QCD contributions to $(g-2)/2$:
  - hadronic vacuum polarization;
  - hadronic "light-by-light" scattering.





Magnetic Moments in QED (+ EW + BSM)

Electromagnetic Vertex



$$= e_R \bar{u}(p') \left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2) \right] u(p)$$

- Static quantities—electric charge and magnetic moment—obtained as $q \to 0$.
- Magnetic moment $\mu = \frac{e\hbar}{2m} 2\left[F_1(0) + F_2(0)\right]$.
- By definition of e_R, $F_1(0) = 1$.
- So $a = F_2(0)$: as Prateek discussed, algebraically intensive methods can be automated.

Four-loop QED



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Eighth-order QED contribution to the anomalous magnetic moment of the muon

T. Kinoshita, B. Nizic,* and Y. Okamoto†

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(Received 27 September 1989)

We report a calculation of the eighth-order QED contribution to the muon anomalous magnetic moment $a_\mu^{(8)}$ coming from 469 Feynman diagrams, all of which contain electron loops of vacuum-polarization type and/or light-by-light scattering type. Our result is $126.92(41)(\alpha/\pi)^4$. The error represents the estimated accuracy (90% confidence limit) of the required numerical integration. We also report an estimate of the tenth-order contribution to a_μ. Combining these with the lower-order results and the latest theoretical value for the electron anomaly a_e, we find that the QED contribution to the muon anomaly is given by $a_\mu^{\mathrm{QED}} = 1\,165\,846\,947(46)(28)\times 10^{-12}$, where the first error is an estimate of theoretical uncertainty and the second reflects the measurement uncertainty in α. Including the hadronic and electroweak contributions, the best theoretical prediction for a_μ available at present is $a_\mu^{\mathrm{theory}} = 116\,591\,920(191)\times 10^{-11}$, where the error comes predominantly from the hadronic contribution.

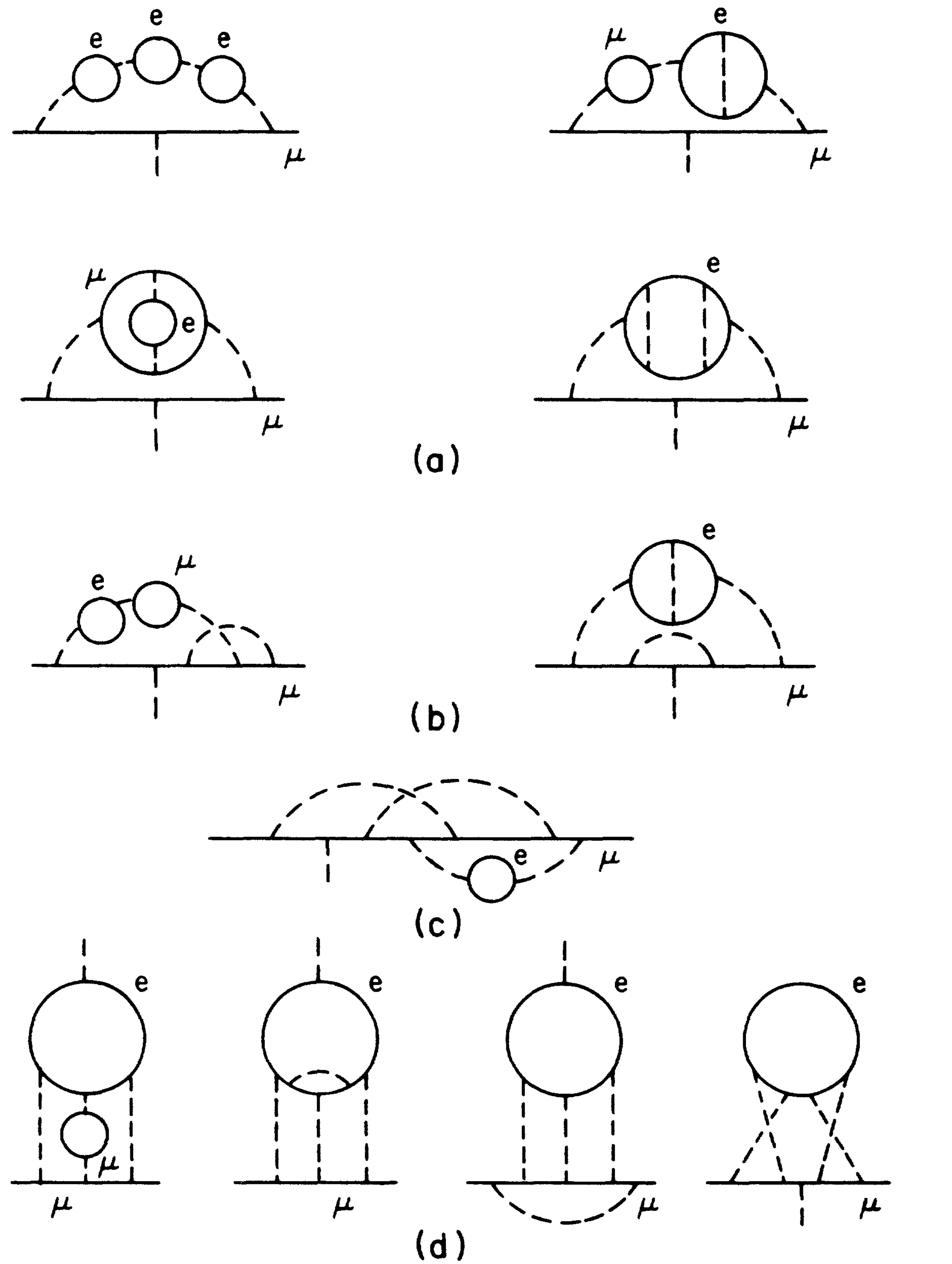


FIG. 1. Typical eighth-order vertex diagrams from the four groups contributing to a_μ.

$\mathrm{Tr}\ \gamma^{\mathrm{odd}} = 0$

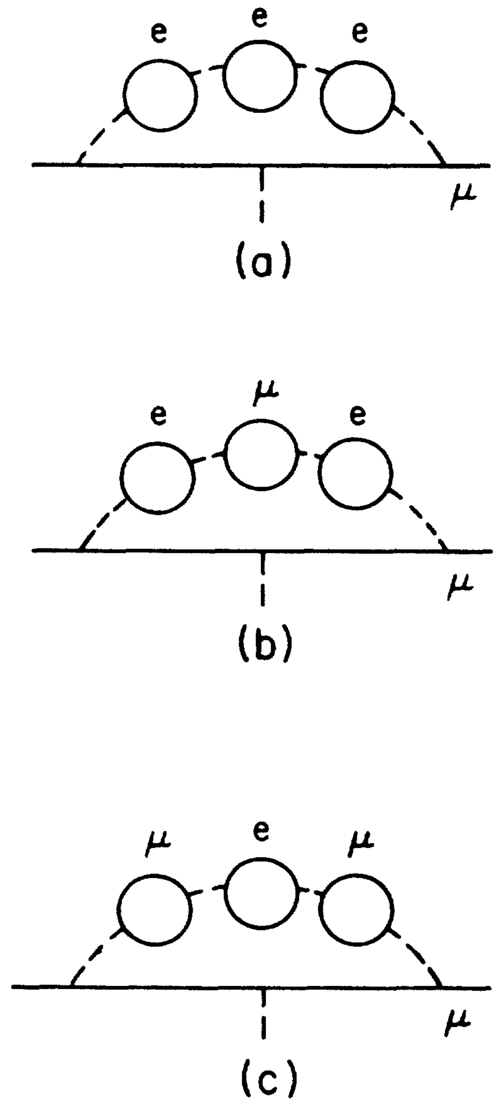


FIG. 2. Three of the diagrams contributing to subgroup I(a).

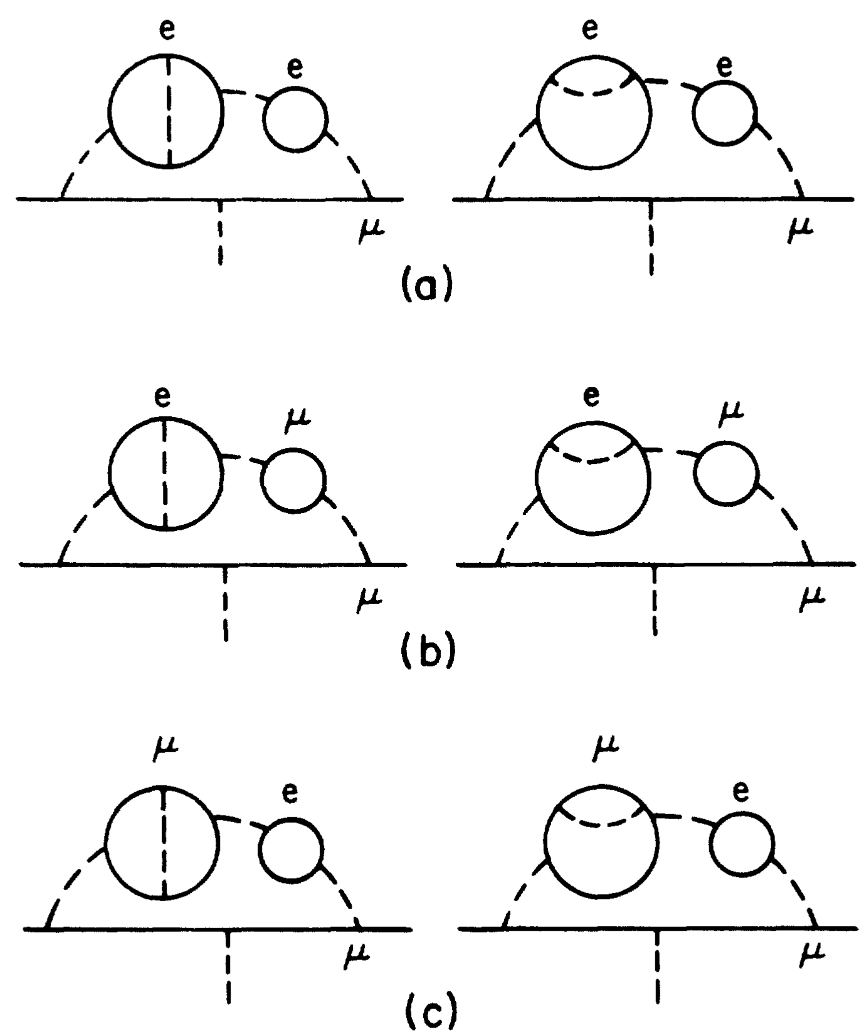


FIG. 3. Six of the diagrams contributing to subgroup I(b).

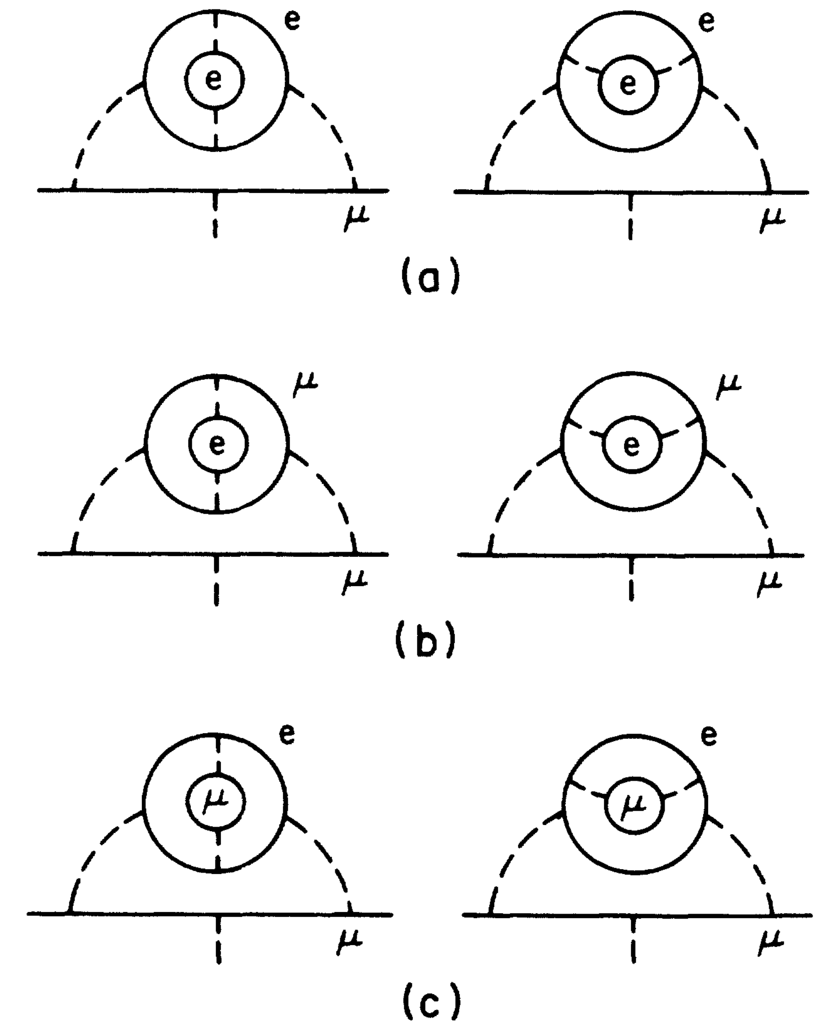


FIG. 4. Six of the diagrams contributing to subgroup I(c).

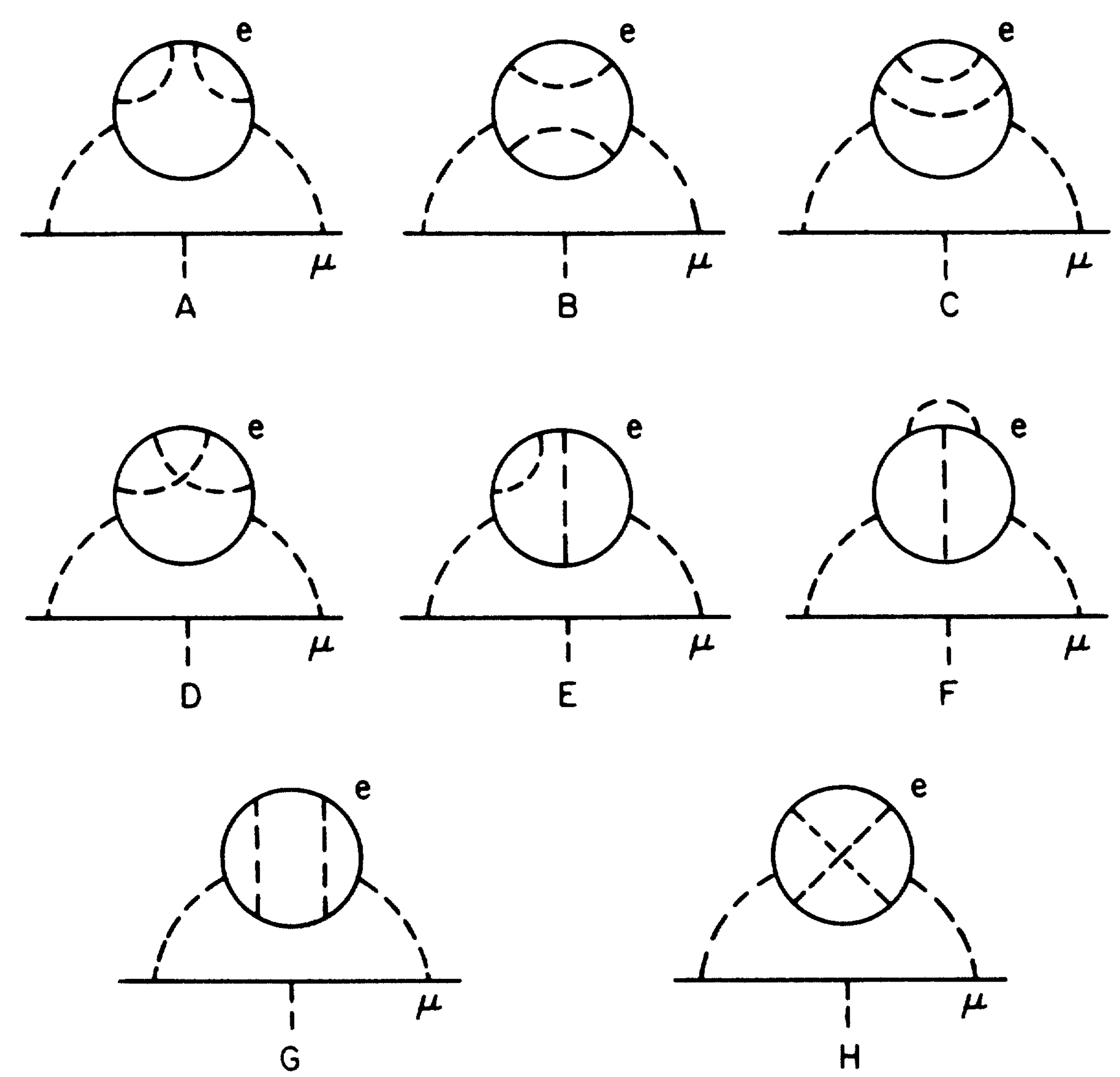


FIG. 5. Eighth-order vertices obtained by insertion of sixth-order (single electron loop) vacuum-polarization diagrams in a second-order muon vertex.

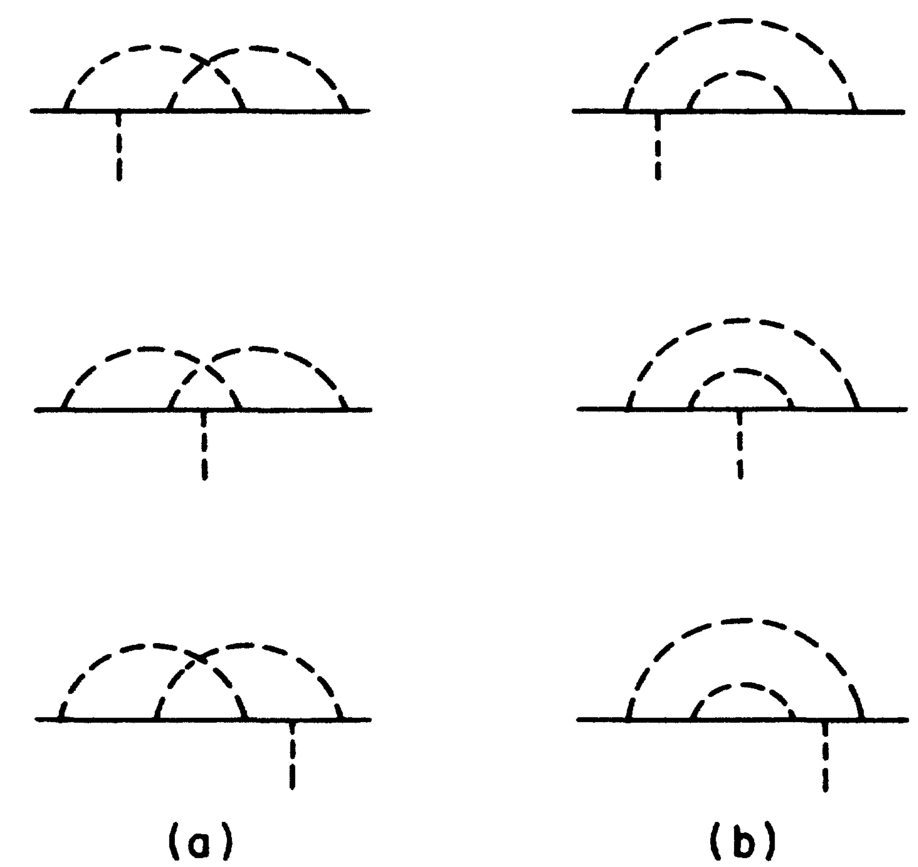


FIG. 6. (a) Fourth-order vertex diagrams with crossed photon lines. (b) Fourth-order vertex diagrams in which photon lines do not cross.

