

The Cosmic Microwave Background: How It Works

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Academic Lecture Series

Fermilab

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General Relativity

- Metric geometry

$$ds^2 = g_{\mu\nu} dx_\mu dx_\nu \quad g_{\mu\nu} = \begin{pmatrix} g_{\eta\eta} & g_{\eta x} & g_{\eta z} & g_{\eta z} \\ g_{\eta x} & g_{xx} & g_{xy} & g_{xz} \\ g_{\eta z} & g_{xy} & g_{yy} & g_{yz} \\ g_{\eta z} & g_{xz} & g_{yz} & g_{zz} \end{pmatrix}$$

- 10 free functions reduced by 4 to 6 by coordinate freedom
- Can decompose according to helicity (2scalar+2vector+2tensor)

$$g_{\mu\nu} = a[\eta]^2 (\bar{g}_{\mu\nu} + h_{\mu\nu}^S + h_{\mu\nu}^V + h_{\mu\nu}^T) \quad dt = a[\eta] d\eta$$

$$\bar{g}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad h_{\mu\nu}^S = -2 \begin{pmatrix} \Phi & 0 \\ 0 & \Psi I \end{pmatrix} \quad h_{\mu\nu}^V = \begin{pmatrix} 0 & \mathbf{h}^\perp \\ \mathbf{h}^\perp & \mathbf{0} \end{pmatrix} \quad h_{\mu\nu}^T = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{H}^{\perp\text{tr}} \end{pmatrix}$$

$$\nabla \cdot \mathbf{h}^\perp = 0 \quad \nabla \cdot \mathbf{H}^{\perp\text{tr}} = \text{Tr}[\mathbf{H}^{\perp\text{tr}}] = 0$$

- Dynamics: Einstein's Eq's: $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

scalar	$\nabla^2 \Psi$	$= 4\pi a^2 G \left(\delta\rho - 3 \frac{a'}{a} \nabla^{-2} (\nabla \cdot \mathbf{S}) \right)$	elliptical
scalar	$\nabla^2 (\Phi - \Psi)$	$= -12\pi a^2 G \frac{a'}{a} \nabla^{-2} \nabla \cdot (\nabla \cdot \Pi^{\text{tr}})$	elliptical
vector	$\nabla^2 \mathbf{h}^\perp - 6 \left(\frac{a'}{a} \right)^2 \mathbf{h}^\perp$	$= 16\pi a^2 G \mathbf{S}^\perp$	elliptical
tensor	$\ddot{\mathbf{H}}^{\perp\text{tr}} + 2 \frac{a'}{a} \dot{\mathbf{H}}^{\perp\text{tr}} - \nabla^2 \mathbf{H}^{\perp\text{tr}}$	$= 8\pi a^2 G \Pi^{\perp\text{tr}}$	hyperbolic

$$T^{\mu\nu} = \begin{pmatrix} \rho & \mathbf{S} \\ \mathbf{S} & \Pi \end{pmatrix}$$

Cosmic Relics:

- Photons: The 2.725K CMBR
- Neutrinos: (difficult to see directly) expect $T_\nu=1.955\text{K}$
- Baryons: (origin of baryon anti-baryon asymmetry unknown)
- Dark Matter: (origin unknown)
- Scalar Perturbation: inhomogeneities
- ?Tensor Perturbations: gravitational radiation
- Dark Energy (origin unknown – only important recently?)

Λ CDM Model

Thermal:

$$\rho_{\text{tot}} \cong \frac{1}{c^2} \Lambda + \frac{3 H_0^2}{8 \pi G} \frac{\Omega_{m0}}{a^3} + \frac{\pi^2}{30} \frac{(k_B T_{\gamma 0})^4}{(\hbar c)^3 c^2} \frac{2 + \frac{7}{8} \left(\frac{4}{11}\right)^{\frac{4}{3}} (2 N_{\nu}^{\text{eff}})}{a^4}$$

$$\rho_{\text{tot}} \cong -\Lambda + \frac{\pi^2}{90} \frac{(k_B T_{\gamma 0})^4}{(\hbar c)^3} \frac{2 + \frac{7}{8} \left(\frac{4}{11}\right)^{\frac{4}{3}} (2 N_{\nu}^{\text{eff}})}{a^4}$$

$$\rho_b \cong \frac{3 H_0^2}{8 \pi G} \frac{\Omega_{b0}}{a^3} \quad \Omega_{b0}^{\text{BBN}} h^2 = 0.0215 \pm 0.0025 \text{ (PDB)} \quad h \equiv \frac{H_0}{100 \text{ km/sec/Mpc}}$$

$$T_{\gamma b} \cong \frac{T_{\gamma 0}}{a^{\sqrt{\frac{1}{2} g_*[a]}}} \quad T_{\gamma 0} = 2.72548 \pm 0.00057 \text{ K}$$

Inhomogeneities:

$$\Phi[\mathbf{x}, t] = \int d^3 \mathbf{k} e^{i \mathbf{k} \cdot \mathbf{x}} \tilde{\Phi}[\mathbf{k}, t] \quad \lim_{t \rightarrow 0} \langle \tilde{\Phi}[\mathbf{k}, t] \tilde{\Phi}[\mathbf{k}', t] \rangle = A_S k^{n_S - 4} \delta^{(3)}[\mathbf{k} - \mathbf{k}']$$

$$\mathbf{H}^{\perp \text{tr}}[\mathbf{x}, t] = \int d^3 \mathbf{k} e^{i \mathbf{k} \cdot \mathbf{x}} \tilde{\mathbf{H}}^{\perp \text{tr}}[\mathbf{k}, t] \quad \lim_{t \rightarrow 0} \langle \tilde{\mathbf{H}}^{\perp \text{tr}}[\mathbf{k}, t] \cdot \tilde{\mathbf{H}}^{\perp \text{tr}}[\mathbf{k}', t]^{\dagger} \rangle = A_T k^{n_T} (I - \hat{\mathbf{k}} \otimes \hat{\mathbf{k}}) \delta^{(3)}[\mathbf{k} - \mathbf{k}']$$

Parameters: $T_{\gamma 0}, H_0, \Lambda, \Omega_{m0}, \Omega_{b0}, \Omega_0, N_{\text{eff}}, A_S, A_T, n_S, n_T, T$

How to Describe the CMBR?

Microscopic Description

$$E_a[\mathbf{x}, t] \propto \int d\nu e^{i2\pi\nu t} \int d^2\hat{c} e^{i2\pi\nu \hat{c} \cdot \mathbf{x}} \tilde{E}_a[\hat{c}, \nu]$$

in a small frequency bin:

$$\left\langle \begin{pmatrix} \tilde{E}_x & \tilde{E}_x^* & \tilde{E}_x & \tilde{E}_z^* \\ \tilde{E}_z & \tilde{E}_x^* & \tilde{E}_z & \tilde{E}_z^* \end{pmatrix} \right\rangle \propto \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}$$

- I intensity
- Q, U linear polarization
- V circular polarization

By definition: I, Q, U, V real

Schwartz Inequality: $I^2 \geq Q^2 + U^2 + V^2$

Elliptically Polarized: $I^2 = Q^2 + U^2 + V^2$

Linearly Polarized: $I^2 = Q^2 + U^2 \quad V=0$

Circularly Polarized: $I=|V| \quad Q=U=0$

Unpolarized: $Q=U=V=0$

Light are a collection of electromagnetic waves.

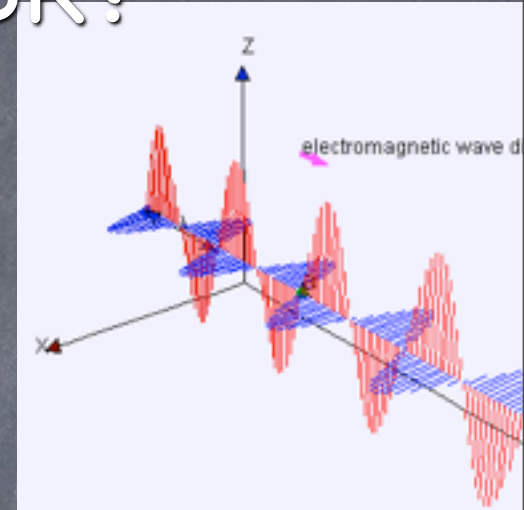
- There could in principle be a lot of information in all the detailed correlations of the EM field.
- The interesting information is usually only in the time averaged 2nd moments of the E fields.

Expectation:

CMBR slightly linearly polarized

$$I^2 \gg Q^2 + U^2 \gg V^2$$

Rees (1968)



How to Describe the CMBR?