FIG. 7. Fourth-order muon self-energy diagrams containing no vacuum-polarization loops.

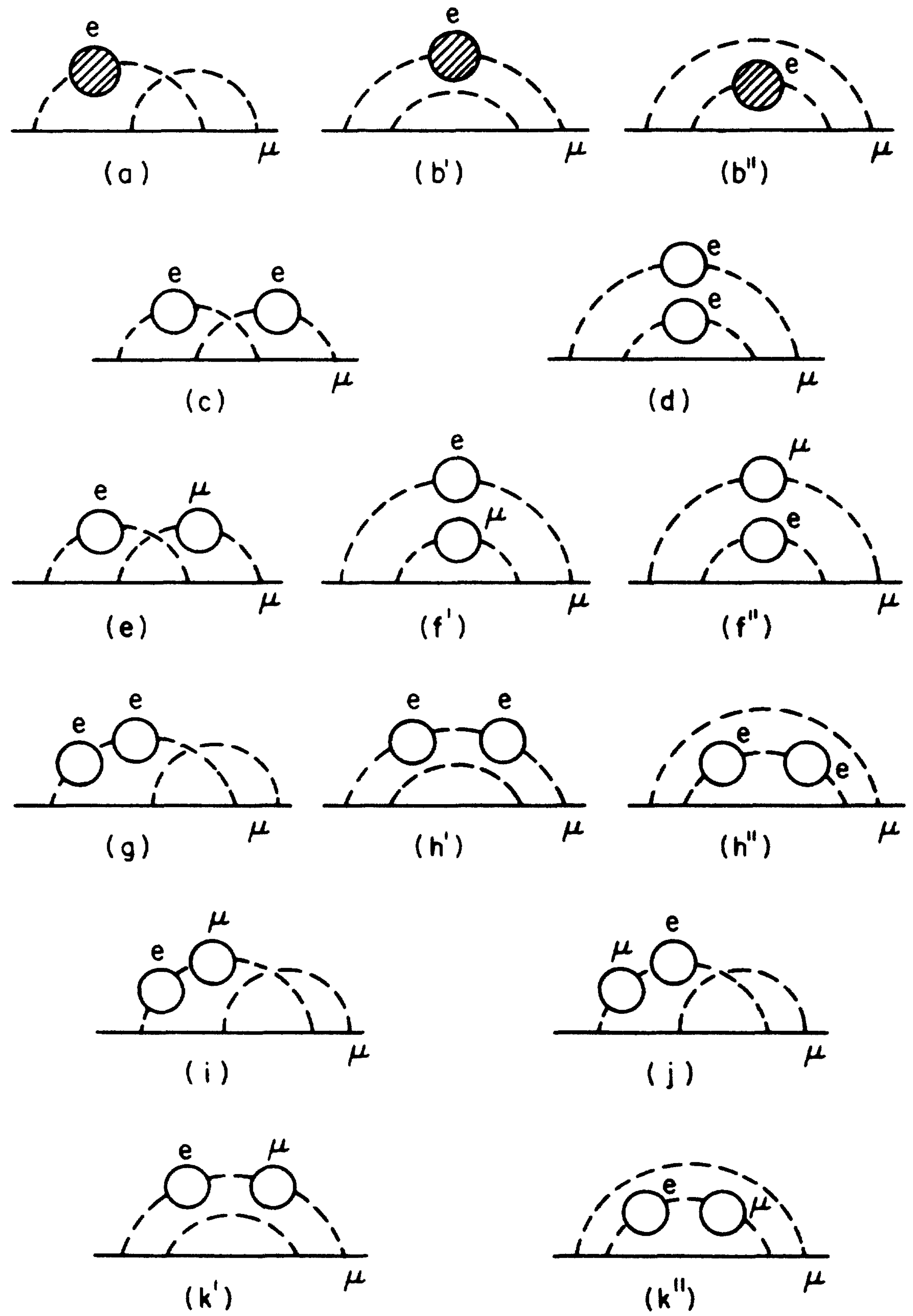


FIG. 8. Eighth-order muon self-energy diagrams obtained from the fourth-order diagrams of Fig. 7 by inserting vacuum-polarization loops. Seven more diagrams related to *a*, *e*, *g*, *i*, *j*, *k'*, and *k''* by time reversal are not shown. Shaded circles represent the sum of all fourth-order vacuum-polarization loops.

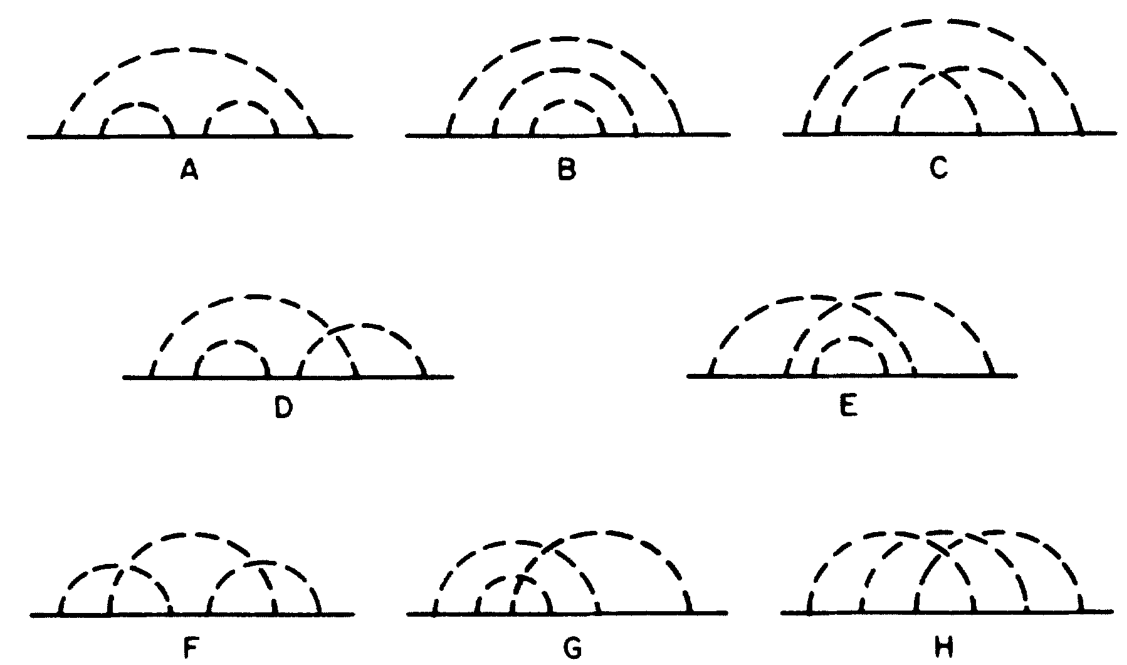


FIG. 9. Muon self-energy diagrams of the three-photon-exchange type. Two mcre diagrams related to *D* and *G* by time reversal are not shown.

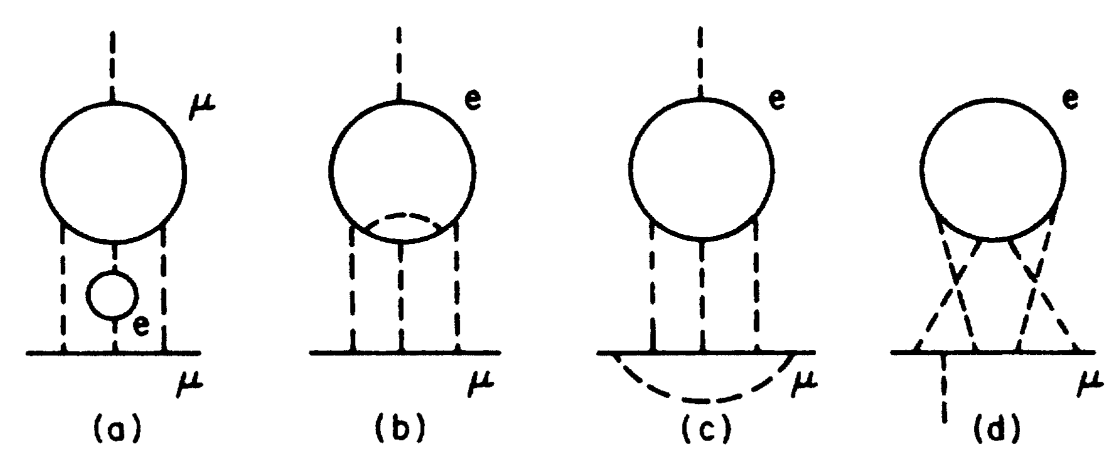


FIG. 10. Representative diagrams of each subgroup of group IV.

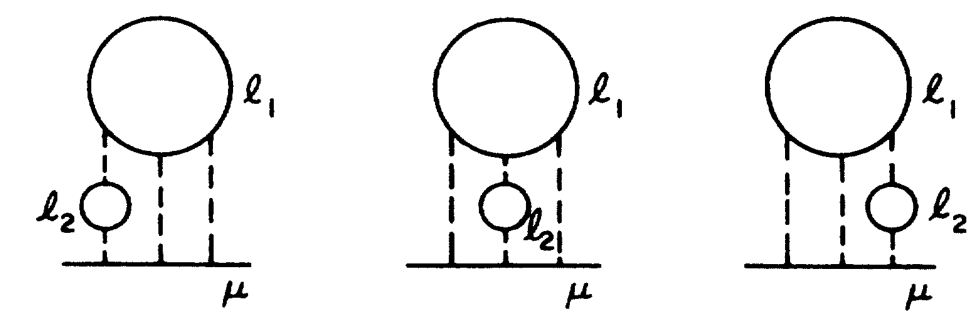


FIG. 11. Self-energy diagrams representing the external-vertex-summed integrals of subgroup IV(a).

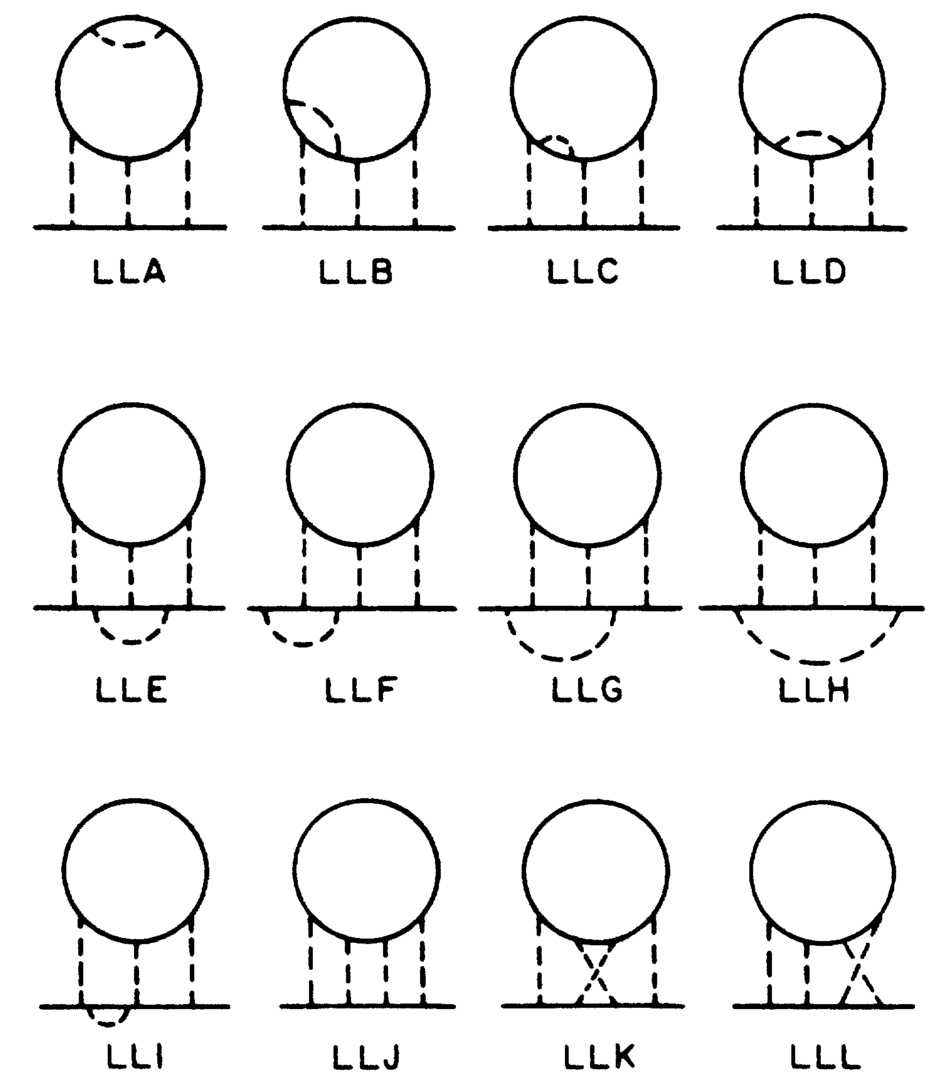


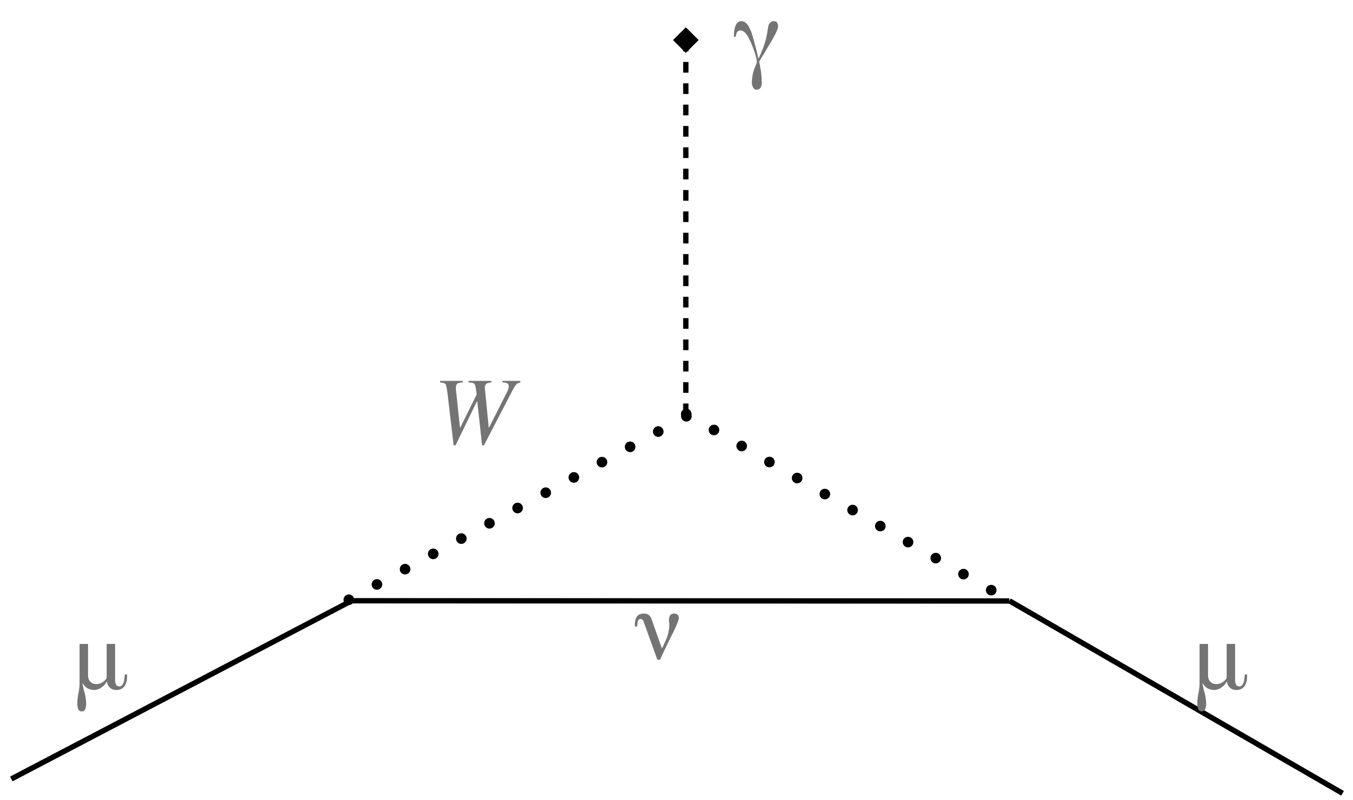
FIG. 12. Self-energy diagrams representing the external-vertex-summed integrals of subgroups IV(b), IV(c), and IV(d).

Croatian Renormalization

- Bene Nižić: It is time to go for beer!
  - Chorus: Oh! Why is it time to go for beer?
- Bene: Renormalization works the way they say it does! Four #^$@*&% loops!!!
  - Chorus: Four loops!?! Gee minus two?!?
- Bene: Yes, Yuko and I isolated all the infinities and renormalized the electric charge. The infinite pieces in the magnetic moment all canceled!!! Amazing!!! Four loops!!!
  - Chorus: It's time to go for beer!

Further Corrections

- Electroweak (to two loops, recall m^2/M^2):
    - similar diagrams with Z and H;
    - additional diagrams with Ws:



- For BSM: compute diagrams with new particles in loop (1 or 2 loops enough).

- Higher order QED at O(e^{10})—5 loops:

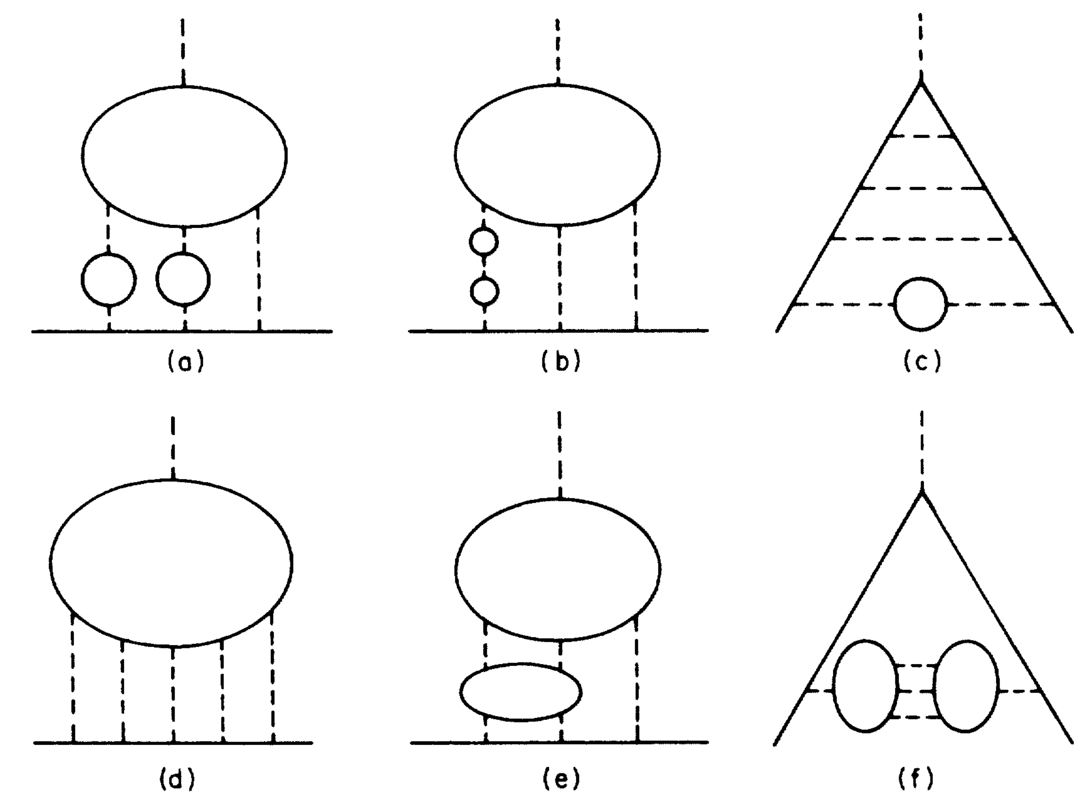


FIG. 13. Some tenth-order diagrams. (a) and (b) are generated by inserting two electron vacuum-polarization loops in a sixth-order diagram containing a light-by-light scattering subdiagram. There are 36 diagrams of these types. (c) is generated by inserting an electron vacuum-polarization loop in an electron-loop-free eighth-order diagram. There are 2072 diagrams belonging to this group. (d) contains a six-point electron loop. This group appears for the first time in the tenth order and consists of 120 diagrams. (e) and (f) contain two light-by-light scattering subdiagrams.