Macroscopic Description

On cosmological length and times-scales (millions to billions of light-years):

$$I[\hat{c}, \nu, \mathbf{x}, t], \quad Q[\hat{c}, \nu, \mathbf{x}, t], \quad U[\hat{c}, \nu, \mathbf{x}, t], \quad V[\hat{c}, \nu, \mathbf{x}, t]$$

- as we shall see $V=0$ is a good approximation.

- Spatial Fourier transform, e.g.

$$I[\hat{c}, \nu, \mathbf{x}, t] = \sum_{\mathbf{k}} e^{i \mathbf{k} \cdot \mathbf{x}} \tilde{I}[\hat{c}, \nu, \mathbf{k}, t]$$

- Angular decomposition: spherical harmonics

$$\tilde{I}[\hat{c}, \nu, \mathbf{k}, t] = \sum_{\ell} \sum_m Y_{(\ell, h)}[\hat{c}] \tilde{I}_{(\ell, h)}[\nu, \mathbf{k}, t]$$

- For each \mathbf{k} , align "North Pole" of $Y_{(\ell, h)}$ to \mathbf{k} direction then h gives helicity

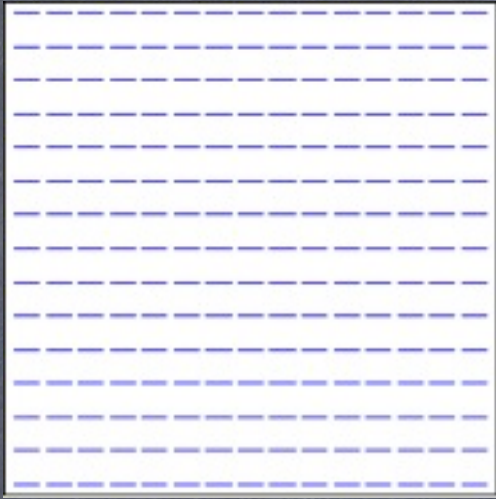
- as we shall see $I_{(\ell, h)}=0$ for $|h|>2$ is a good approximation.

- A simple $Y_{(\ell, h)}$ decomposition of Q, U is not the best!

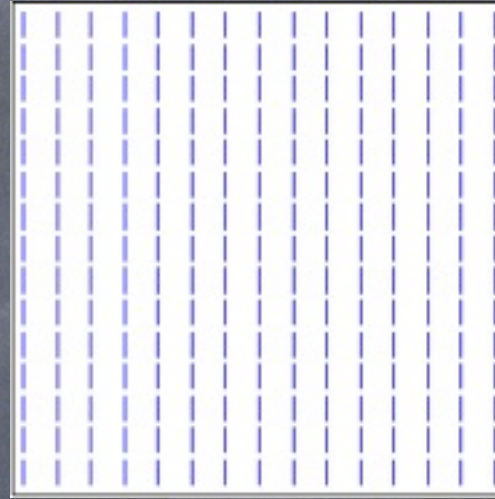
Graphical Representation of Linear Polarization

2d Symmetric Traceless Tensors

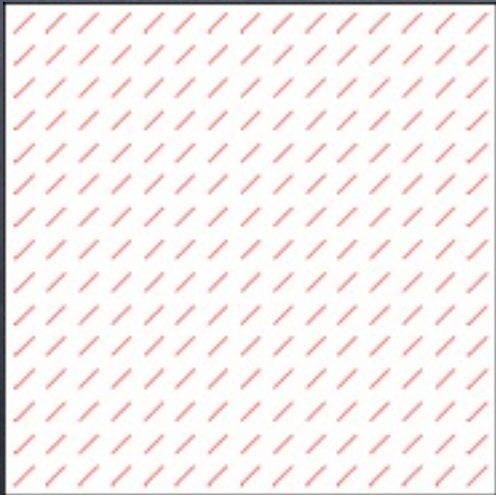
$$Q > 0$$



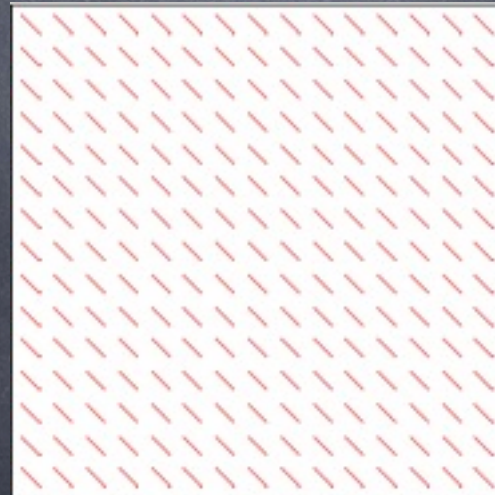
$$Q < 0$$



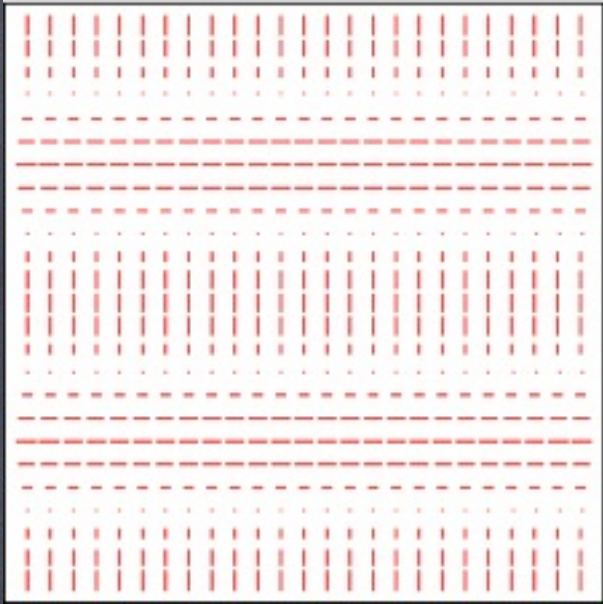
$$U > 0$$



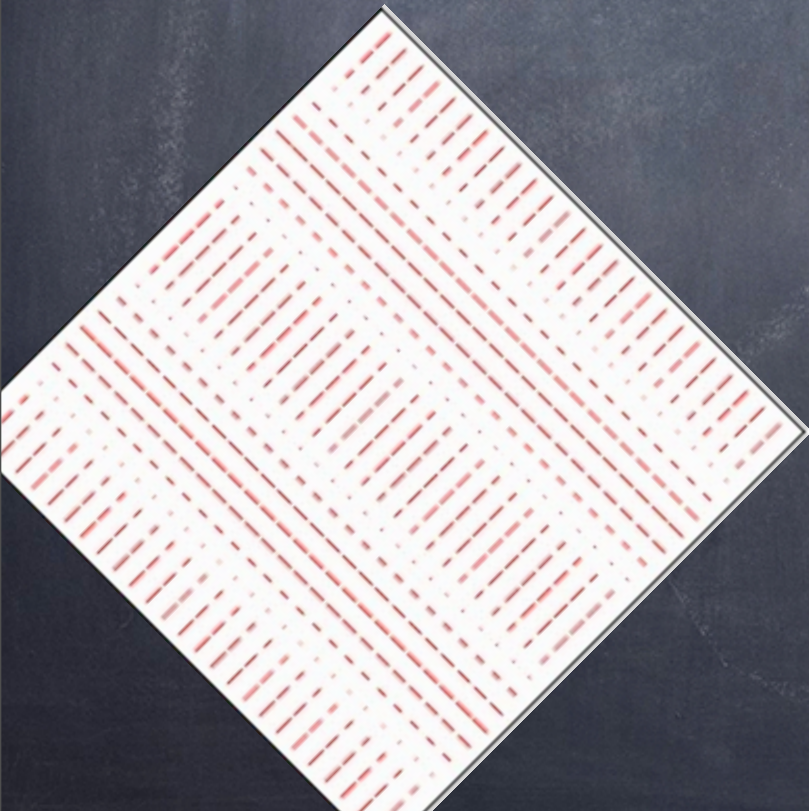
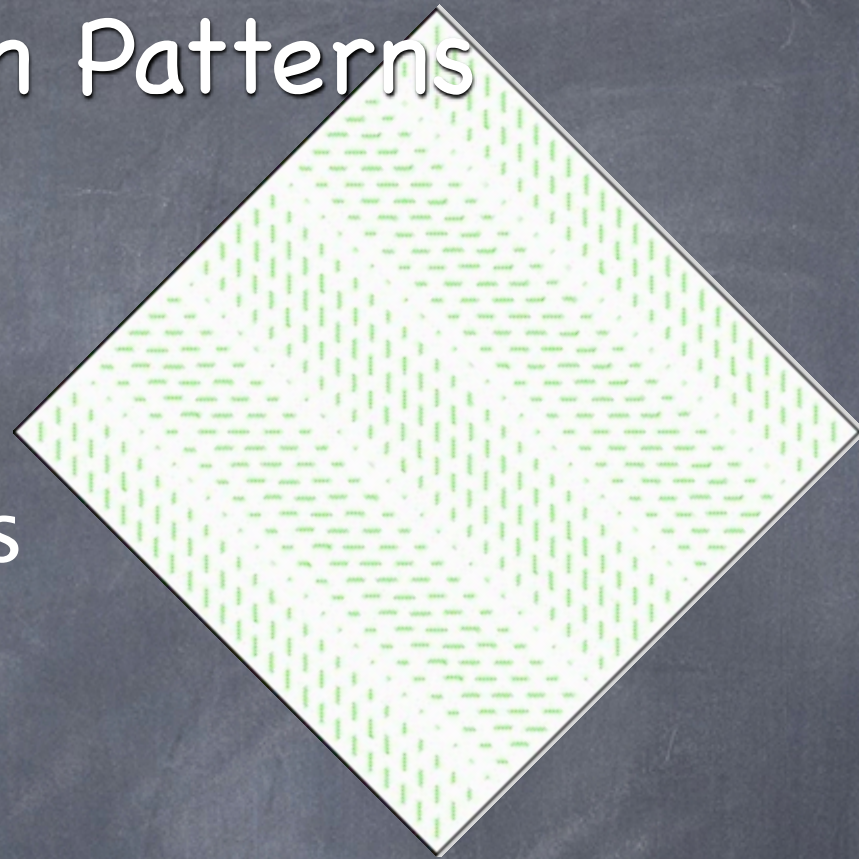
$$U < 0$$



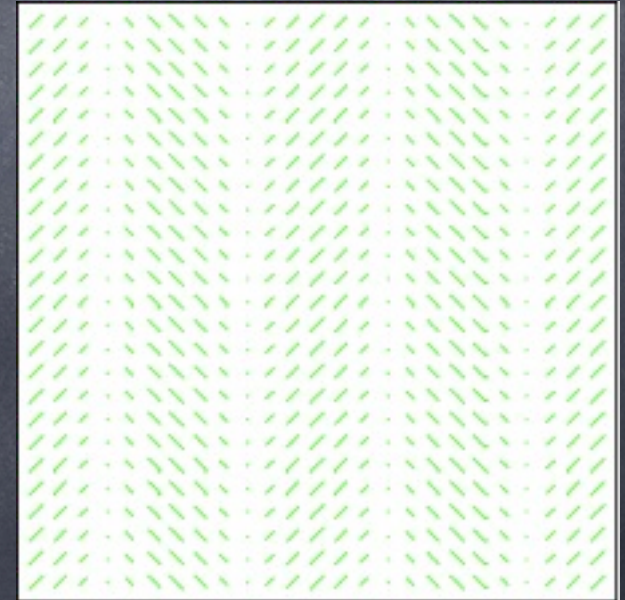
Linear Polarization Patterns



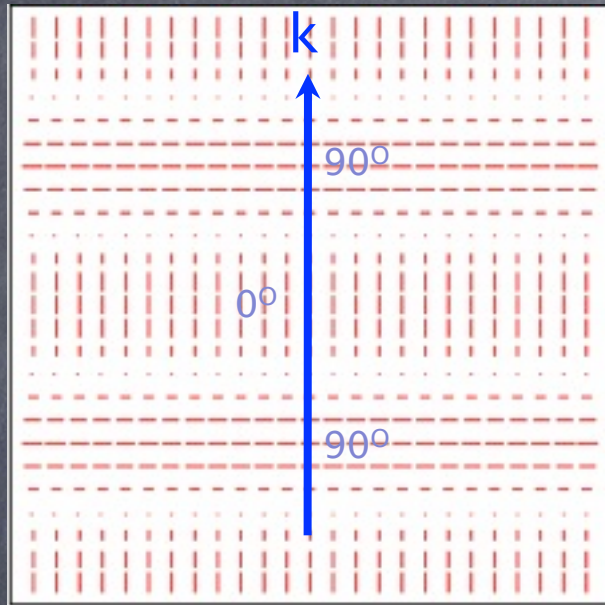
Q patterns



U patterns



Linear Polarization Patterns



0°–90° pattern

scalar pattern
gradient pattern
E-mode

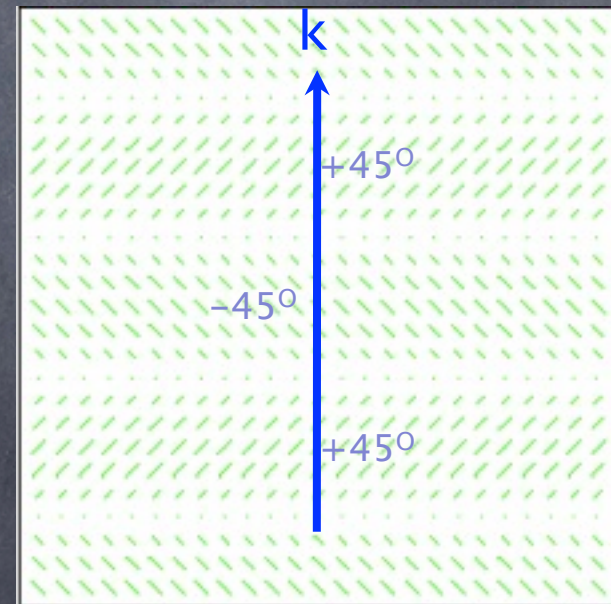
Stebbins 1996

Kaminokowski, Kosowsky, Stebbins 1997

Seljak, Zaldarriaga 1997

pseudo-scalar pattern
curl pattern
B-mode

$\pm 45^\circ$ pattern



General E- B- Mode Decomposition

- in any 2-D Riemannian manifold one has 2 covariant tensors:
 - metric g_{ab} and Levi-Civita symbol $\epsilon_{ab} = \sqrt{\text{Det}[g_{ab}]} \{ \{0,1\}, \{-1,0\} \}$
 - contracting a vector with ϵ_{ab} rotates by 90°
 - contracting a tensor with ϵ_{ab} rotates eigenvectors 45°
- starting with any (scalar) function f
 - construct corresponding E- and B- mode vectors
 - E-mode: covariant derivative: $f_{;a}$ B-mode: rotate by 90° : $f_{;b} \epsilon^b_a$
 - construct corresponding E- and B- mode traceless symmetric tensors
 - E-mode: 2nd derivative - trace: $f_{;ab} - \frac{1}{2}(\nabla^2 f) \delta_{ab}$
 - B-mode: symmetrically rotate by 45° : $\frac{1}{2}(f_{;ac} \epsilon^c_b + f_{;bc} \epsilon^c_a)$
- One can construct E-mode and B-mode tensors of any rank this way!

E- B- Mode Spherical Harmonics

- E- B- mode decomposition applied to complete scalar basis gives complete tensor basis!
- on (direction) 2-sphere use spherical harmonic basis: $Y_{(l,m)}$
- gives E- B- mode basis for symmetric traceless tensors on sphere

$$Y^E_{((l,m)ab)} \propto Y_{((l,m);ab)} - \frac{1}{2}(\nabla^2 Y_{((l,m))}) \delta_{ab} \quad Y^B_{(l,m)ab} \propto \frac{1}{2}(Y_{((l,m);ac}\epsilon^c_b + Y_{(l,m);bc}\epsilon^c_a)$$

$$Y^E_{(0,m)ab} = Y^B_{(0,m)ab} = Y^E_{(1,m)ab} = Y^B_{(1,m)ab} = 0$$

- these can be used to describe linear polarization:

$$P_{ab} = \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix}$$

$$= \sum_k e^{i k \cdot x} \sum_{\ell} \sum_h \left(\begin{pmatrix} I_{(\ell,h)} & +i V_{(\ell,h)} \\ -i V_{(\ell,h)} & I_{(\ell,h)} \end{pmatrix} Y_{(\ell,h)} + E_{(\ell,h)} Y^E_{(\ell,h)} + B_{(\ell,h)} Y^B_{(\ell,h)} \right)$$

- Equivalent formulation uses spin-weighted spherical harmonic functions $Y_{(s,l,m)}$

$$Q + i U = \sum_k e^{i k \cdot x} \sum_{\ell} \sum_h Y_{(2,\ell,h)}$$

How to Describe the CMBR?