- Compute (a) + (b) & estimate others.

Magnetic Moments in the SM with QCD

Adding the Contributions

$$10^{11}\,\frac{\alpha}{2\pi} = 116\,140\,973.30$$

- Adding the standard-model contributions [*cf*. Andreas Höcker, arXiv:1012.0055]:

$$\begin{array}{rrl}
10^{11} a_\mu = & 116\,584\,718.09(0.15) & \text{4-loop QED} \\
+ & 194.8 & \text{1-loop EW} \\
- & 39.1(1.0) & \text{2-loop EW, with } M_H = 125\ \text{GeV} \\
+ & 6\,923(42) & \text{LO HVP from } R(e^+e^- \to \text{hadrons}) \\
- & 97.9(0.9) & \text{NLO HVP} \\
+ & 105(26) & \text{HL} \times \text{L from Glasgow consensus} \\
\hline
= & 116\,591\,804(42)(26) & \text{Total (shift for knowing Higgs mass is } +2)
\end{array}$$

- The discrepancy is enormous: in these units, 285(63)(49), while EW is only $195_{\text{1loop}} - 40_{\text{2loop}}$.

- Experiment, HVP, and HL×L all have to move 2σ to resolve the tension.

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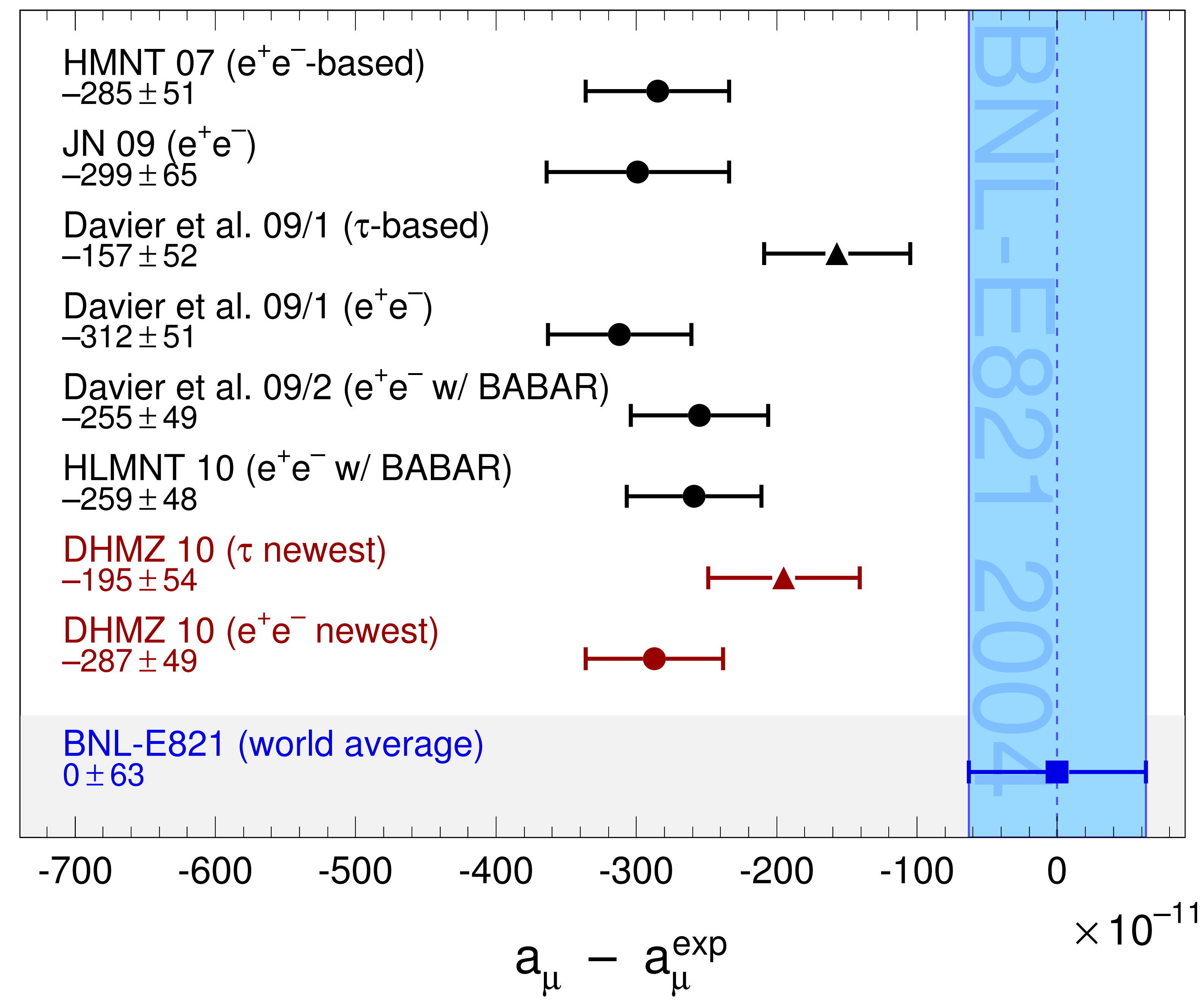
Results and Forecasts for a_μ

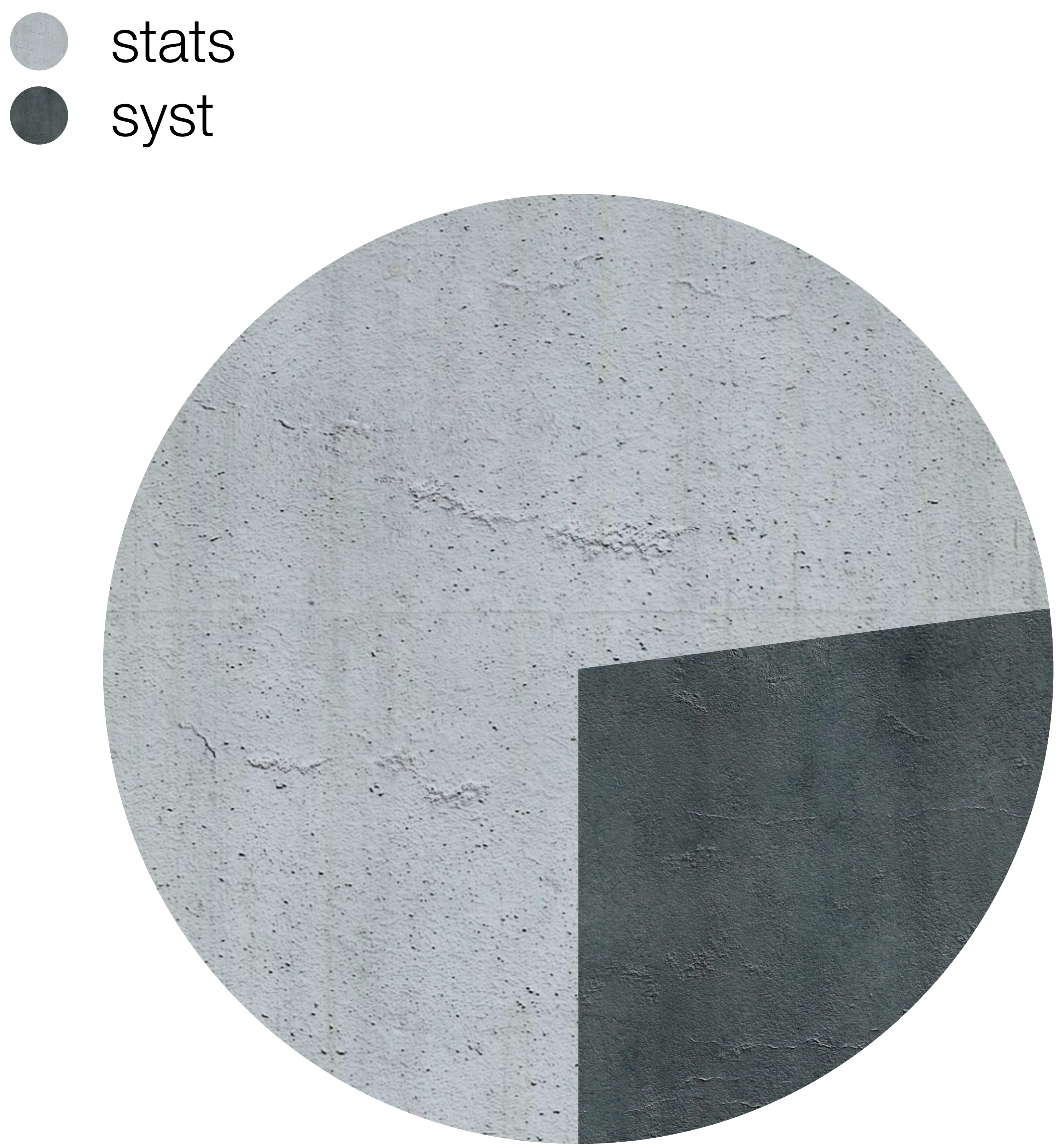
how	$10^{11}a_\mu$	$10^{11}\times$error
E821 μ^+	116 592 04–	90
E821 μ^-	116 592 15–	90
E821 $\mu^\pm$	116 592 089	63
SM(τ)	116 591 894	54
SM(e^+e^-)	116 591 802	49
HVP (lo)	6 923	42
HL×L	105	26
E989 μ^+	116 59– ——	**16**



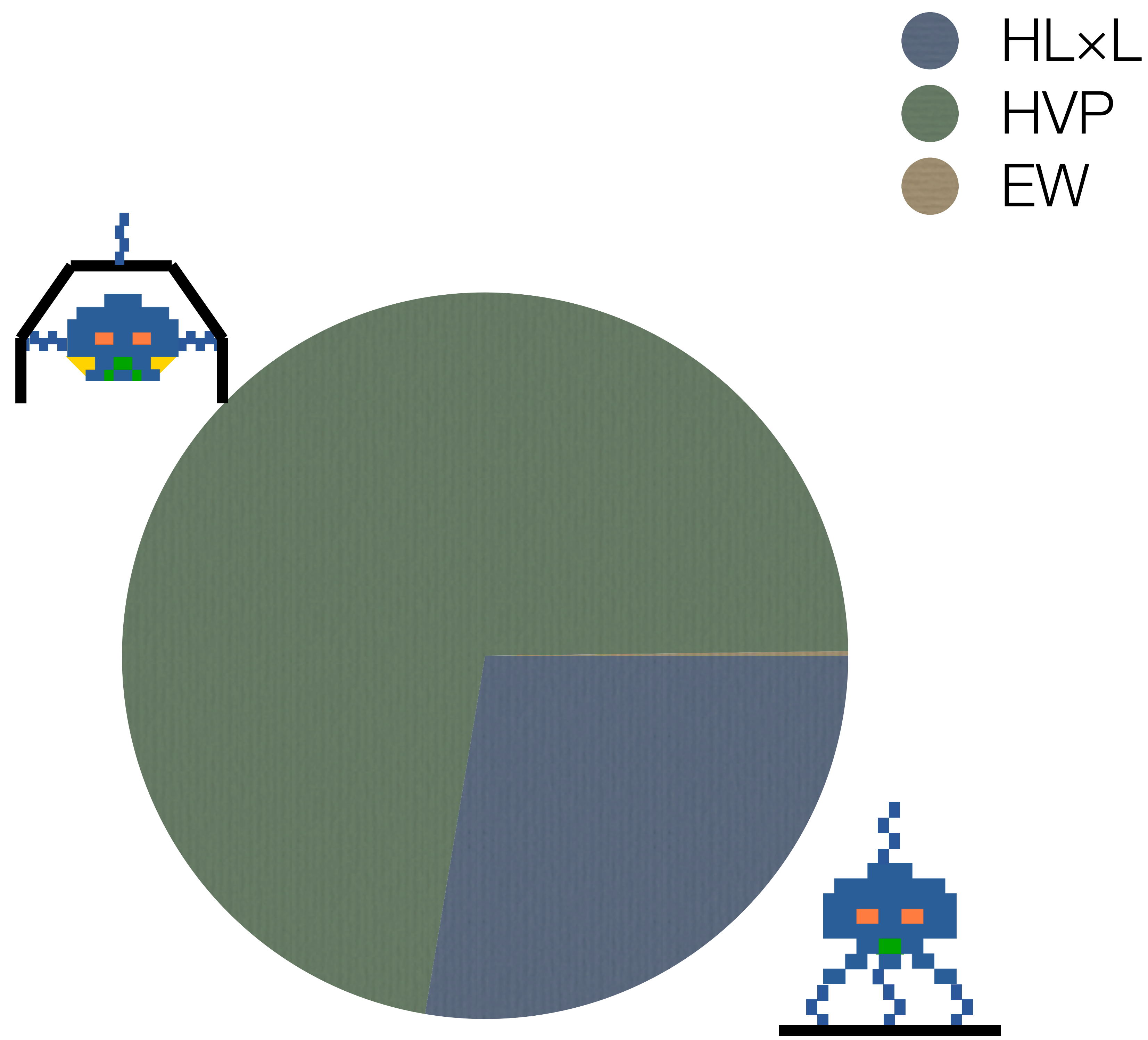
- SM values and compilation from Andreas Höcker, arXiv:1012.0055

Error Budgets for Muon $(g-2)$

error ∝ perimeter; area ∝ weight in sum in quadrature



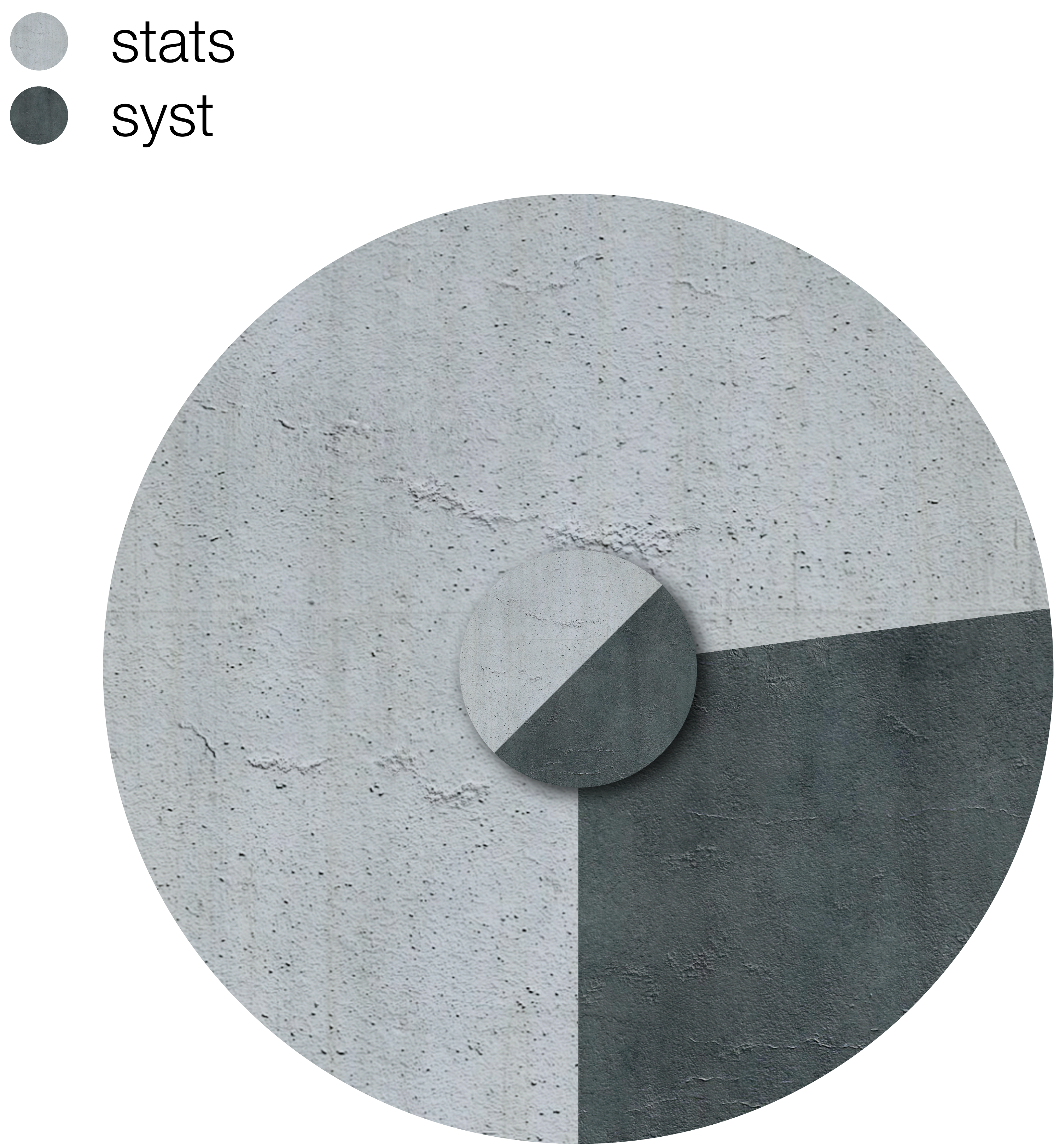
BNL E821 → FNAL E989



Standard Model Calculation

Error Budgets for Muon $(g-2)$

error $\propto$ perimeter; area $\propto$ weight in sum in quadrature



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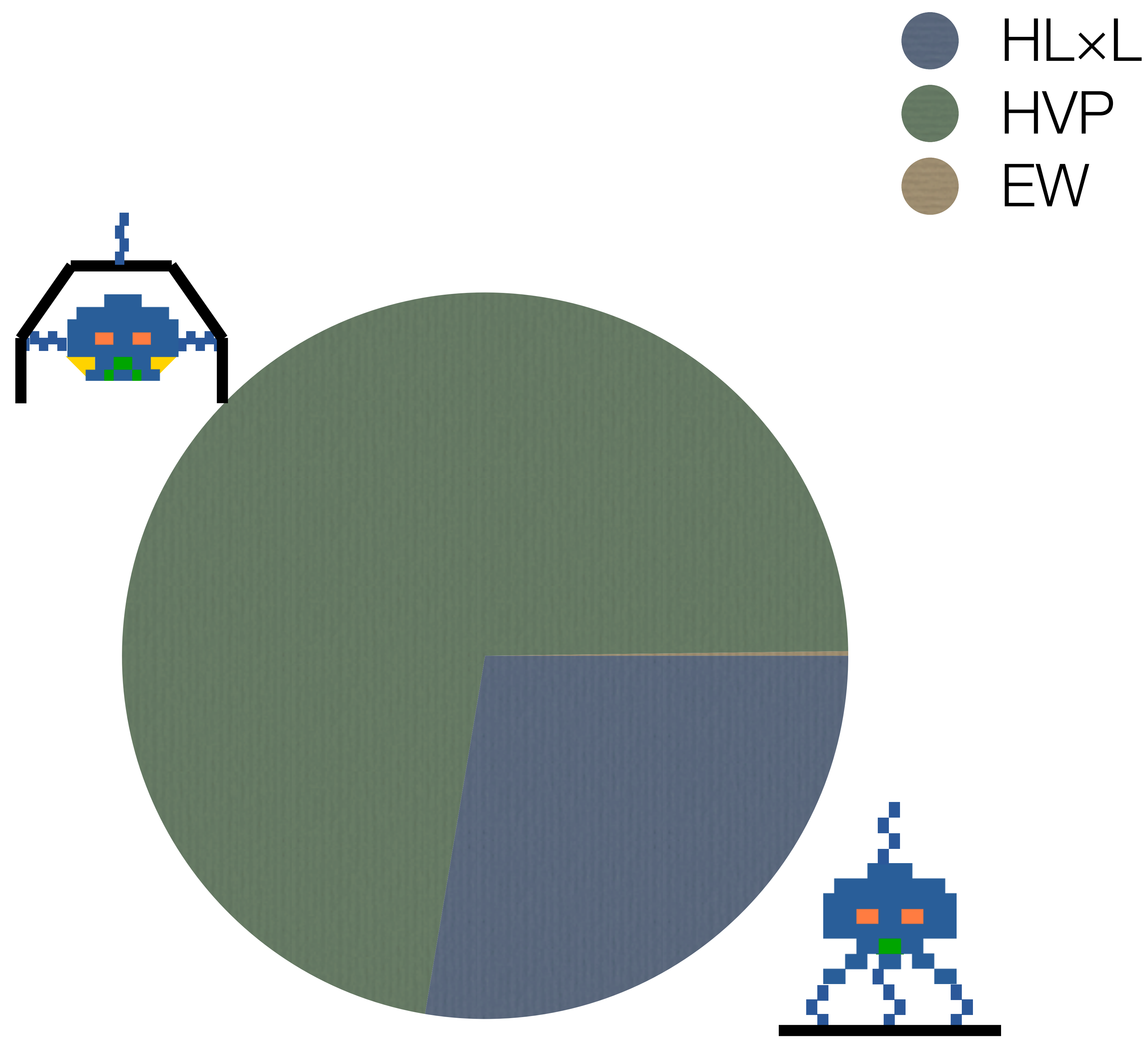