Intensity and Units

In astronomy $I[\hat{c}, \nu, x, t]$ usually has units: ergs/cm²/sec/steradian/Hz

- recall Poynting energy flux $S = E \times B / (8\pi) = |E|^2 / (8\pi)$ (Gaussian CGS units)
- radio astronomy: often convenient to define a Rayleigh Jeans Brightness temperature

$$kT_{RJ} = \frac{1}{2}(c/\nu)^2 I$$

- this gives the thermodynamic temperature if $h\nu \ll kT$,
- theoretically it is most convenient to use the quantum mechanical occupation number

$$n^T[\nu] = \frac{1}{2}(c/\nu)^2 I / (h\nu) = kT_{RJ} / (h\nu)$$

- for a blackbody $n = n_{BB}[\nu, T] = 1 / (e^{(h\nu)/(kT)} - 1)$ N.B.
- one can multiply E, B, V by $\frac{1}{2}(c/\nu)^2 / (h\nu)$ to put them in dimensionless occupation number units: n^T, n^E, n^B, n^V

Spectral Decomposition

One may also decompose the spectrum of each component $X=I,E,B,V$:

- $n^X_{(\ell,h)}[v,k,t] = \sum_p (-1)^p/p! n^X_{(\ell,h,p)}[k,t] \partial^p n_{BB}[v,T]/\partial(\ln v)^p$
- this is a (generalized) Fokker Planck expansion about a blackbody.
- $p=0$ corresponds to a pure blackbody - only $n^T_{(0,0,0)} = 1 \neq 0$
- $p=1$ is spectral deviation from temperature shift
 - Doppler, gravitational redshifts, etc.
 - all 1st order anisotropies and polarizations will have this form
- $p=2$ arises from a mixture temperatures shifts
 - it only arises to 2nd order in perturbations theory (small)
 - Thermal Sunyaev-Zel'dovich (SZ) effect:
 - hot plasma ($v_{e,rms} = (m_p/m_e)^{1/2} v_{p,rms} = 0.1 c$) thermal

How to Describe the CMBR?

Summary

- Mode decomposed each Stokes parameter w/ “quantum numbers”
 - k spatial dependence
 - h helicity: $=0$ scalar, $=1$ vector, $=2$ tensor
 - ℓ angular wavenumber
 - p spectral mode

Statistical Description of CMBR

- Assume CMBR can be described as a realization of statistical distribution
- Assume statistical homogeneity and isotropy
- These assumptions severely restricts form of 2-point statistics
 - translation symmetry requires different \mathbf{k} modes uncorrelated
 - rotational symmetry requires different h modes uncorrelated

$$\langle n^X_{(\ell,h,p)}[\mathbf{k},t] n^Y_{(\ell',h',p')}[\mathbf{k}',t']^* \rangle = C^{XY}_{(\ell,\ell';h,p,p')}[|\mathbf{k}|;t,t'] \delta_{\mathbf{k},\mathbf{k}'} \delta_{h,h'}$$

Statistical Description of Observed CMBR

- We only get to measure CMBR from one vantage point at one time

$$\begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix} = \frac{1}{2}(c/v)^2/(hv) \sum_p (-1)^p/p! \partial^p n_{BB}[v,T]/\partial(\ln v)^p \sum_\ell \sum_m$$

$$\left(\begin{pmatrix} n^T_{(\ell,m,p)} & +i n^V_{(\ell,m,p)} \\ -i n^V_{(\ell,m,p)} & n^T_{(\ell,m,p)} \end{pmatrix} Y_{(\ell,m)} + n^E_{(\ell,m,p)} Y^E_{(\ell,m)} + n^B_{(\ell,m,p)} Y^B_{(\ell,m)} \right)$$

- where $n^X_{(\ell,m,p)} = \sum_k \sum_h D^{\ell}_{mh}[\mathbf{k}] n^X_{(\ell,h,p)}[\mathbf{k},t]$

- since the \mathbf{k} 's are isotropically distributed our sky is isotropic:

$$\int d^2\hat{\mathbf{c}} D^{\ell}_{mh}[\hat{\mathbf{c}}] D^{\ell'}_{m'h}[\hat{\mathbf{c}}] = 4\pi \delta_{\ell,\ell'} \delta_{m,m'}$$

$$\langle n^X_{(\ell,m,p)} n^Y_{(\ell',m',p')}^* \rangle = C^{XY}_{(\ell;p,p')} \delta_{\ell,\ell'} \delta_{m,m'}$$

$$\text{where } C^{XY}_{(\ell;p,p')} = \sum_k \sum_h C^{XY}_{(\ell,\ell';h;p,p')}[|\mathbf{k}|;t_0,t_0]$$

Statistical Description of Observed CMBR

- To first order we only observe $p=1$: $C^{XY}_\ell = C^{XY}_{(\ell;1,1)}$
- Circular polarization damped
- possible modes:
 - parity even: C^{TT}_ℓ , C^{EE}_ℓ , C^{BB}_ℓ , C^{TE}_ℓ
 - parity odd: C^{TB}_ℓ , C^{EB}_ℓ

Boltzmann Equation

- Dynamics determined by free-streaming and scattering

- $D_t n^X = C^X$

- $\partial_t n^X[\hat{\mathbf{c}}, v, \mathbf{x}, t] + c \hat{\mathbf{c}} \cdot \nabla n^X[\hat{\mathbf{c}}, v, \mathbf{x}, t] + (\partial_t \hat{\mathbf{c}}) \cdot \nabla_{\hat{\mathbf{c}}} n^X[\hat{\mathbf{c}}, v, \mathbf{x}, t] + (\partial_t \ln v) \partial_{\ln v} n^X[\hat{\mathbf{c}}, v, \mathbf{x}, t] = C^X[\hat{\mathbf{c}}, v, \mathbf{x}, t]$

- only Thompson (non-relativistic Compton) scattering is important!

- absorption and emission unimportant

- $d\sigma[\hat{\mathbf{c}}, \hat{\mathbf{c}}'; v, v'] / (d^2 \hat{\mathbf{c}}' dv') = 3/16\pi \sigma_T (1 + \hat{\mathbf{c}} \cdot \hat{\mathbf{c}}') \delta[v - v']$

- $S^X[\hat{\mathbf{c}}, v, \mathbf{x}, t] = 3/16\pi c \sigma_T n_e[\mathbf{x}, t] \sum_Y \int d^2 \hat{\mathbf{c}}' (1 + \hat{\mathbf{c}} \cdot \hat{\mathbf{c}}') n^Y[\hat{\mathbf{c}}', v, \mathbf{x}, t]$

- lensing term $(\partial_t \hat{\mathbf{c}}) \cdot \nabla_{\hat{\mathbf{c}}} n^X[\hat{\mathbf{c}}, v, \mathbf{x}, t]$ is 2nd order

- $\partial_t \ln v = -\hat{\mathbf{c}} \cdot \nabla \Phi + \partial_t \Phi + \hat{\mathbf{c}} \cdot \partial_t \mathbf{H}^{\perp \mp} \cdot \hat{\mathbf{c}}$ independent of v

Boltzmann Equation

$$\dot{\tilde{a}}_{(l,m)}^{\text{T}} + ik \left(f_{(l+1,m)}^{\text{T}} \tilde{a}_{(l+1,m)}^{\text{T}} + f_{(l,m)}^{\text{T}} \tilde{a}_{(l-1,m)}^{\text{T}} \right) = -\dot{\tau} \left(\left(1 - \delta_{l0} - \frac{1}{10} \delta_{l2} \right) \tilde{a}_{(l,m)}^{\text{T}} + \frac{\sqrt{3}}{20} \delta_{l2} \tilde{a}_{(2,m)}^{\oplus} \right)$$

$$\dot{\tilde{a}}_{(l,m)}^{\text{V}} + ik \left(f_{(l+1,m)}^{\text{T}} \tilde{a}_{(l+1,m)}^{\text{V}} + f_{(l,m)}^{\text{T}} \tilde{a}_{(l-1,m)}^{\text{V}} \right) = -\dot{\tau} \left(\left(1 - \frac{1}{2} \delta_{l2} \right) \tilde{a}_{(l,m)}^{\text{V}} \right)$$

$$\dot{\tilde{a}}_{(l,m)}^{\oplus} + ik \left(f_{(l+1,m)}^{\text{P}} \tilde{a}_{(l+1,m)}^{\oplus} + f_{(l,m)}^{\text{P}} \tilde{a}_{(l-1,m)}^{\oplus} + i f_{(l,m)}^{\oplus\otimes} \tilde{a}_{(l,m)}^{\otimes} \right) = -\dot{\tau} \left(\left(1 - \frac{3}{5} \delta_{l2} \right) \tilde{a}_{(l,m)}^{\oplus} + \frac{2\sqrt{3}}{5} \delta_{l2} \tilde{a}_{(2,m)}^{\text{T}} \right)$$

$$\dot{\tilde{a}}_{(l,m)}^{\otimes} + ik \left(f_{(l+1,m)}^{\text{P}} \tilde{a}_{(l+1,m)}^{\otimes} + f_{(l,m)}^{\text{P}} \tilde{a}_{(l-1,m)}^{\otimes} - i f_{(l,m)}^{\oplus\otimes} \tilde{a}_{(l,m)}^{\oplus} \right) = -\dot{\tau} \tilde{a}_{(l,m)}^{\otimes}$$

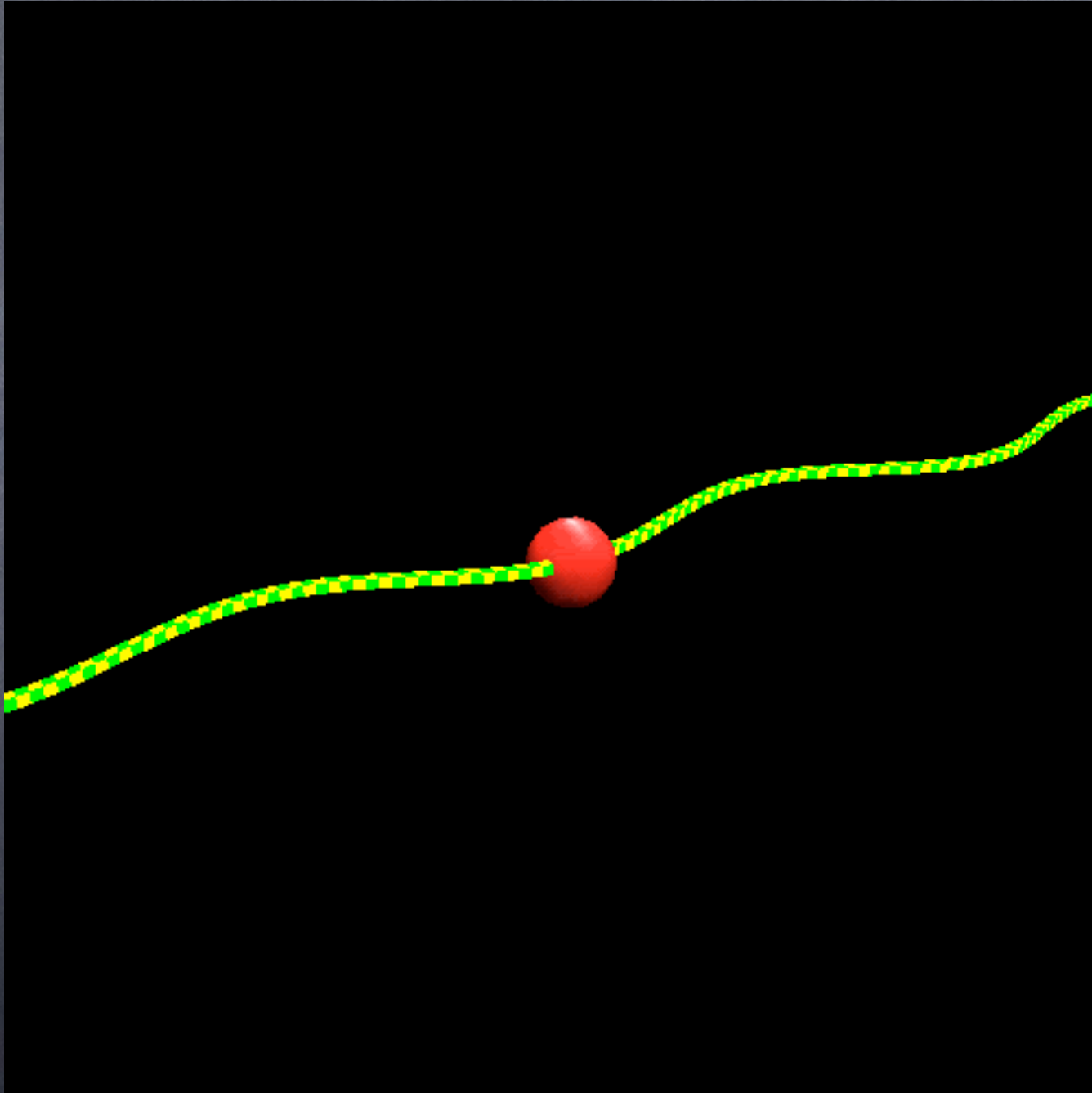
$$f_{(l,m)}^{\text{T}} = \sqrt{\frac{l^2 - m^2}{4l^2 - 1}}$$

$$f_{(l,m)}^{\text{P}} = \sqrt{\frac{(l^2 - m^2)(l^2 - 4)}{l^2(4l^2 - 1)}}$$

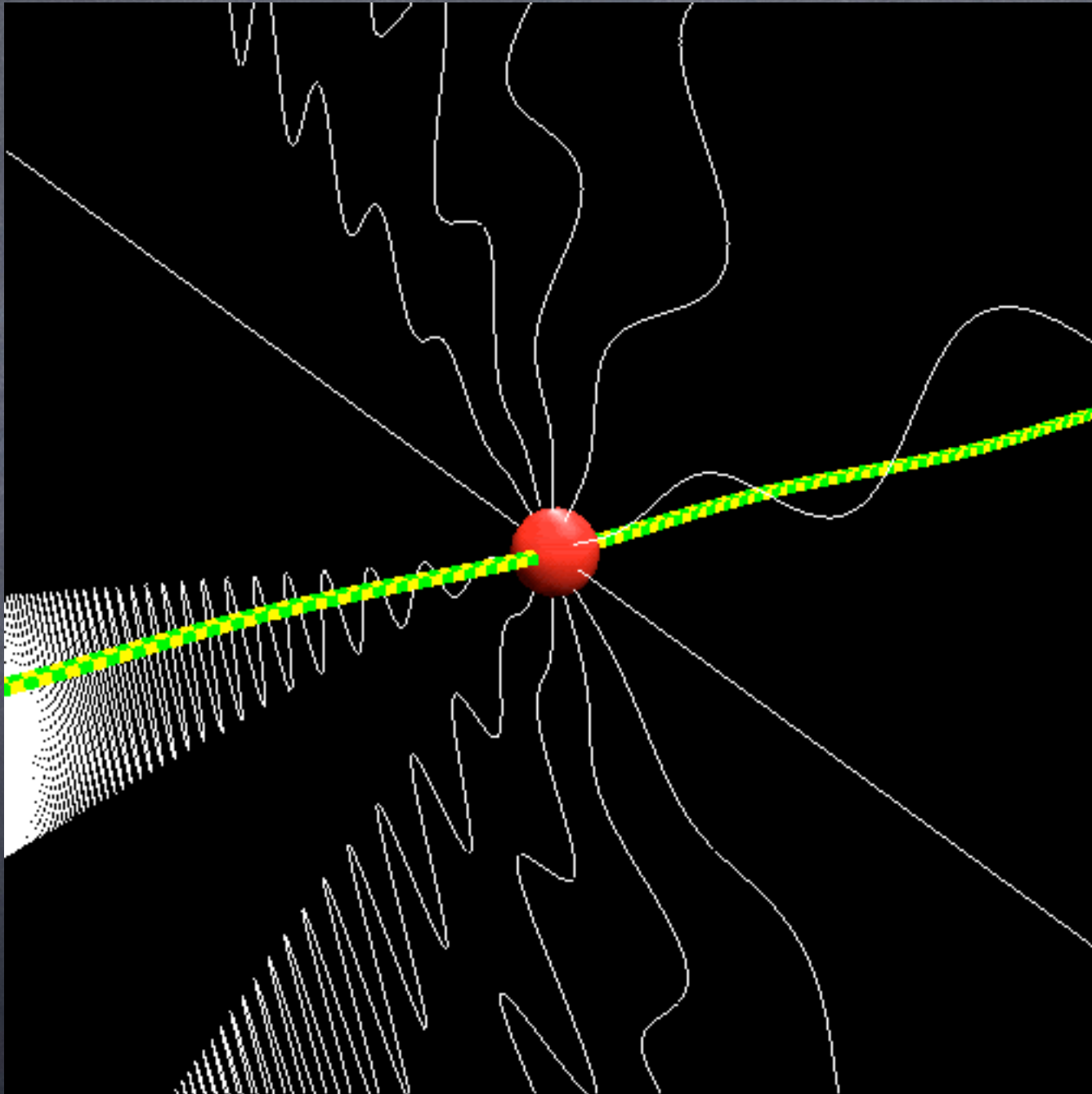
$$i f_{(l,m)}^{\oplus\otimes} = \frac{2im}{l(l+1)}$$