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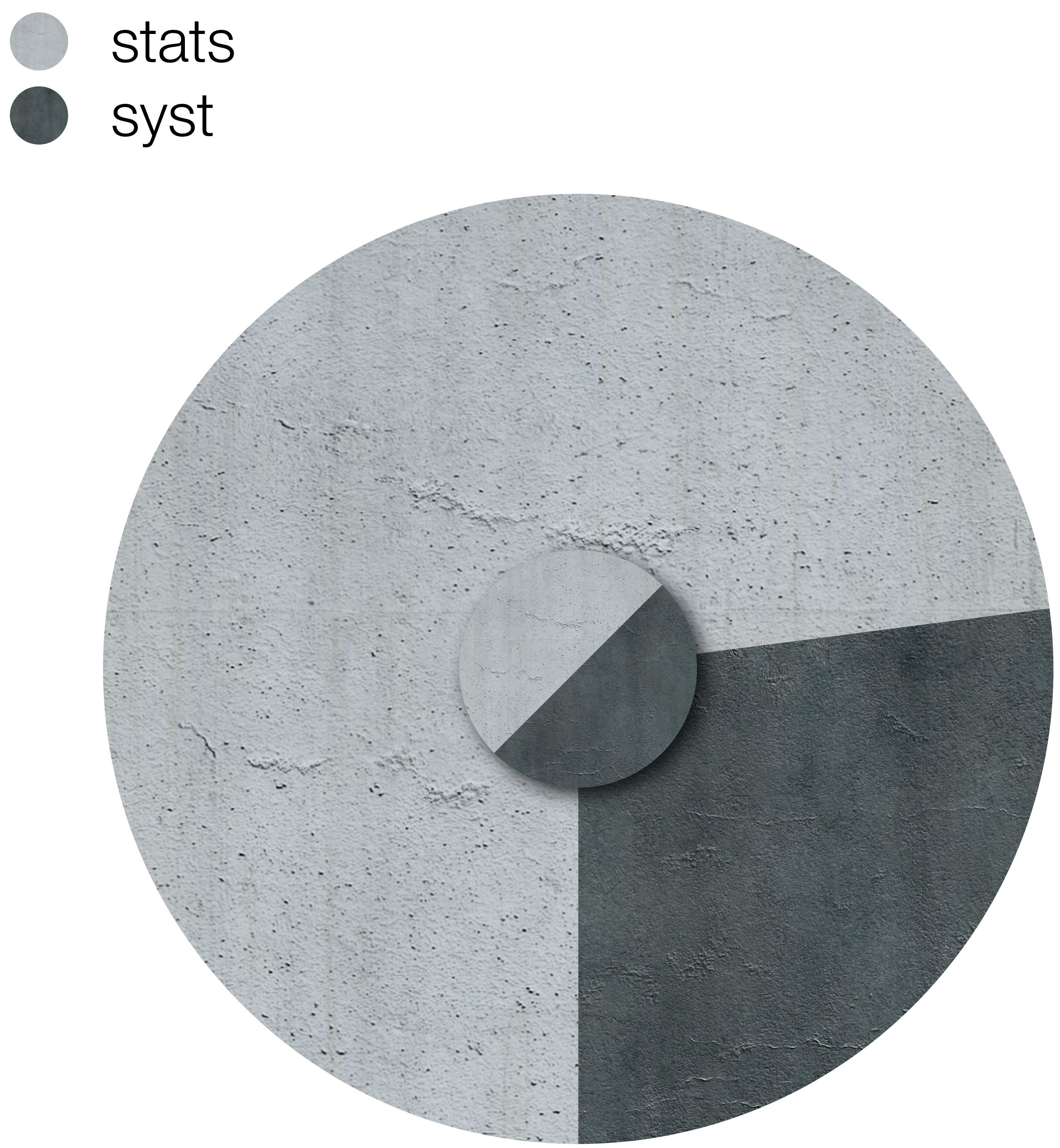

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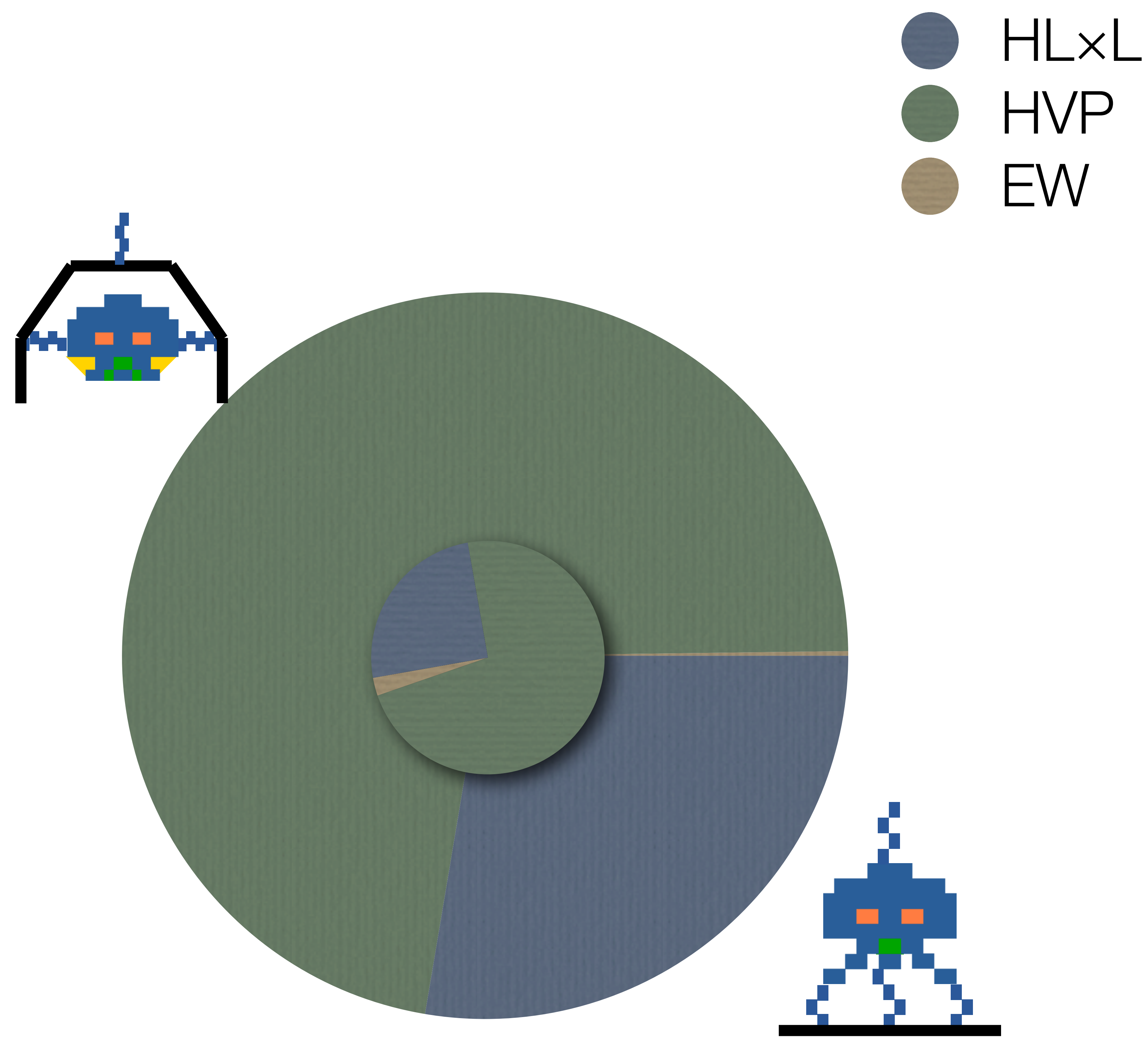


Standard Model Calculation

Explaining the Anomalous Anomaly BSM

Explanations beyond the Standard Model

Bill Marciano

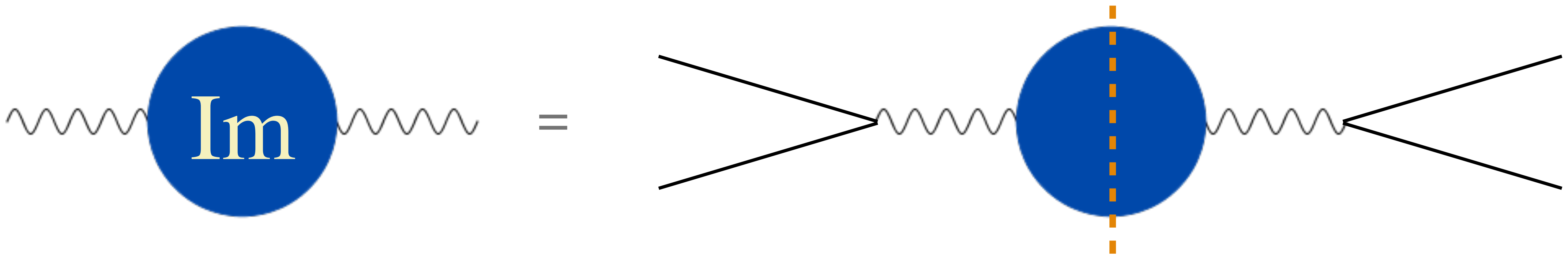
- Discrepancy in $10^{11}a_\mu$ is 285 ± 80 [Höcker, arXiv:1012.0055].
- Generic susy is $\mathrm{sign}(\mu)\ 260\ (\tan\beta/8)\ (200\ \mathrm{GeV}/M_{\mathrm{susy}})^2$; “fits like a glove”.
- Multi-Higgs models; extra dimensions, ….
- Dark photon with $M_{\mathrm{A}} \approx 10\text{–}150\ \mathrm{MeV}$ and $\alpha' = 10^{-8}$:
  - would be seen the first weekend of planned searches at JLab or Mainz.
- Insanely light Higgs, $M_{\mathrm{H}} < 10\ \mathrm{MeV}$ [Kinoshita & Marciano (1990)]:
  - Why doesn’t everyone know why every decade of M_{H} is ruled out?

Hadronic Contributions and their Constraints

HVP from $e^+e^- \to$ hadrons *vs.* hadronic τ decay

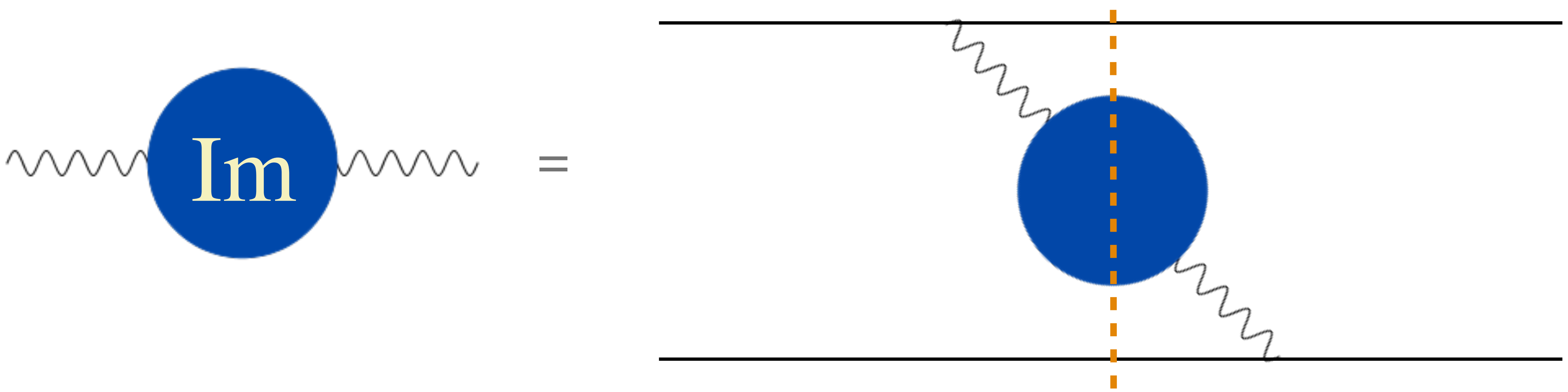
F. Jegerlehner

- The cross section for $e^+e^- \to$ hadrons contains the needed vacuum polarization:



– radiative corrections

- The partial width for $\tau \to$ hadrons contains W VP (related to γ VP by isospin):



$\oplus$ isospin corrections

- Jegerlehner & Szafron [arXiv:1101.2872] find that energy-dependence of mixing in the 2×2 ϱ-γ propagator can resolve the discrepancy. See also Benayoun *et al.*, arXiv:0907.5603.

Hadronic Vacuum Polarization

- Integral over space-like momenta [Blum, hep-lat/0212018 (PRL)]:

$$a_\mu^{\mathrm{HVP}} = \frac{\alpha}{2\pi}\int_0^\infty dt\,\frac{64t^2}{(t+\sqrt{t^2+4t})^4\sqrt{t^2+4t}}\,2\pi\alpha\left[\Pi(m_\mu^2 t)-\Pi(0)\right]$$

where $t = q^2/m_\mu^2$ (Euclidean—or Weinberg's—conventions).

- Integral over time-like momenta $s = -q^2 > 0$:

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{4m_\pi^2}^\infty ds\,K(s)R(s) \qquad R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

- Split (both) integrals into data (experimental or numerical) portion & pQCD portion.

- Vacuum polarization function $\Pi(q^2)$ is defined by ($J_{\rm em}$ for quarks only)

$$\Pi^{\mu\nu}(q^2) = (q^\mu q^\nu - \delta^{\mu\nu} q^2)\Pi(q^2) = \int d^4x\, e^{iq\cdot x} \langle J^\mu_{\rm em}(x) J^\mu_{\rm em}(0)\rangle$$

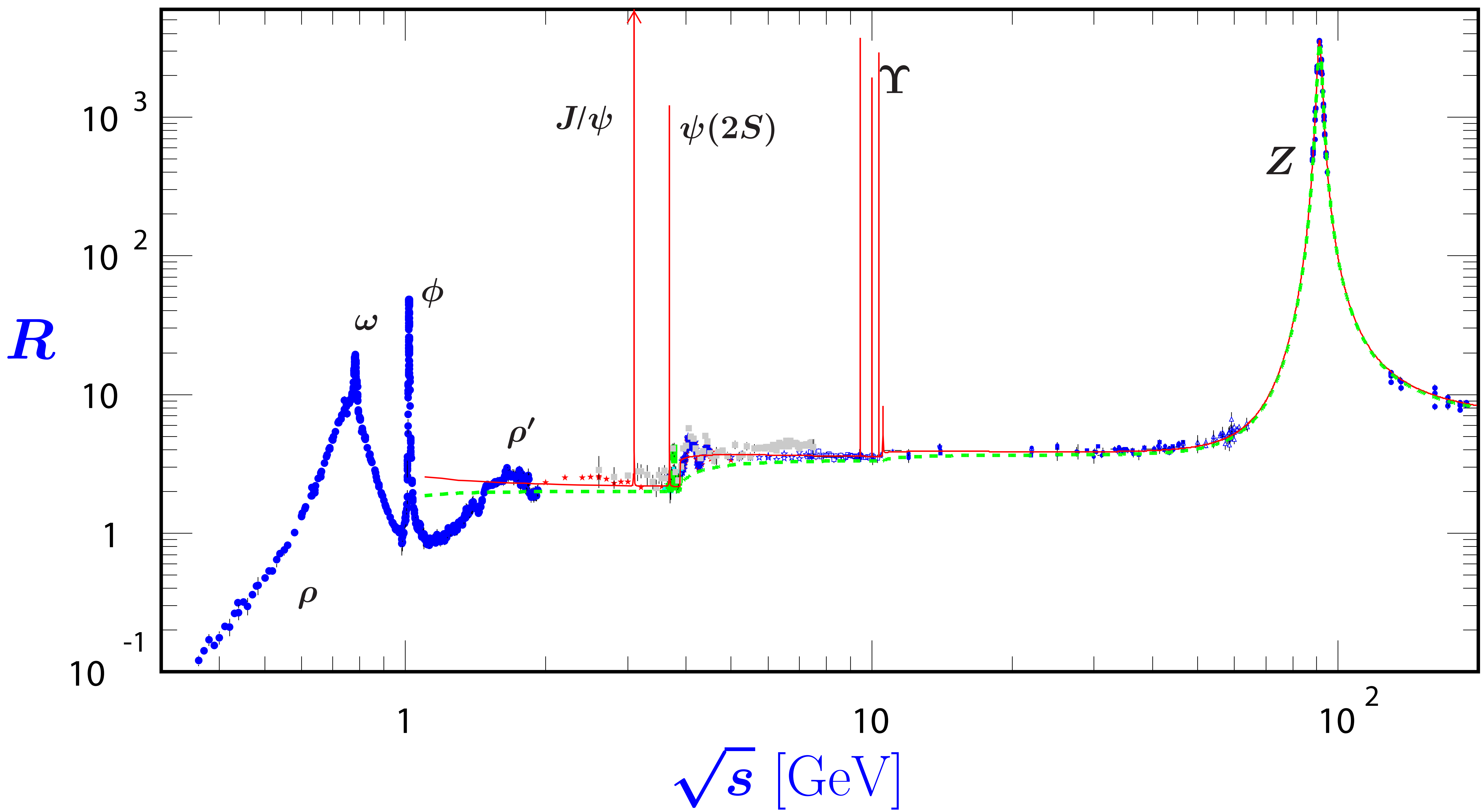
which is very smooth: space-like q^2!!!

- At time-like q^2, dispersion relations can relate this function to its imaginary part, and then the optical theorem to the total cross section:

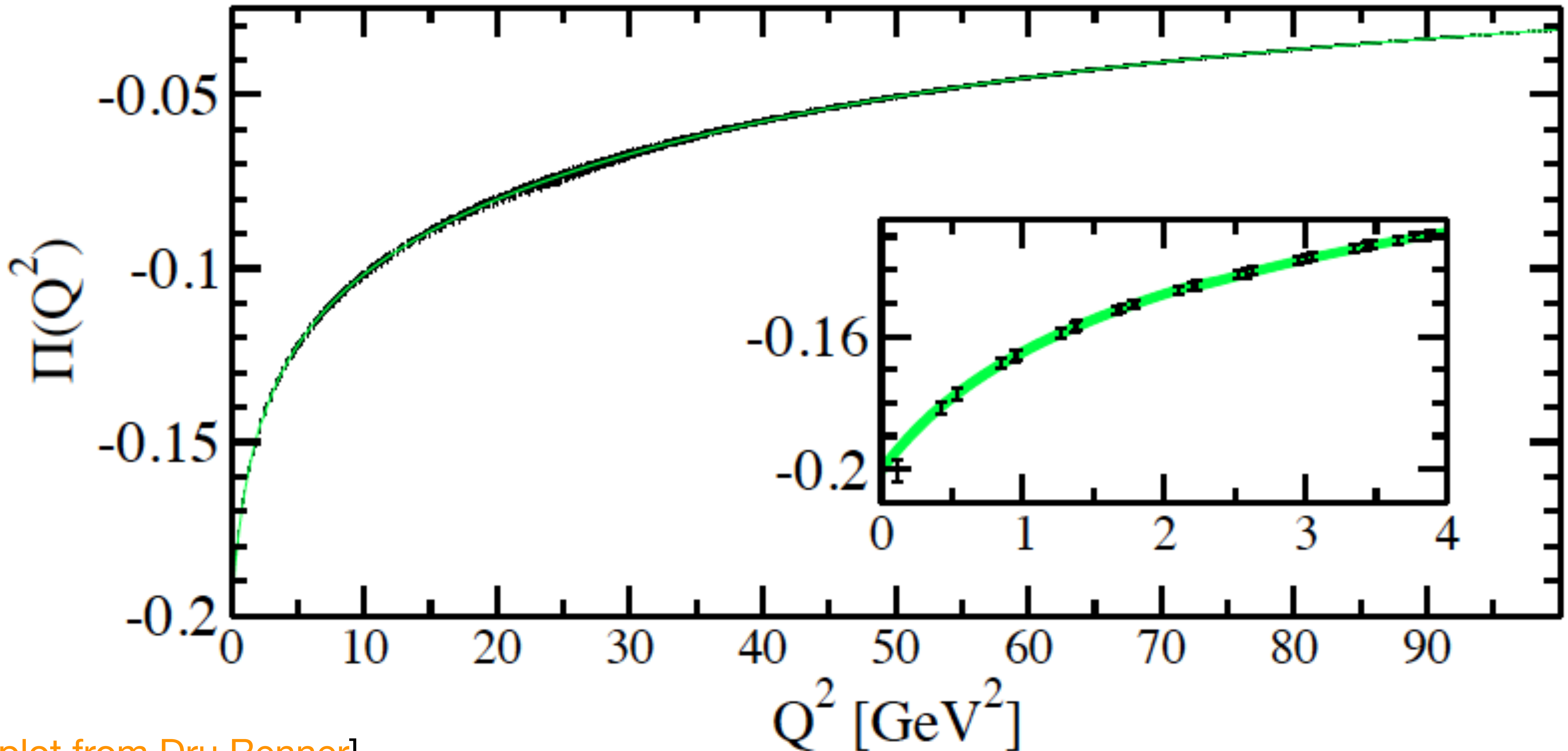
$$\Pi(q^2) - \Pi(0) = \frac{q^2}{\pi}\int_0^\infty ds \frac{\Im \Pi(-s)}{s(s+q^2+i0^+)} = \frac{q^2}{\pi}\int_0^\infty ds \frac{\alpha(s)R(s)}{3s(s+q^2+i0^+)}$$

take jagged resonance regions from experiment; rest from pQCD.

PDG: $e^+e^- \to$ hadrons



Lattice QCD: Hadronic $\Pi(q^2)$



[plot from Dru Renner]

Hadronic Light-by-light Amplitude

- The contribution to $(g\text{–}2)$ is [*e.g.*, arXiv:0901.0306]

$$a_\mu^{\mathrm{HL\times L}} = \frac{e^2}{24m_\mu}\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\,\mathcal{K}_{\mu\lambda\nu\rho\sigma}(p,k_1,k_2,k_3)\,\frac{\partial}{\partial q_\mu}\Pi^{\lambda\nu\rho\sigma}(q,k_1,k_2,k_3)\bigg|_{k_2=k_1-k_3-q,\;q=0}$$

where QED readily yields

$$\mathcal{K}_{\mu\lambda\nu\rho\sigma}(p,k_1,k_2,k_3) = \frac{\mathrm{tr}\{[i\not{p}-m_\mu]\sigma_{\mu\lambda}[i\not{p}-m_\mu]\gamma_\nu[i(\not{p}+\not{k}_1)-m_\mu]\gamma_\rho[i(\not{p}+\not{k}_3)-m_\mu]\gamma_\sigma\}}{k_1^2k_2^2k_3^2[(p+k_1)^2+m_\mu^2][(p+k_3)^2+m_\mu^2]}$$

and QCD not-so-readily provides

$$\Pi^{\lambda\nu\rho\sigma}(q,k_1,k_2,k_3) = \int d^4x_1\,d^4x_2\,d^4x_3\,e^{-i(k_1x_1-k_2x_2-k_3x_3)}\left\langle J^\lambda_{\mathrm{em}}(0)J^\nu_{\mathrm{em}}(x_1)J^\rho_{\mathrm{em}}(x_2)J^\sigma_{\mathrm{em}}(x_3)\right\rangle$$

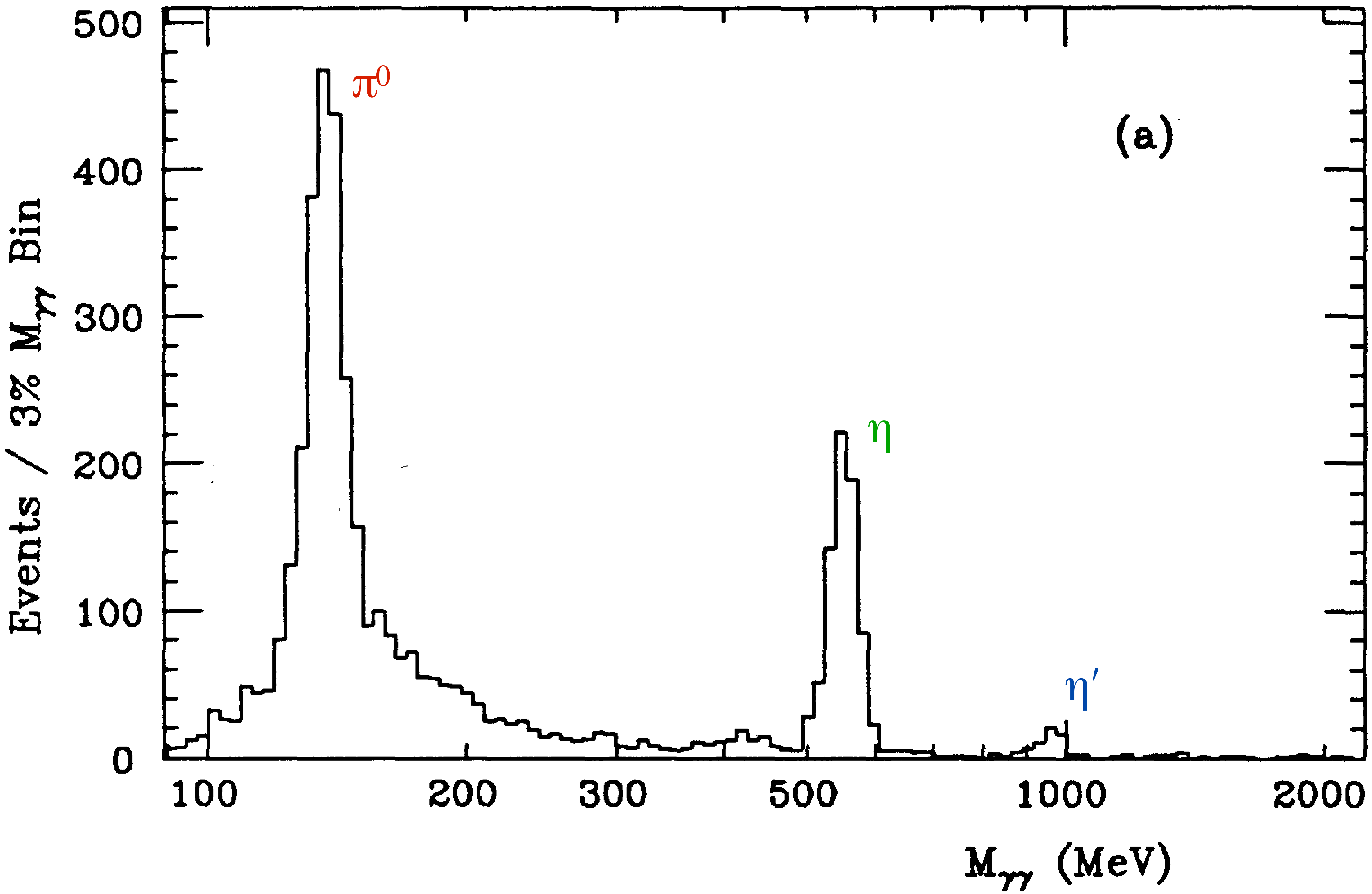
What Do Data Say about HL×L?

Fred Jegerlehner

- HL×L contains a $\gamma \to \gamma^*\gamma^*\gamma^*$ amplitude, which can be related—by analyticity and optical theorem—to cross sections for $\gamma^{(*)}\gamma^{(*)} \to$ hadrons.
- Crystal Ball (1988) $\gamma\gamma \to$ hadrons spectrum shows clear peaks for π, η, & η' but nothing else.
- Primakoff effect ($\gamma\mathrm{N} \to \pi^0 \to \gamma\gamma$) yields pion part of $\gamma\gamma\gamma\gamma^*$.
- Central π^0 production in e^+e^- (CELLO, CLEO, BaBar, …) yield pion part of $\gamma^{(*)}\gamma^*\gamma\gamma$.
- Axial-vector mesons require off-shell photon(s) (Lee-Yang theorem): data are "sparse".
- Scalar mesons seen in $\gamma\gamma \to \pi\pi$; tensor mesons needed too….
- Need to connect data with 0, 2, or 4 photons off shell to amplitude with 3 off shell: models inevitably enter: they should be compatible with measurements mentioned here.

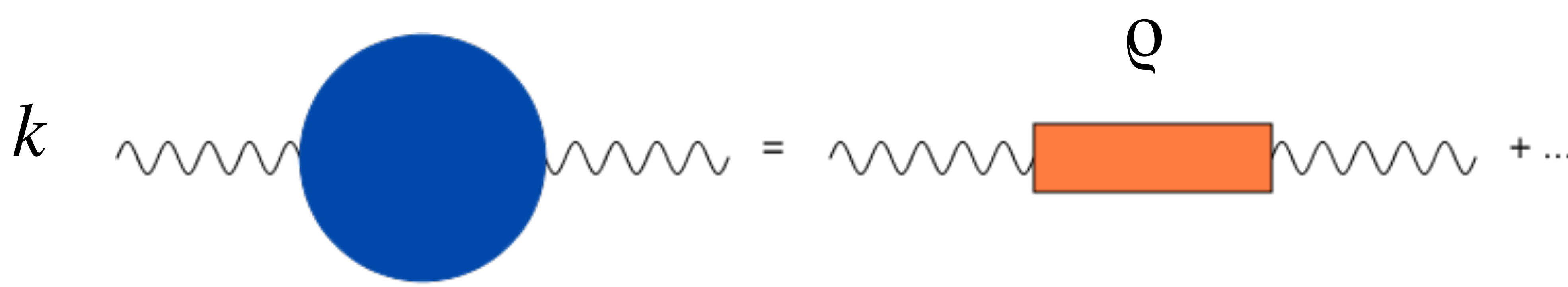
Crystal Ball (1988): π^0, η, and η' in $\gamma\gamma \to \gamma\gamma$

SLAC-PUB-4580, Fig. 2 (see also Fig. 8)

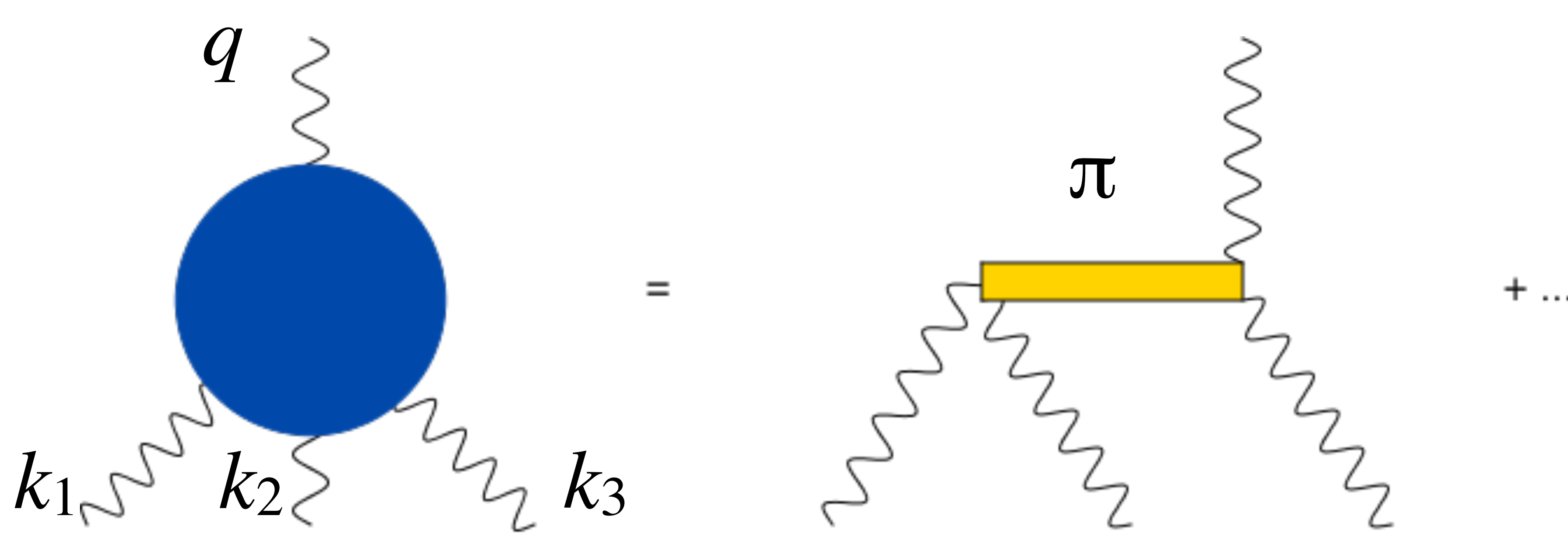


Dominant contributions

- Hadronic vacuum polarization is dominated by the rho meson (VMD):



- Hadronic light-by-light amplitude is dominated by π (and η, η′) exchange (normalized by the anomaly; well described by Wess-Zumino Lagrangian)



- Of course, the uncertainty is dominated by the other contributions … .

Estimates of HL×L from Models of QCD

Apology

- Most of the following slides follow the dreadful format “so-and-so gave a nice talk in which he* showed this nice plot”.

- Just without the nice plots.

- * At this workshop, all speakers were “he”.

Glasgow Consensus

Prades, de Rafael, Vainshtein [arXiv:0901.0306]

- Combining several ingredients (covered below), PRV find $10^{11} a_\mu^{\mathrm{HL\times L}} = 105 \pm 26$:
  - $10^{11} a_\mu^{\mathrm{HL\times L}}(\pi, \eta, \eta') = 114 \pm 13$ [MV ≈ (ENJL+OPE) ± max.ENJL];
  - $10^{11} a_\mu^{\mathrm{HL\times L}}(a_1, \text{etc.}) = 15 \pm 10$ [MV ± 10×MV];
  - $10^{11} a_\mu^{\mathrm{HL\times L}}(\text{scalars}) = -7 \pm 7$ [ENJL ± inflated ENJL];
  - $10^{11} a_\mu^{\mathrm{HL\times L}}(\text{dressed } \pi \text{ loop}) = -19 \pm 19$ [ENJL ± inflated ENJL];
  - add error estimates in quadrature.

Extended Nambu–Jona-Lasinio & Chiral Quark Models

Hans Bijnens (work with Pallante & Prades)

- The chiral quark model has a pion field (χPT) constituent-like quark field:
  - quark captures short-distance QCD, but freezes out at long distances;
  - pion captures long-distance constraints of chiral symmetry;
  - need great care to avoid double counting of long & short (>1 invariant!).
- NJL adds to this four-quark interactions whose bubble sums generate non-NG mesons.
- Thus, combo incorporates obviously needed ingredients: pion & other meson exchange + quark loop.
- Hayakawa, Kinoshita, Sanda: meson models, VMD, hidden local symmetry.

Chiral approach and resonance dominance

Andreas Nyffeler

- The BPP and HKS papers simplify the pion exchange amplitude

$$\mathcal{A} \propto F_{\pi\gamma^*\gamma^*}\left((q_1+q_2)^2, q_1^2, q_2^2\right) \frac{1}{(q_1+q_2)^2 - m_\pi^2} F_{\pi\gamma^*\gamma}\left((q_3+q_4)^2, q_3^2, 0\right)$$

with $F_{\pi\gamma^*\gamma^*}\left((q_1+q_2)^2, q_1^2, q_2^2\right) \approx F_{\pi\gamma^*\gamma^*}\left(m_\pi^2, q_1^2, q_2^2\right)$.

- Off-shell effects should enter. How large are they?
- Can be estimated only using resonance models, and in a model calculation of HL×L, this is not an essentially new ingredient: estimates $10^{11}a_\mu^{\text{HL×L}}$(off shell) ≈ 35–40.
- NB: *magnetic susceptibility* $\langle\bar{q}\sigma_{\mu\nu}q\rangle_{F_{\mu\nu}}$ constrains meson exchanges [Belyaev & Kogan, 1984]; can be calculated in lattice gauge theory.

Using Constraints from Operator Product Expansion

Arkady Vainshtein; Kiril Melnikov

- In the limit $k_1^2 \approx k_2^2 \gg k_3^2 \gg \Lambda_{\text{QCD}}^2$, the OPE relates $\text{FT}\langle VVVV\rangle$ to $\text{FT}\langle AVV\rangle$ [hep-ph/0312226]:
  - fixes normalization of pseudoscalar and axial-vector exchanges in these kinematics;
  - in particular, $\lim_{q^2 \gg \Lambda^2} F_{\pi\gamma^*\gamma^*}(q^2, q^2) = \dfrac{8\pi^2 f_\pi^2}{N_c q^2}$ matches low-energy normalization from anomaly;
  - facilitates introduction of a model *function* to interpolate between limits (in contrast to model Lagrangians of other approaches);
  - MV choose an Ansatz; you could choose yours.
- Despite any limitations of MV's Ansatz, it should be clear that model Lagrangians in other approaches should satisfy their OPE constraint.

Two-loop Chiral Perturbation Theory

Michael Ramsay-Musolf

- Notes that χPT provides useful, model-independent constraint of pion contribution:
  - pion pole term yields $\ln^2$; single $\ln$ from $\pi \to e^+e^-$; last LEC from lattice
  - BR($\pi \to e^+e^-$) from KTeV 2007 should reduce uncertainty in single $\ln$.
- Resonances built up from higher-order contributions:
  - MRM + students computing full 2-loop χPT HL×L.
- Pion loops will need further LECs from pion charge radius and pion polarizability.
- This seems like a hard way to gain real improvement, but I think these calculations could guide chiral extrapolation of QED+QCD method.