$$\partial_{+}\tau = c \sigma_{\text{T}} n_{\text{e}}$$

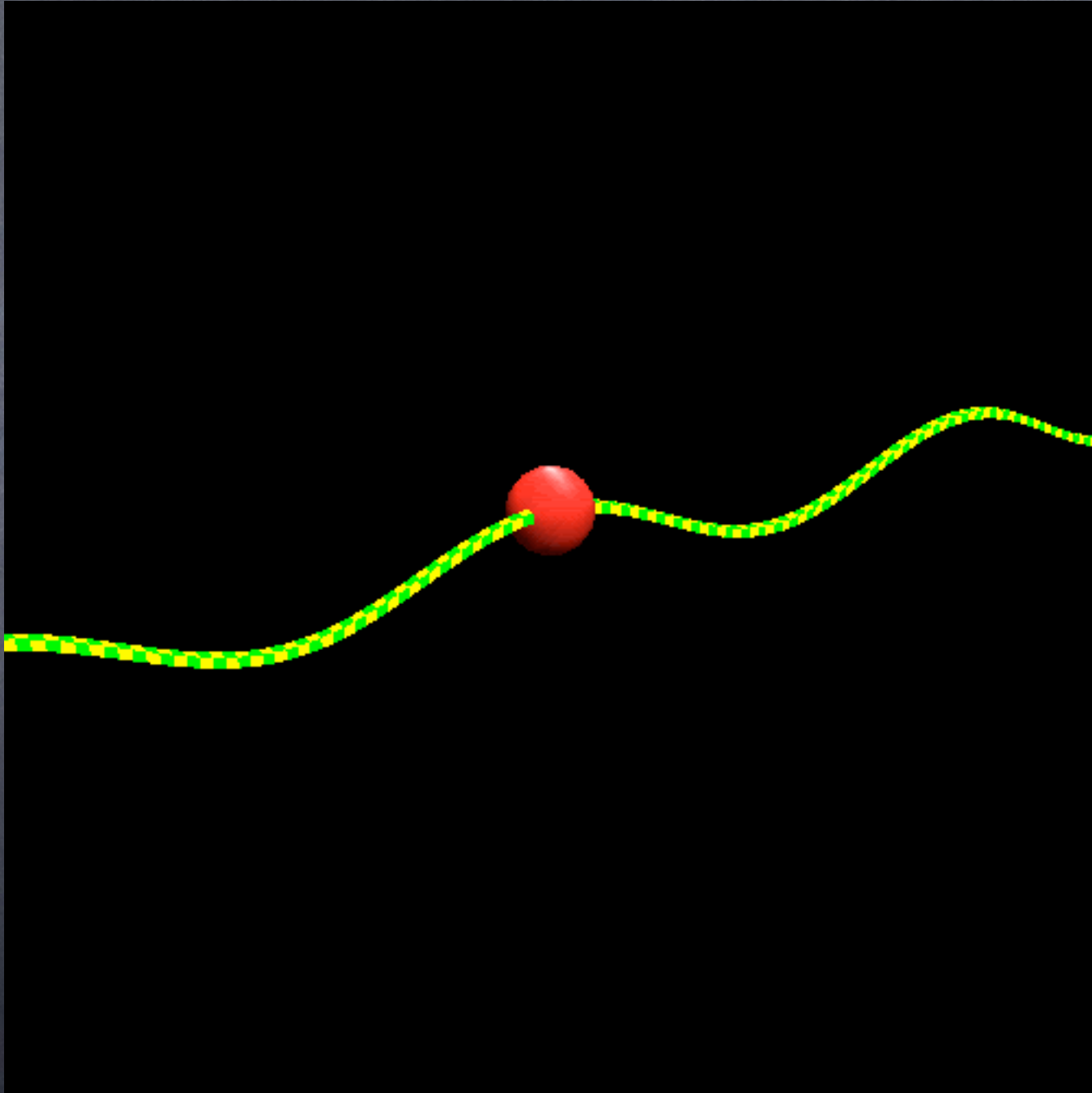
Thomson Scattering



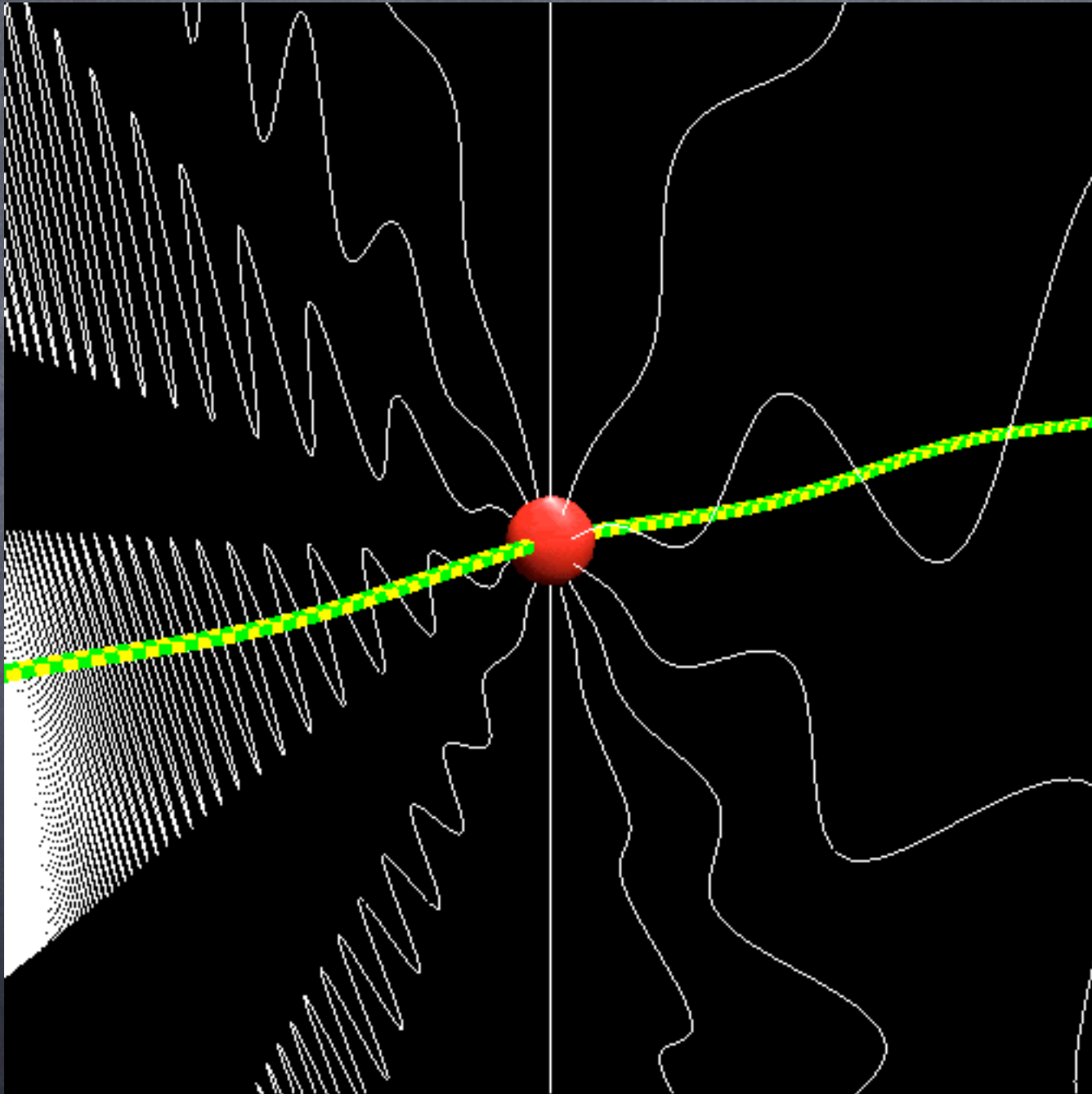
Thomson Scattering



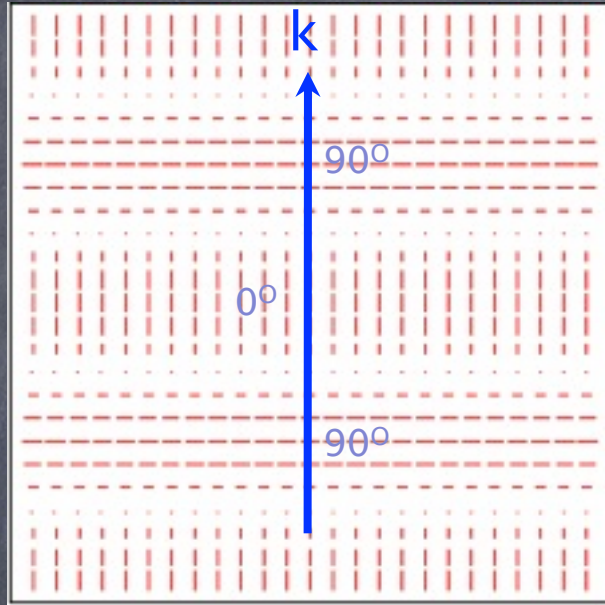
Thomson Scattering