Schwinger-Dyson Equations (DSE)

Richard Williams

- Start with (exact) Dyson-Schwinger eq'ns for dressed propagators, vertex, 4-pt function.
- Introduce "model" functions (*e.g.*, Maris-Tandy) that satisfy—
  - Ward identities;
  - good agreement with phenomenology in other applications;
  - good agreement with lattice calculations (in Landau gauge).
- Keep large N_c part in DSE resummation (i.e., neglect non-planar and 2- & 3-gluon vtx).
- Results: $10^{11}a_\mu^{\mathrm{HVP}} = 6700$ & $10^{11}a_\mu^{\mathrm{HL\times L}} = 217 \pm 91$ [arXiv:1012.3886] or 147 ± 91 [this talk?]; compare: $10^{11}a_\mu^{\mathrm{HVP}} = 6923 \pm 42$ [data] & $10^{11}a_\mu^{\mathrm{HL\times L}} = 105 \pm 26$ [consensus, arXiv:0901.0306].

Compilation of Models: Consensus?

Andreas Nyffeler

Contribution	BPP	HKS, HK	KN	MV	BP, MdRR	PdRV	N, JN	FGW
π^0, η, η'	85 ± 13	82.7 ± 6.4	83 ± 12	114 ± 10	–	114 ± 13	99 ± 16	84 ± 13
axial vectors	2.5 ± 1.0	1.7 ± 1.7	–	22 ± 5	–	15 ± 10	22 ± 5	–
scalars	-6.8 ± 2.0	–	–	–	–	-7 ± 7	-7 ± 2	–
π, K loops	-19 ± 13	-4.5 ± 8.1	–	–	–	-19 ± 19	-19 ± 13	–
π, K loops +subl. N_C	–	–	–	0 ± 10	–	–	–	–
other	–	–	–	–	–	–	–	0 ± 20
quark loops	21 ± 3	9.7 ± 11.1	–	–	–	2.3	21 ± 3	107 ± 48
Total	83 ± 32	89.6 ± 15.4	80 ± 40	136 ± 25	110 ± 40	105 ± 26	116 ± 39	191 ± 81

BPP = Bijnens, Pallante, Prades '95, '96, '02; HKS = Hayakawa, Kinoshita, Sanda '95, '96; HK = Hayakawa, Kinoshita '98, '02; KN = Knecht, Nyffeler '02; MV = Melnikov, Vainshtein '04; BP = Bijnens, Prades '07; MdRR = Miller, de Rafael, Roberts '07; PdRV = Prades, de Rafael, Vainshtein '09; N = Nyffeler '09, JN = Jegerlehner, Nyffeler '09; FGW = Fischer, Goecke, Williams '10, '11 (used values from arXiv:1009.5297v2 [hep-ph], 4 Feb 2011)

Lattice QCD

arXiv:1203.1204, arXiv:1209.3468

Lattice Gauge Theory

K. Wilson, *PRD* **10** (1974) 2445

- Invented to understand asymptotic freedom without the need for gauge-fixing and ghosts [Wilson, hep-lat/0412043].

- Gauge symmetry on a spacetime lattice:

  - mathematically rigorous definition of QCD functional integrals;

$$\langle \bullet \rangle = \frac{1}{Z} \int \mathcal{D}U \, \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-S\right) [\bullet]$$

  - enables theoretical tools of statistical mechanics in quantum field theory and provides a basis for constructive field theory.

- Lowest-order strong coupling expansion demonstrates confinement.

Numerical Lattice QCD

- Nowadays "lattice QCD" usually implies a numerical technique, in which the functional integral is integrated numerically on a computer.

- A big computer.

- Some compromises:

  - finite human lifetime ⇒ Wick rotate to Euclidean time: $x^4 = ix^0$;

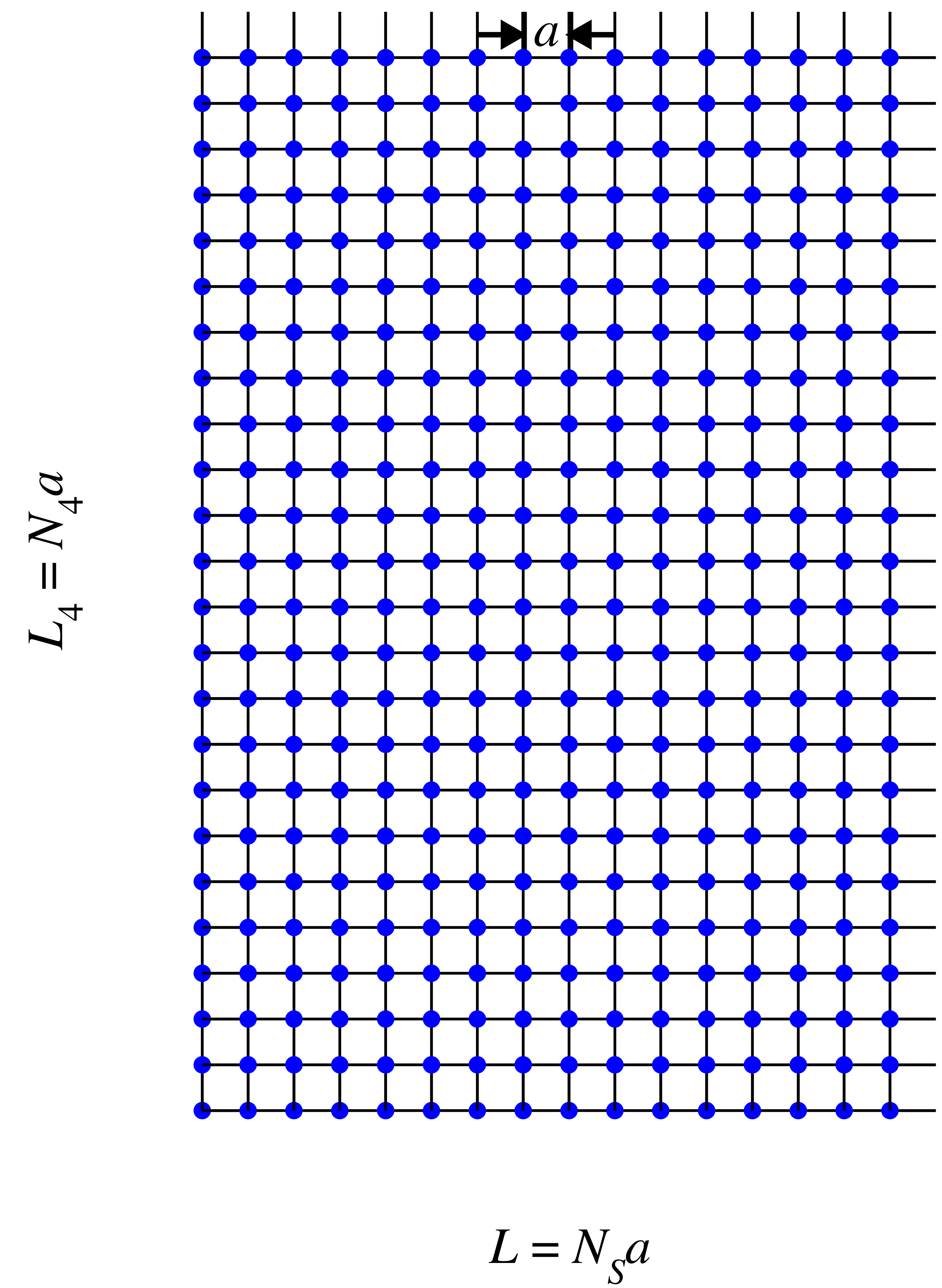
  - finite memory ⇒ finite space volume & finite time extent;

  - finite CPU power ⇒ light quarks often heavier than up and down.

Lattice Gauge Theory

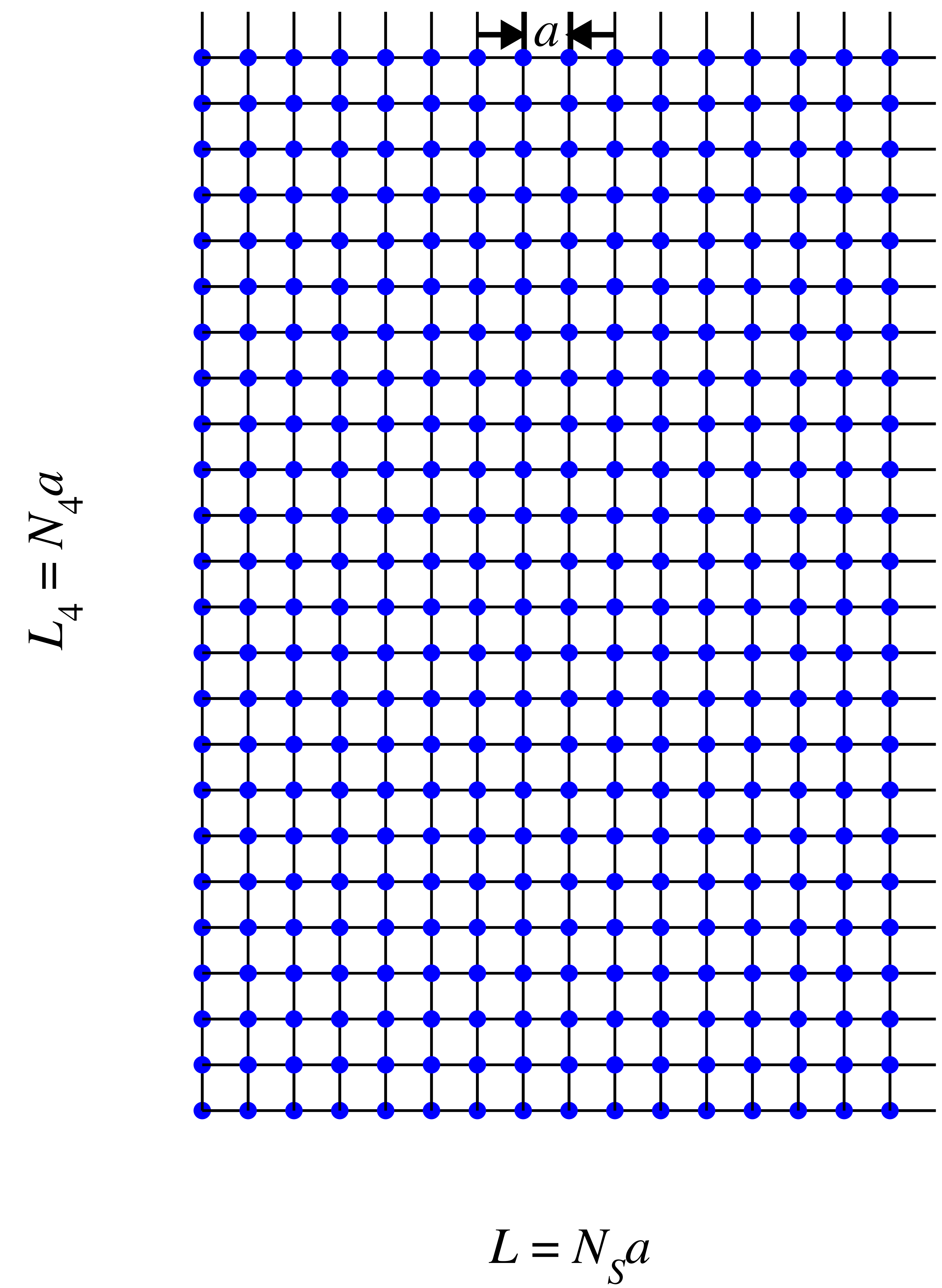
$$\langle \bullet \rangle = \frac{1}{Z} \int \mathcal{D}U \, \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-S\right) [\bullet]$$

- Infinite continuum: uncountably many d.o.f. (⇒ UV divergences);
- Infinite lattice: countably many; used to define QFT;
- Finite lattice: finite dimension ~ 10^8, so compute integrals numerically.



Lattice Gauge Theory

$$\langle \bullet \rangle = \frac{1}{Z} \int \mathcal{D}U \, \underset{\text{hand}}{\boxed{\mathcal{D}\psi \mathcal{D}\bar{\psi}}} \exp\left(-S\right) [\bullet]$$



- Infinite continuum: uncountably many d.o.f. ($\Rightarrow$ UV divergences);
- Infinite lattice: countably many; used to define QFT;
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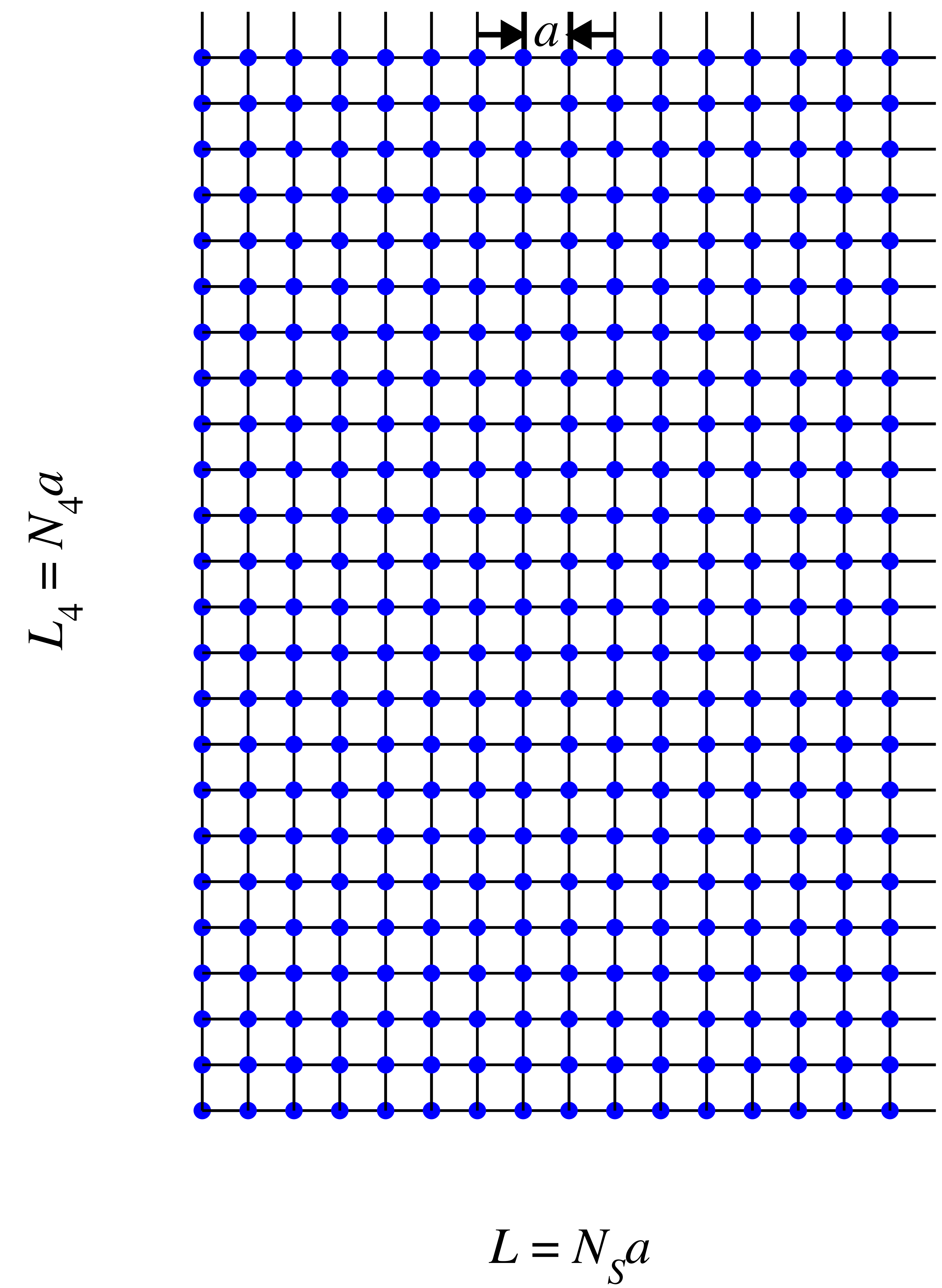
Lattice Gauge Theory

$$\langle \bullet \rangle = \frac{1}{Z}\int \underbrace{\boxed{\mathcal{D}U}}_{\text{MC}}\,\underbrace{\boxed{\mathcal{D}\psi\mathcal{D}\bar{\psi}}}_{\text{hand}} \exp\left(-S\right)\left[\bullet\right]$$



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Some Jargon

- QCD observables (quark integrals by hand):

$$\langle \bullet \rangle = \frac{1}{Z} \int \mathcal{D}U \prod_{f=1}^{n_f} \det(\not{D} + m_f) \exp\left(-S_{\text{gauge}}\right) [\bullet']$$

- *Quenched* means replace $\det$ with 1. (Obsolete.)
- *Un*quenched means not to do that.
- *Partially* quenched (usually) doesn't mean "n_f too small" but $m_{\text{val}} \neq m_{\text{sea}}$, or even $\not{D}_{\text{val}} \neq \not{D}_{\text{sea}}$ ("mixed action").

Some algorithmic issues

e.g., ASK, hep-lat/0205021

- lattice $N_S^3 \times N_4$, spacing a
- memory $\propto N_S^3 N_4 = L_S^3 L_4 / a^4$
- $\tau_\mathrm{g} \propto a^{-(4+z)}$, $z = 1$ or 2.
- $\tau_\mathrm{q} \propto (m_\mathrm{q} a)^{-p}$, $p = 1$ or 2.
- Imaginary time:
  - static quantities

- size $L_S = N_S a$, $L_4 = N_4 a$;
- dimension of spacetime = 4
- critical slowing down
- especially dire with sea quarks
- thermodynamics: $T = 1/N_4 a$

$$\langle \bullet \rangle = \frac{1}{Z} \int \mathcal{D}U \, \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp(-S) \, [\bullet]$$
$$= \mathrm{Tr}\{\bullet \, e^{-\hat{H}/T}\} / \mathrm{Tr}\{e^{-\hat{H}/T}\}$$

Sea Quarks

- Staggered quarks, with rooted determinant, $\mathrm{O}(a^2)$.
- Wilson quarks, $\mathrm{O}(a)$:
  - tree or nonperturbatively $\mathrm{O}(a)$ improved $\Rightarrow \mathrm{O}(a^2)$;
  - twisted mass term—auto $\mathrm{O}(a)$ improvement $\Rightarrow \mathrm{O}(a^2)$.
- Ginsparg-Wilson (domain wall or overlap), $\mathrm{O}(a^2)$:
  - $\not{D}\gamma_5 + \gamma_5\not{D} = 2a\not{D}^2$ implemented w/ $\mathrm{sign}(\not{D}_\mathrm{W})$.

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fast

clean

- Many numerical simulations with sea quarks are called (perhaps misleadingly) “unquenched” or “full QCD.”

  - $n_f = 2$: with same mass, omitting strange sea;

  - $n_f = 3$: may (or may not) imply 3 of same mass;

  - $n_f = 2{+}1$: strange sea + 2 as light as possible for up and down;

  - $n_f = 2{+}1{+}1$: add charmed sea to $2{+}1$.

- “Full QCD” can also mean $m_{\mathrm{val}} = m_{\mathrm{sea}}$, or $\not{D}_{\mathrm{val}} = \not{D}_{\mathrm{sea}}$.

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Computing HVP and HL×L with Lattice Gauge Theory

Lattice QCD for g–2

- With lattice QCD, one can compute FT$\langle V_\mu(x)V_\nu(0)\rangle$ or FT$\langle V_\mu(x)V_\nu(y)V_\rho(z)V_\sigma(0)\rangle$ (from first principles) and convolute the result with QED Feynman diagrams.

- In addition to usual worries (continuum limit, physical pion cloud), need $q \sim m_\mu$, so might expect to need box-size a few times $\pi/m_\mu \sim 6$ fm.

- Structure in Green functions expected at two QCD scales: $m_\pi \approx 1.3m_\mu$ and $m_\rho \approx 7m_\mu$; also need to match onto pQCD regime.

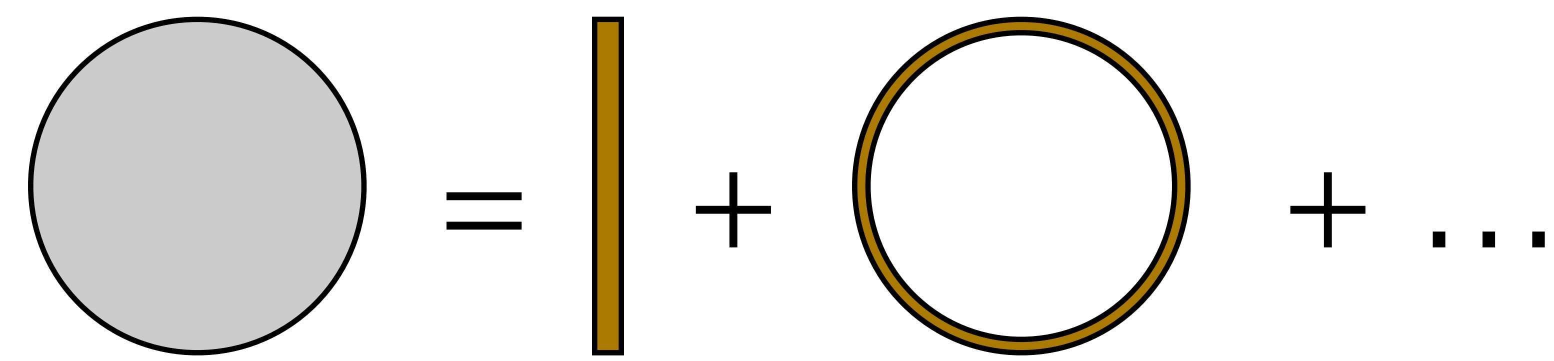
- HVP 2-pt function has 2 (1) form factors; HL×L has 138 (43 by gauge symmetry; 32 in g–2).

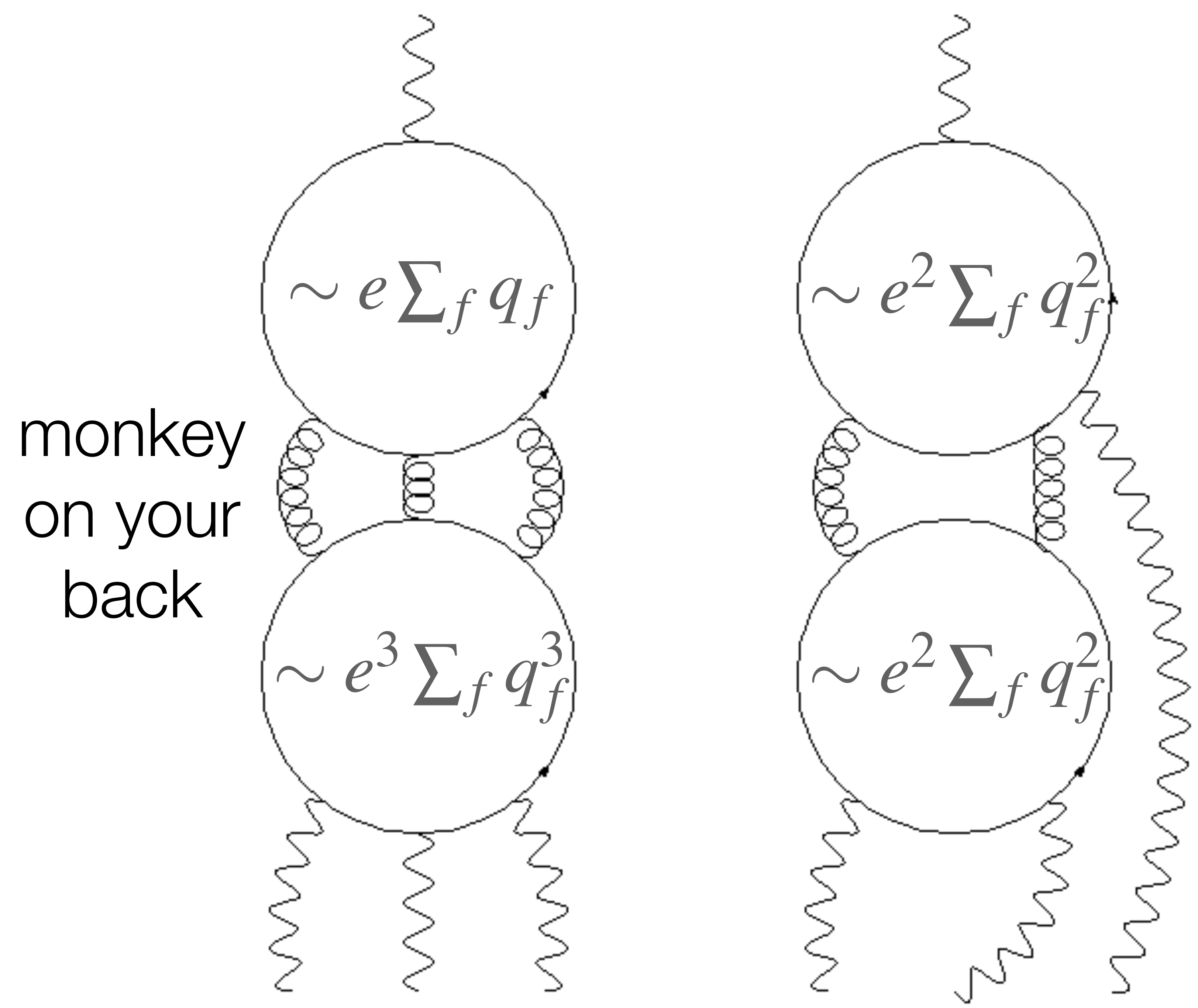
- In the end, need only two numbers, HVP (≈ 7000) to 0.2%, HL×L (≈ 100) to 5%, to match measurement of approved experiment Fermilab E989.

- Probably need cleverness, not just brute force.

Sea Quarks are Necessary for g–2

- Not just for processes sketched in the top figure (for both vacuum polarization and HL×L).
- All fermion lines/loops connected to initial or final state must be treated separately:
  - "disconnected diagrams"—
  - present because photon is flavor singlet;
  - really, really demanding.
- As far as I know, no one has attempted a fully disconnected calculations for HL×L or HVP.





QCD+QED: Direct Calculation of HL×L

Tom Blum

- Computing $\mathrm{FT}\langle VVVV\rangle$ seems difficult and unnecessarily so.
- Need one number: the (hadronic part of the) muon's magnetic form factor at $q^2 = 0$.
- Compute $F_2(0)$ in lattice QCD+QED (QED quenched for now):
  - need subtraction to eliminate some QED renormalization parts;
  - successful in pure QED for muon, not for electron—signal ~ $(m_{\text{leg}}/m_{\text{loop}})2$, noise same;
  - in QCD+QED, muon suffers from the same problem—constituent $m_{\text{loop}} \sim m_{\mu}$.
- Smells like a promising way forward; see also Blum's talk at ⟨Lattice||Experiment⟩.

Two Approaches to Form Factor for $\pi\gamma^{(*)}\gamma^*$

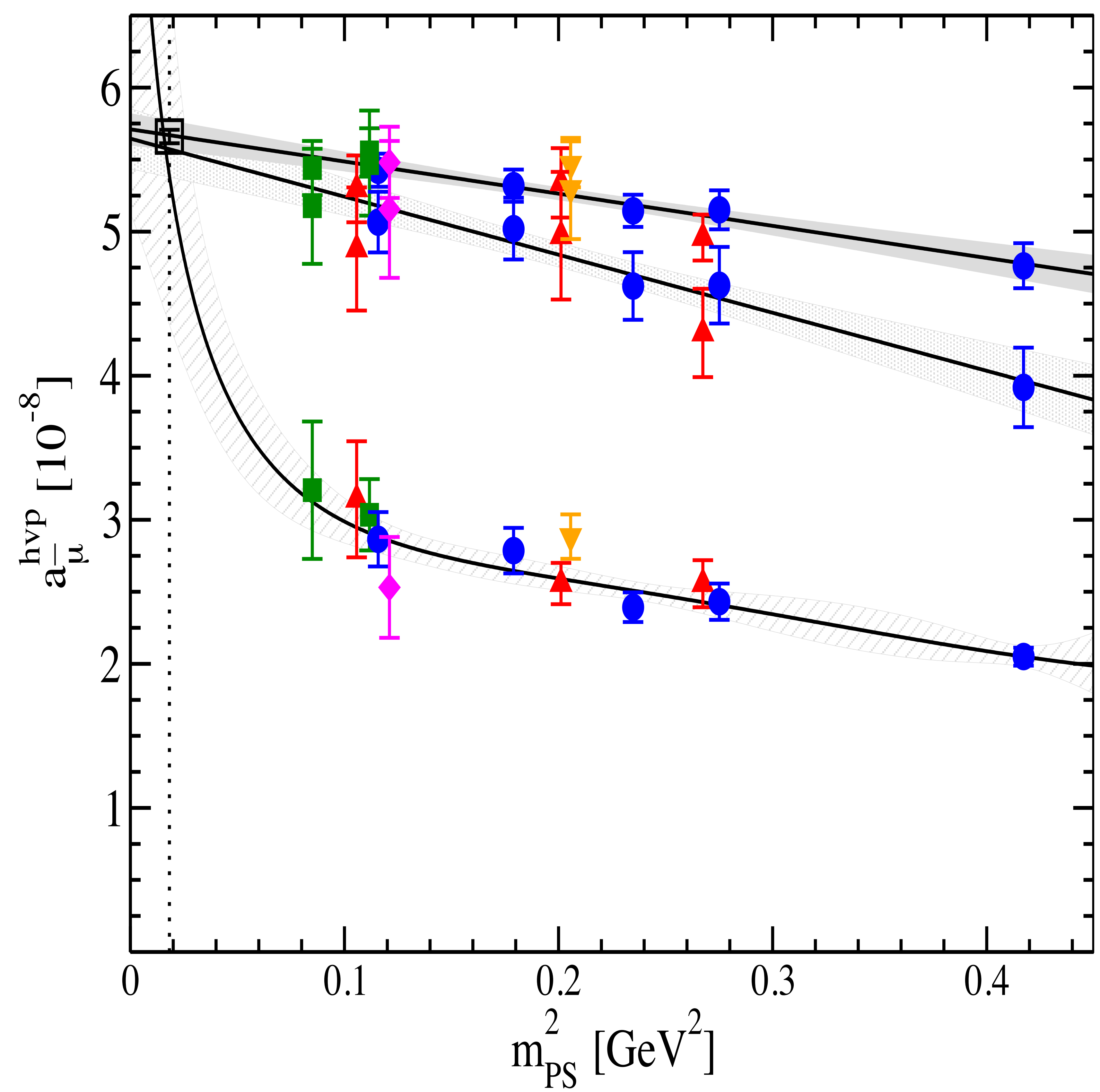
Shoji Hashimoto

- Space-like [arXiv:0912.0253]:
  - standard lattice QCD form factor techniques;
  - ABJ anomaly reproduced (most involved calculation ever) ⇒ precise pion width;
  - limited range of momentum transfer: twisted bc? constrain with unitarity & analyticity?
- Time-like [S. Cohen *et al.*, arXiv:0810.5550]:
  - exploit masses of vector mesons to get to time-like $q^2 = \boldsymbol{p}^2 - m_V{}^2 < 0$;
  - pilot study by JLab group; new preliminary work by JLQCD.

HVP with 2 Twisted-mass Sea Quarks

Karl Jansen

- Lattice calculations of a_μ^{HVP} pioneered by Blum, Blum & Aubin.
- New, and precise, calculation of up-down contribution to HVP (data $10^8 a_\mu^{\mathrm{HVP}} = 5.66 \pm 0.05$):
  - first attempt lacked control of chiral extrapolation: head scratching: resolution:
  - solving this problem: $10^8 a_\mu^{\mathrm{HVP}} = 5.66 \pm 0.11$;
  - agrees with expt and error is only twice;
- Now attack with 2+1+1 flavors of sea quarks!!!



Lattice calculations of HVP

✦ Several independent efforts ongoing

Collaboration	N_f	Fermion action	$a_\mu^{\rm HVP} \times 10^{10}$
Aubin & Blum	2+1	Asqtad staggered	$713(15)_{\rm stat}(31)_{\chi\rm PT}(??)_{\rm other}$
ETMC	2	twisted-mass	$572(16)_{\rm total}$
ETMC (*preliminary*)	2+1+1	twisted-mass	$674(21)_{\rm stat}(18)_{\rm sys}(??)_{\rm disc}$
Edinburgh	2+1	domain-wall	$641(33)_{\rm stat}(32)_{\rm sys}(??)_{\rm disc}$
Mainz	2	$\mathcal{O}(a)$ improved Wilson	$618(64)_{\rm stat+sys}(??)_{\rm disc}$

✦ Use same general method, but introduce different improvements to address some of the most significant sources of systematic uncertainty

[1] Aubin & Blum, Phys.Rev. D75 (2007) 114502
[2] Feng *et al.*, Phys.Rev.Lett. 107 (2011) 081802
[3] Hotzel *et al.*, Lattice 2013
[4] Boyle *et al.*, Phys.Rev. D85 (2012) 074504
[5] Della Morte *et al.*, JHEP 1203 (2012) 055

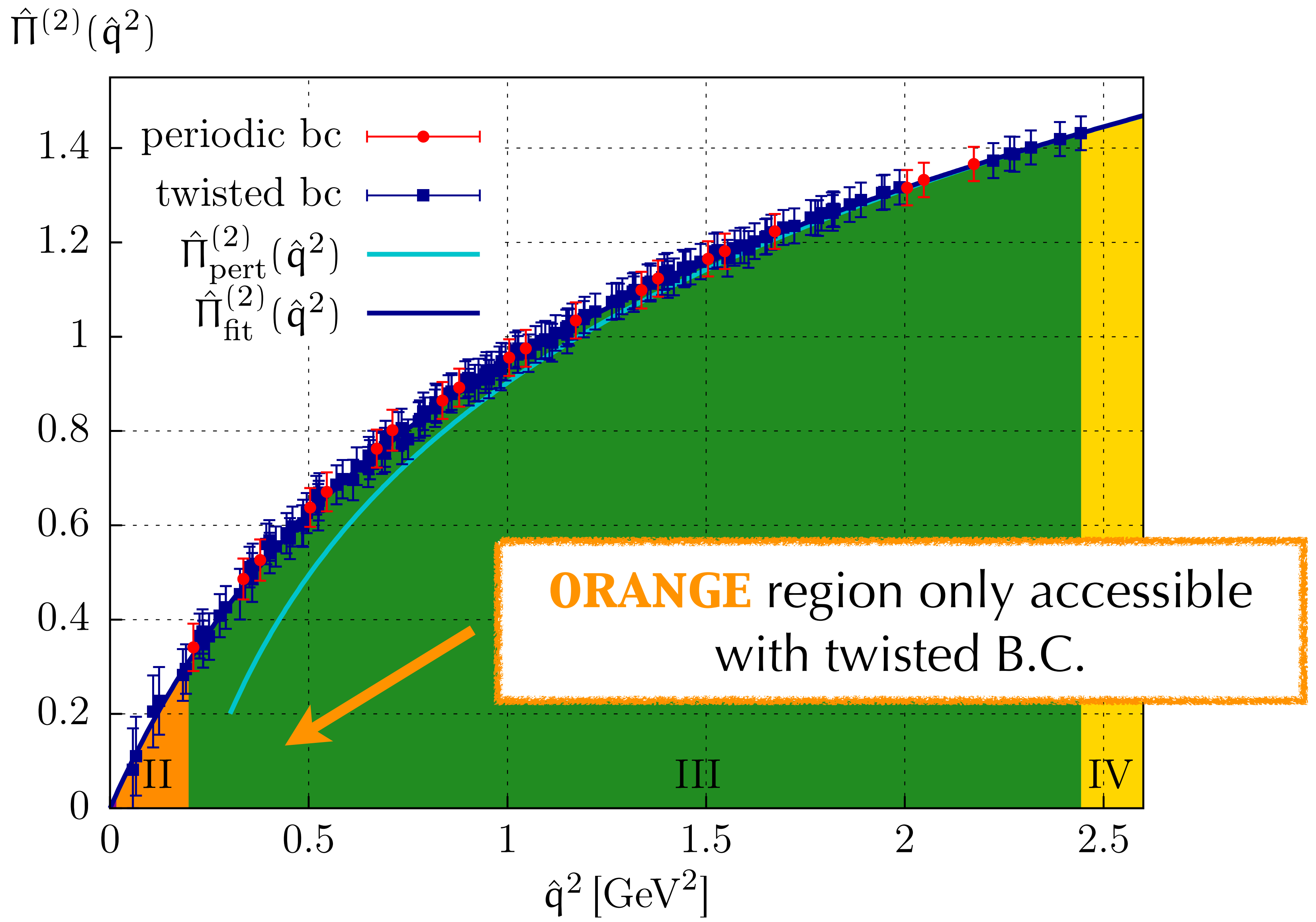
Recent developments

Twisted boundary conditions [Della Morte et al., JHEP 1203 (2012) 055]

- Because of finite spatial lattice size (volume=L^3), simulations with periodic boundary conditions can only access discrete momentum values in units of $(2\pi/L)$ [**RED** points]

  ➡ Lattice data sparse and noisy in low-Q^2 region where contribution to a_μ^{HVP} is largest

- Introduce twisted B.C. for fermion fields to access momenta below $(2\pi/L)$ [**BLUE** points]



Padé approximants [Aubin et al., Phys.Rev. D86 (2012) 054509]

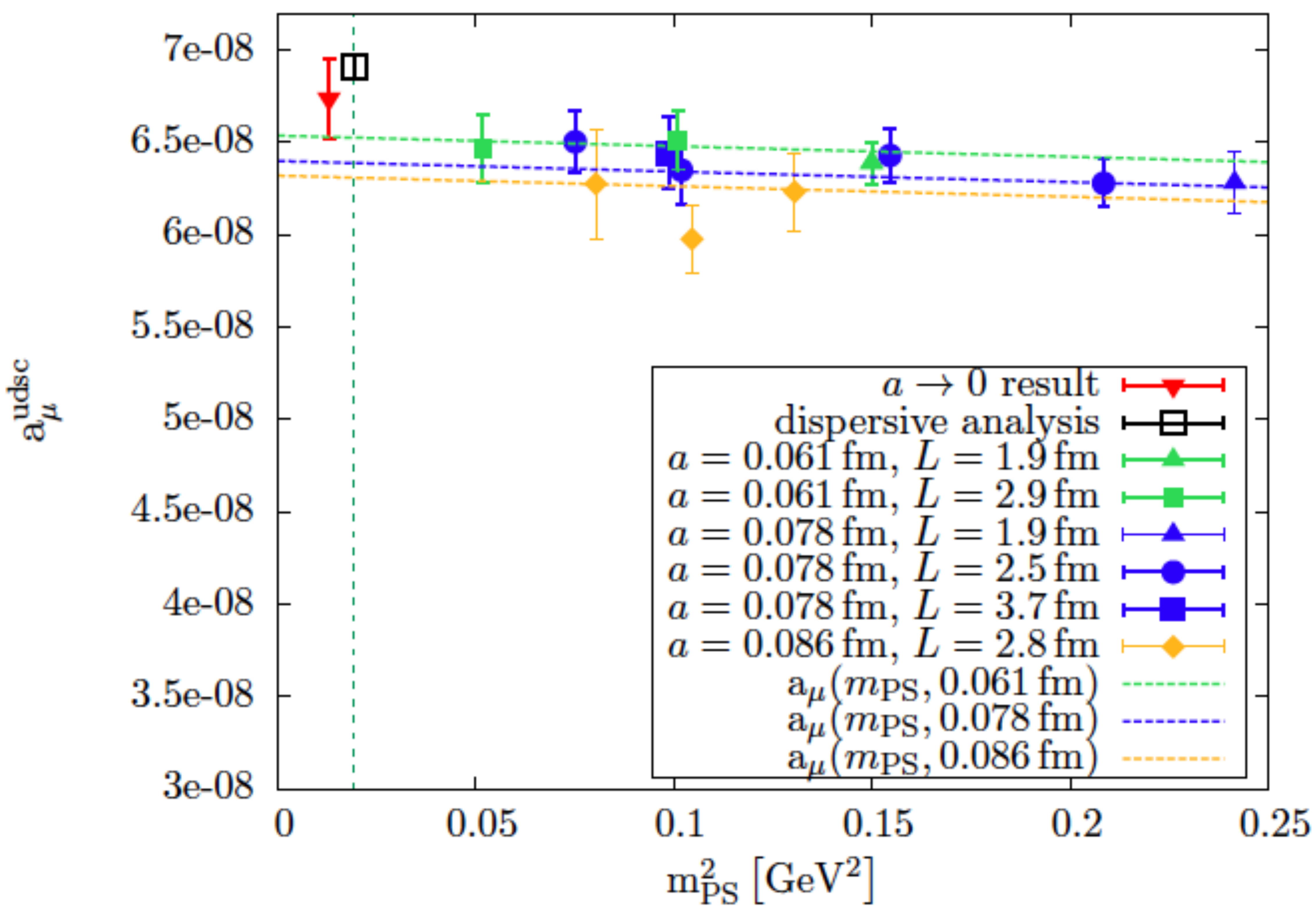
- Even with twisted B.C., contributions to a_μ^{HVP} from $\Pi(Q^2)$ for momenta below the range directly accessible in current lattice simulations are significant

  ➡ Must assume functional form for Q^2 dependence and extrapolate $Q^2 \to 0$

- Use model-independent fitting approach based on analytic structure of $\Pi(Q^2)$ to eliminate systematic associated with vector-meson dominance fits

First four-flavor result (PRELIMINARY)

[Grit Hotzel for ETM Collaboration, Lattice 2013]



$$a_\mu^{HVP} = 6.74(21)_{stat}(18)_{syst} \times 10^{-8}$$

- ✦ Error estimate does not yet include sea-quark mass mistuning (small) or quark-disconnected contributions (as much as ~10%?)