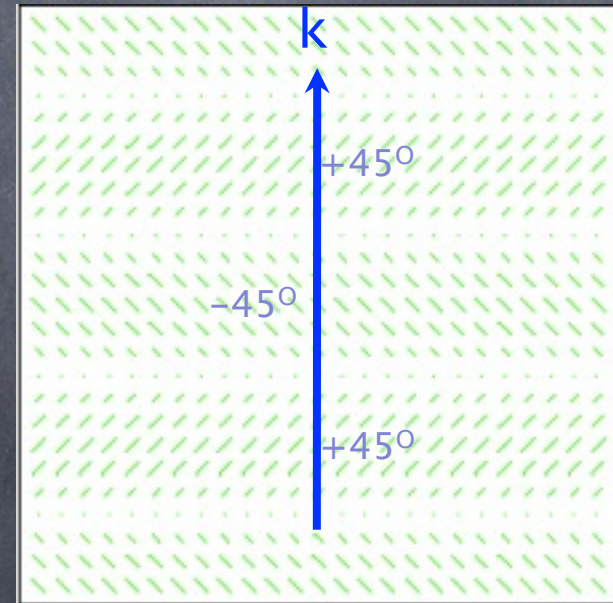
Thomson Scattering



Linear Polarization Patterns

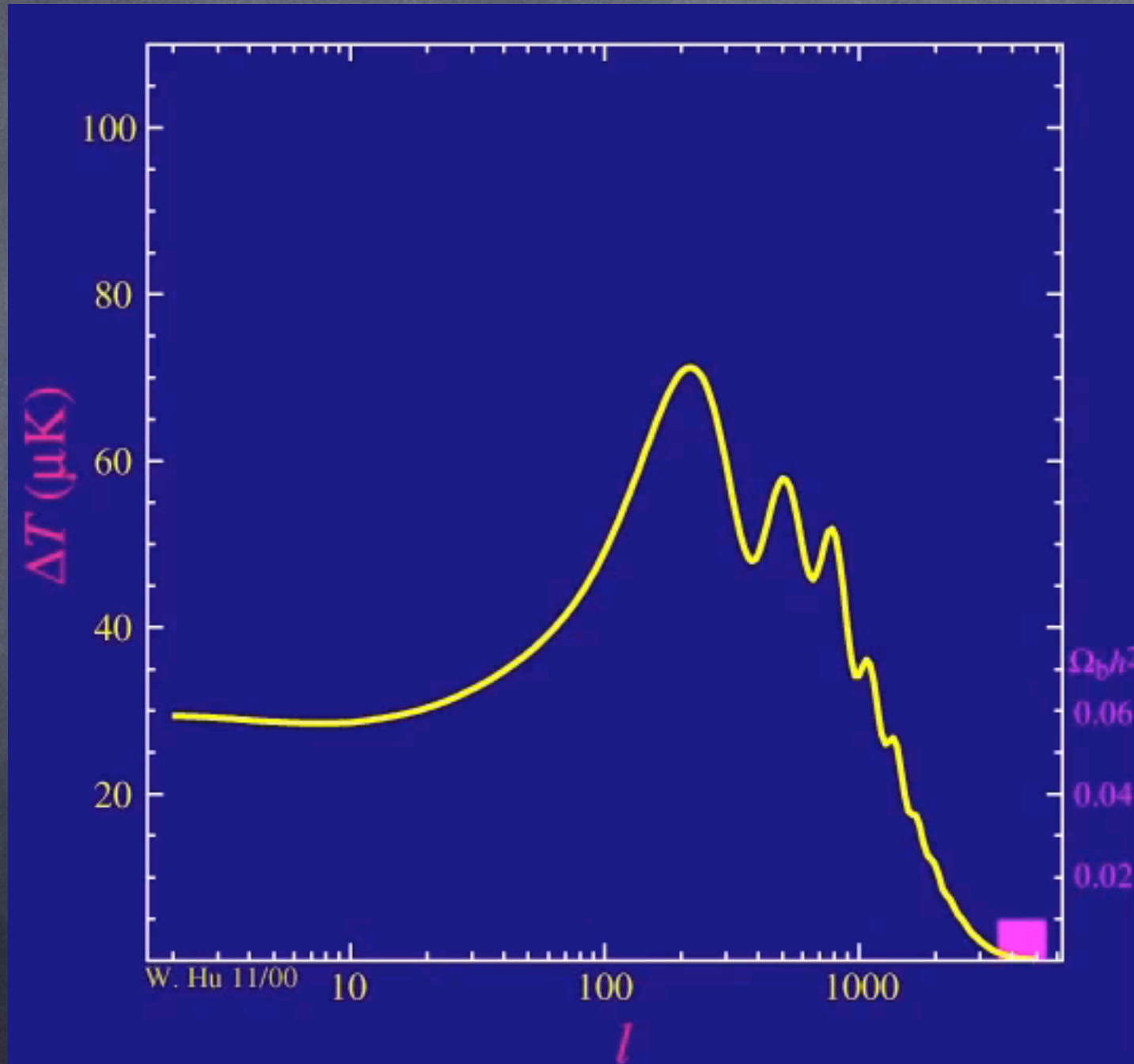


0° – 90° pattern



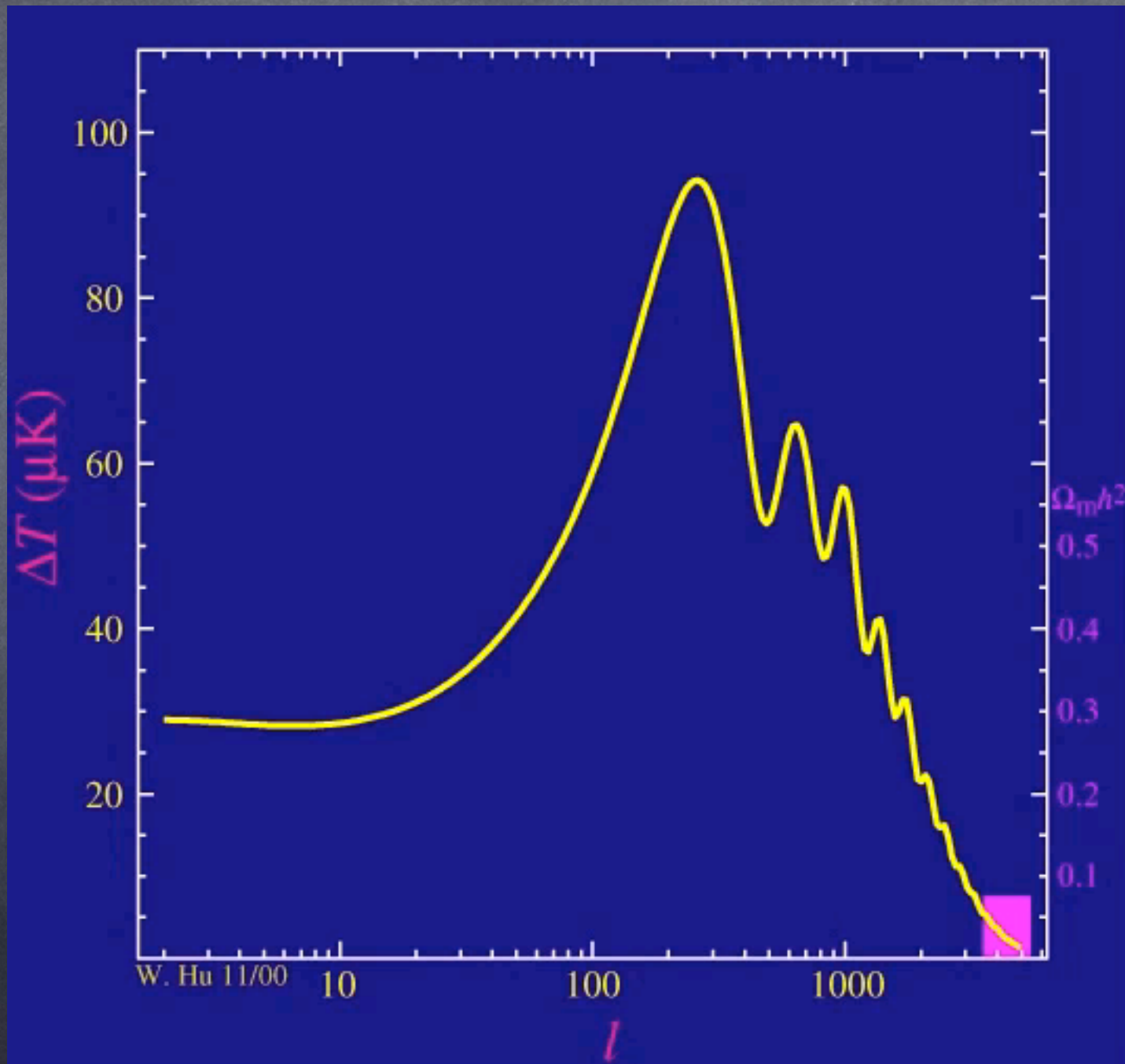
$\pm 45^\circ$ pattern

Baryon Density



<http://background.uchicago.edu/~whu/animbut/anim1.html>