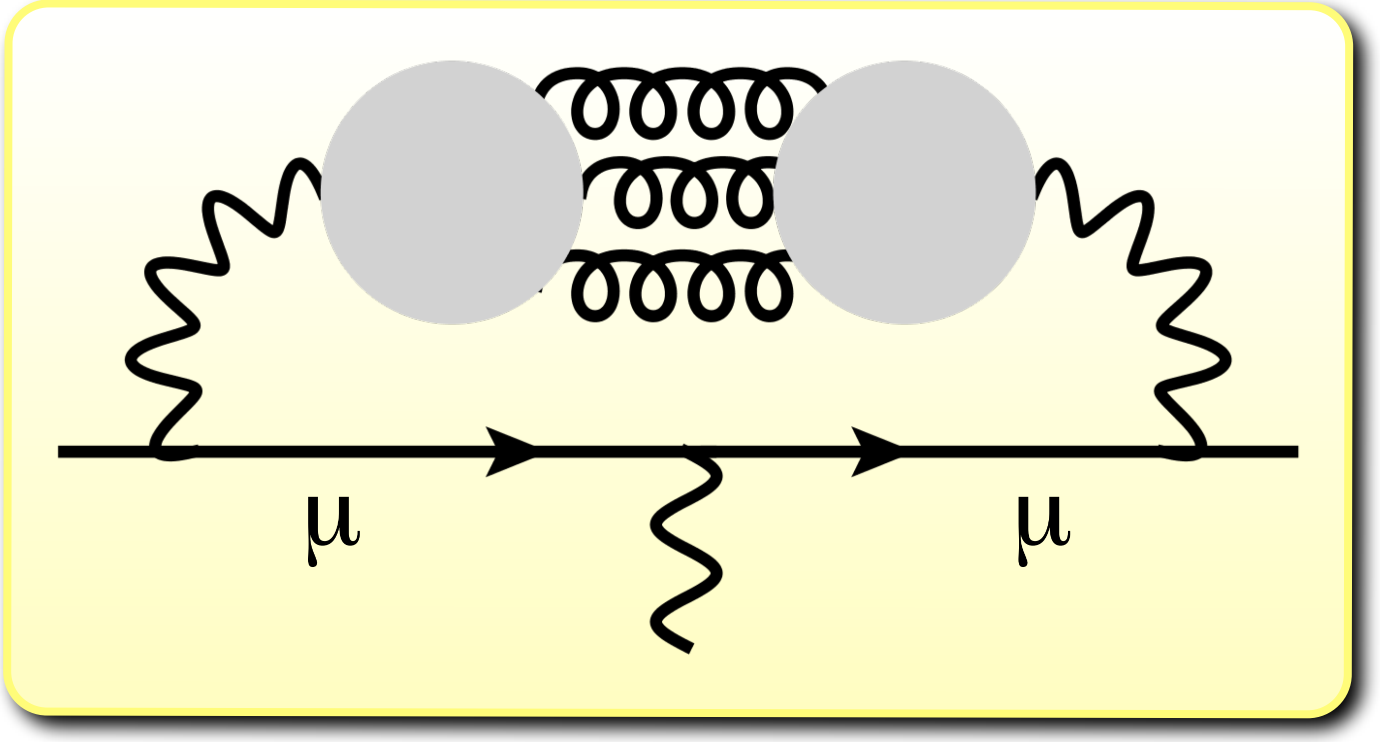
Remaining issues

(1)Chiral extrapolation

- Simulations at the physical pion mass are underway

(2)Quark-disconnected contributions

- Noisy and difficult to compute with good statistical accuracy
- Chiral Perturbation Theory estimate suggests that they could be of O(10%) [Della Morte & Jüttner, JHEP 1011 (2010) 154]



(3)Charm sea-quark contributions

- Simulations with dynamical charm quarks are underway
- Perturbative QCD estimate suggests that charm contribution could be comparable to entire size of HLbL or EW contributions [Bodenstein *et al.*, PRD85 (2012) 014029]

(4)Isospin breaking

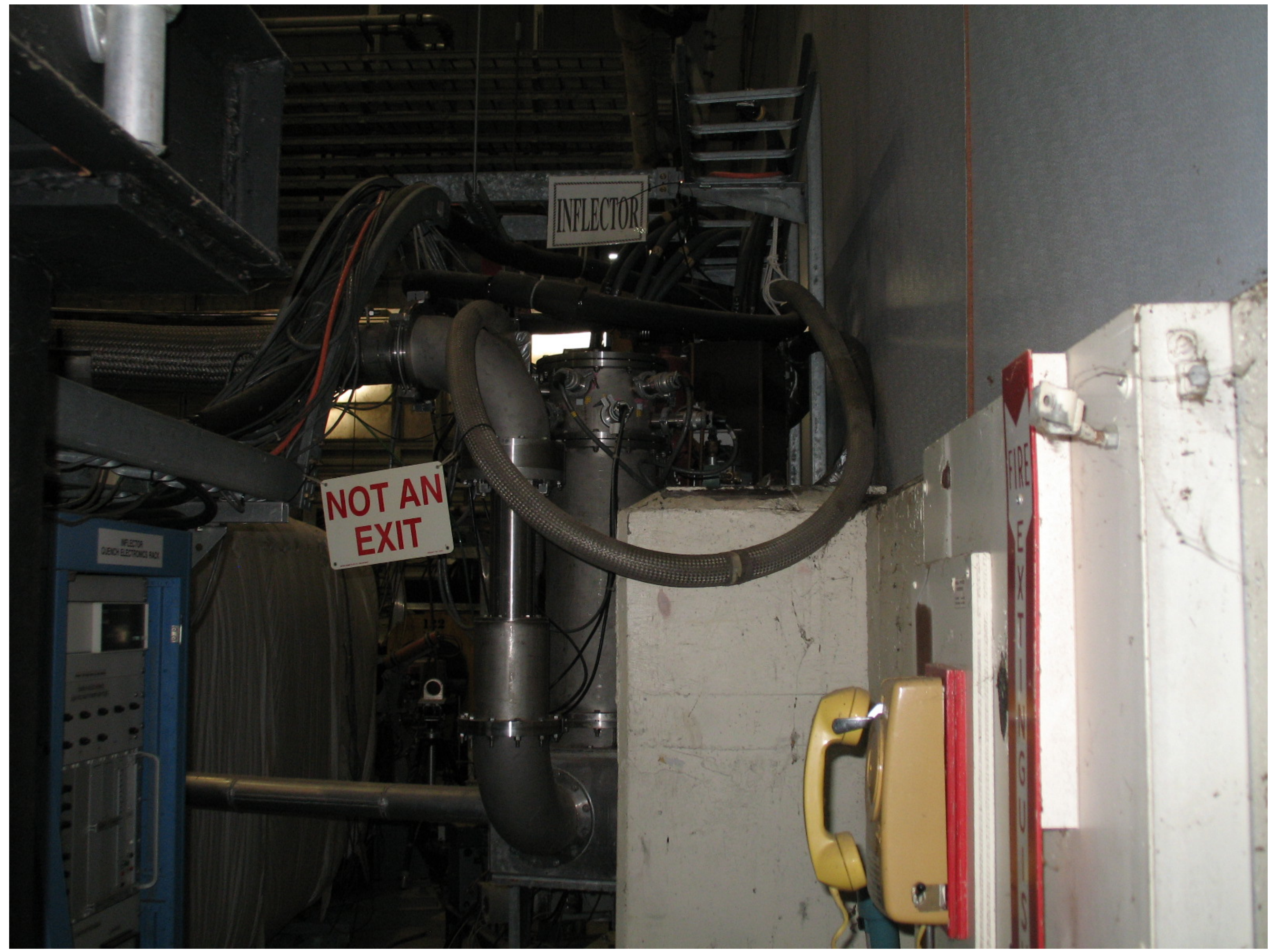
- Will become relevant once the precision reaches the percent level

✦ Can all be addressed straightforwardly with sufficient computing resources

Conclusions and Outlook

Where is the way out?

- Models are faced with several obstacles (my opinion):
  - solidification possible, but …
  - E989 accuracy cannot be met.
- Leaves lattice gauge theory:
  - QCD for HVP;
  - QCD+QED for HL×L.



Needs for *g*–2

ASK

- Let's assume that the monkey-on-your-back topology can be safely neglected (likely).
- Let's assume that the HVP to needed precision comes along with HL×L (not obvious).
- Let's focus on QCD+QED: easier to forecast one number than many form factors.
- BCHIYY find 100% error using 10^{-2} Tflop s^{-1} yr, and planning "reasonable" calculation with 10 Tflop s^{-1} yr. Target 10% (5%) needs—naïvely—a factor of 100 (400) more computing:
    - 1–5 Tflop s^{-1} yr needed.
- *Caveats*: with 100% error it is hard to foresee obstacles both surmountable and unsurmountable. Estimate is, thus, more likely to be over-pessimistic or over-optimistic than accurate.

Resources for g–2

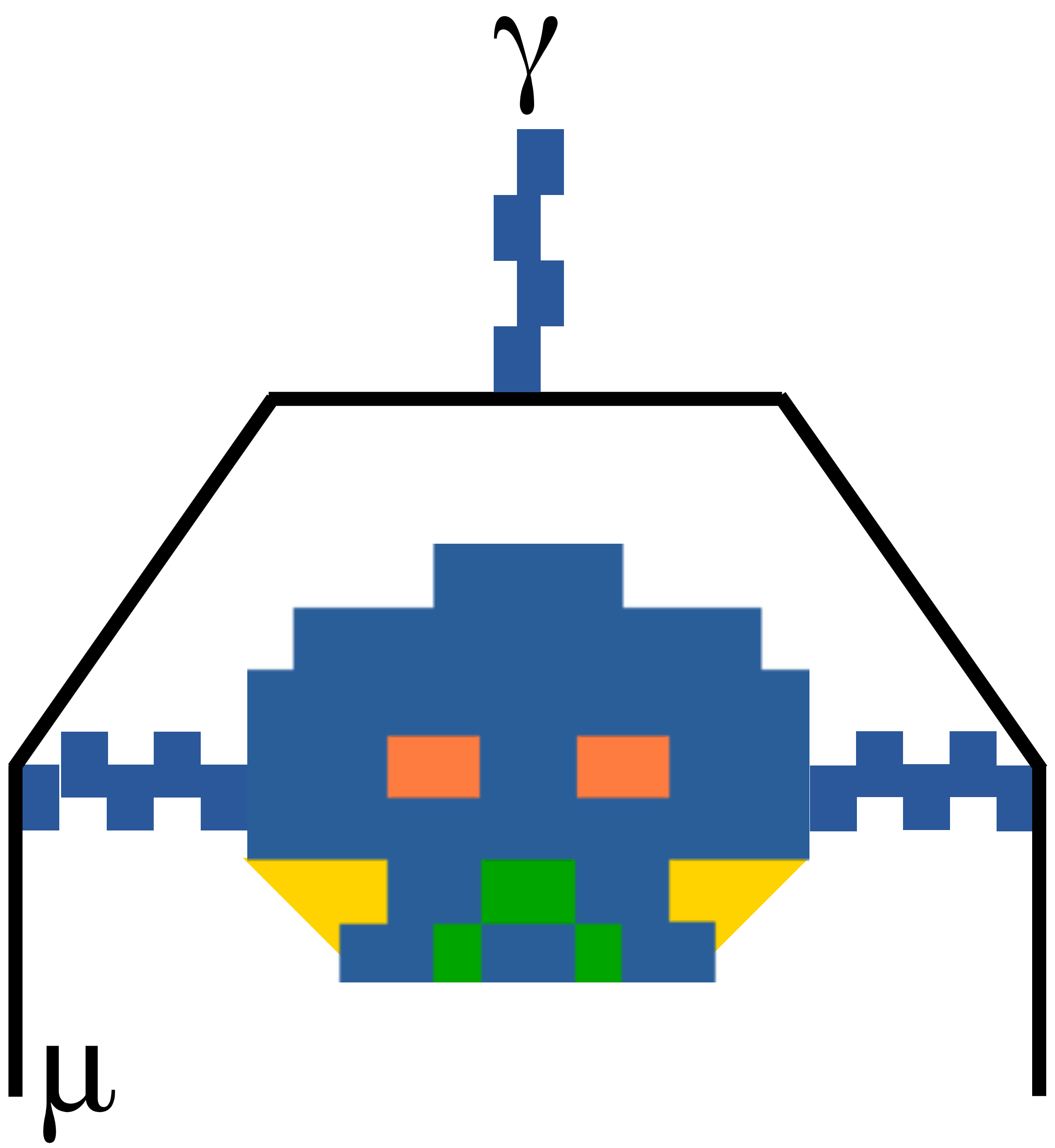
ASK

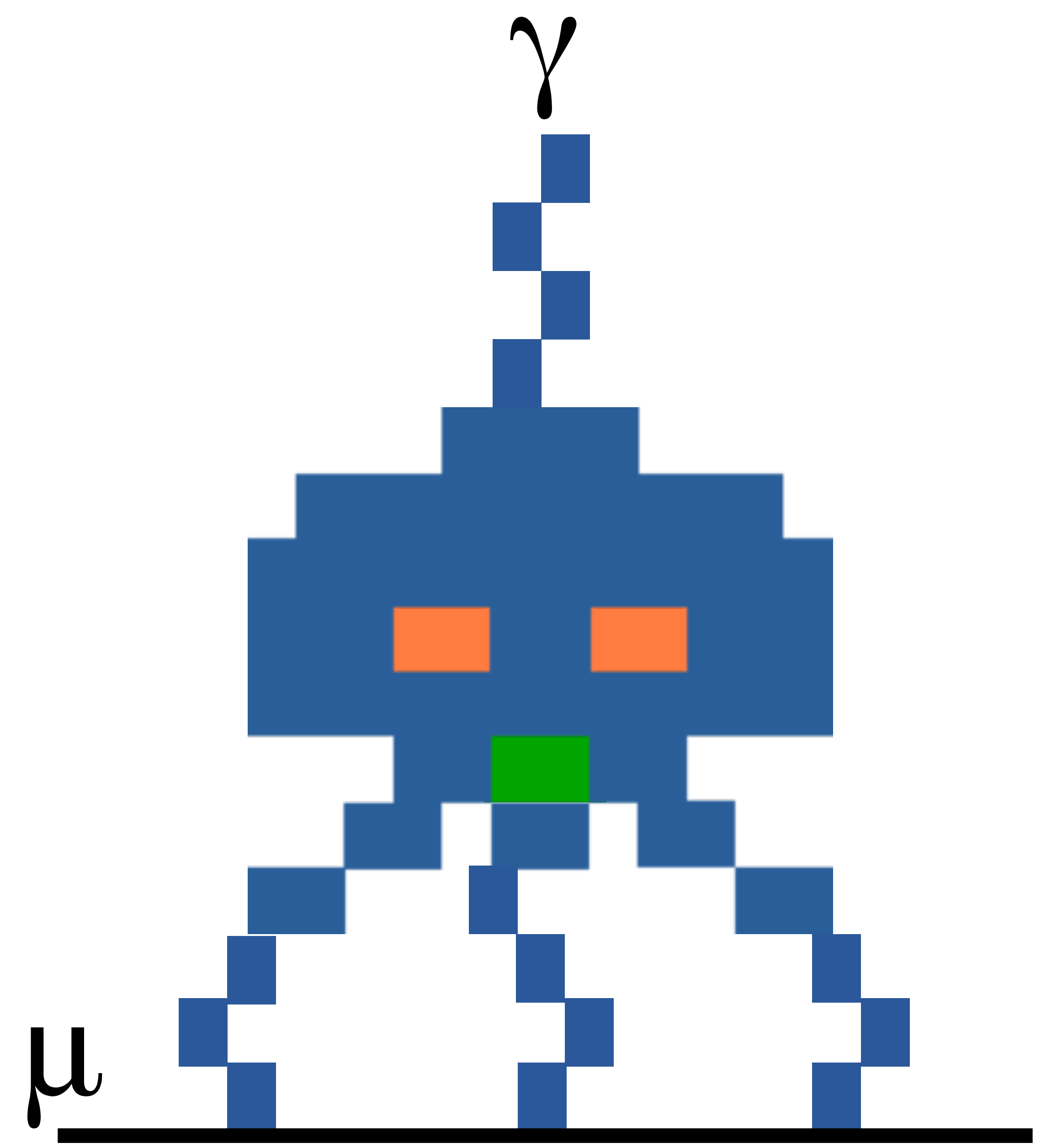
- “Luminosity” formula: resource = f_{g-2} × budget × Moore’s Law; f_{g-2} = fraction for g–2:
  - USQCD Moore’s Law: $2^{t/1.6}$ Tflop s^{-1} ($M)$^{-1}$; (now t = years since 2005.09)
  - USQCD budget experience: 2.9×$2^{t/10.5}$ $M yr^{-1}; (omits Tea Party effects)
  - TB *et al.* are increasing f_{g-2} from 10^{-4} to 10^{-2}.
- Predict resource of 5 Tflop s^{-1} yr in 2016.
- Coincides with forecast of computing need.
- Several groups engaged: perhaps even human resource will be available.

Two-Sentence Summary

- Lattice QCD will compute HVP on the timescale of E989, …
  - … first weighing in on difference between e^+e^- and τ decay, and …
  - … later replacing them & hitting the target set by E989.
- Lattice QCD is the only foreseeable way to solidify and, eventually, reduce the uncertainty in HL×L, but …
  - … it is a research project, not yet a programmatic calculation.

Thank you for your attention!


γ
μ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Extras