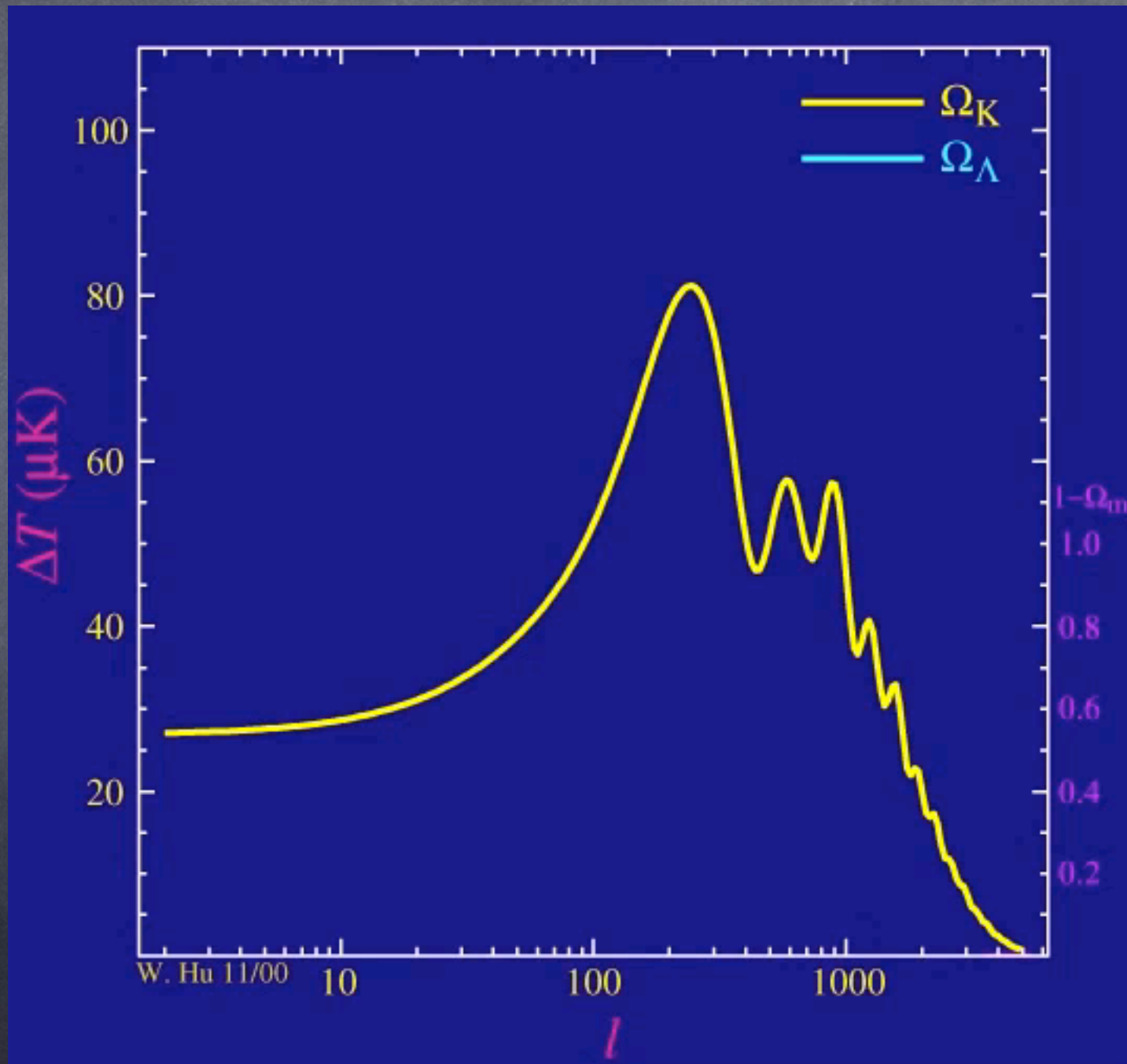
Dark Matter Density



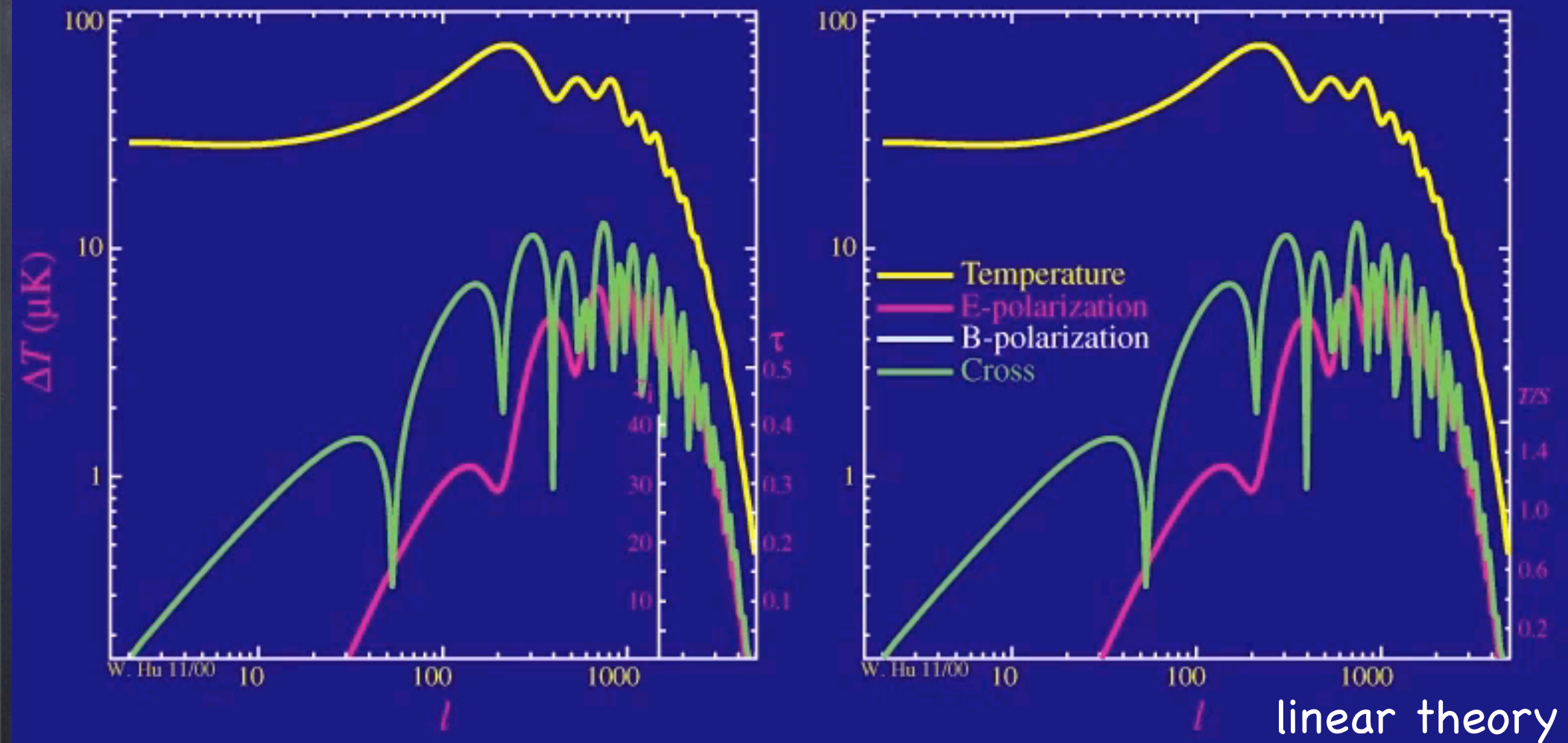
<http://background.uchicago.edu/~whu/animbut/anim2.html>

Curvature & Cosmological Constant

<http://background.uchicago.edu/~whu/animbut/anim3.html>

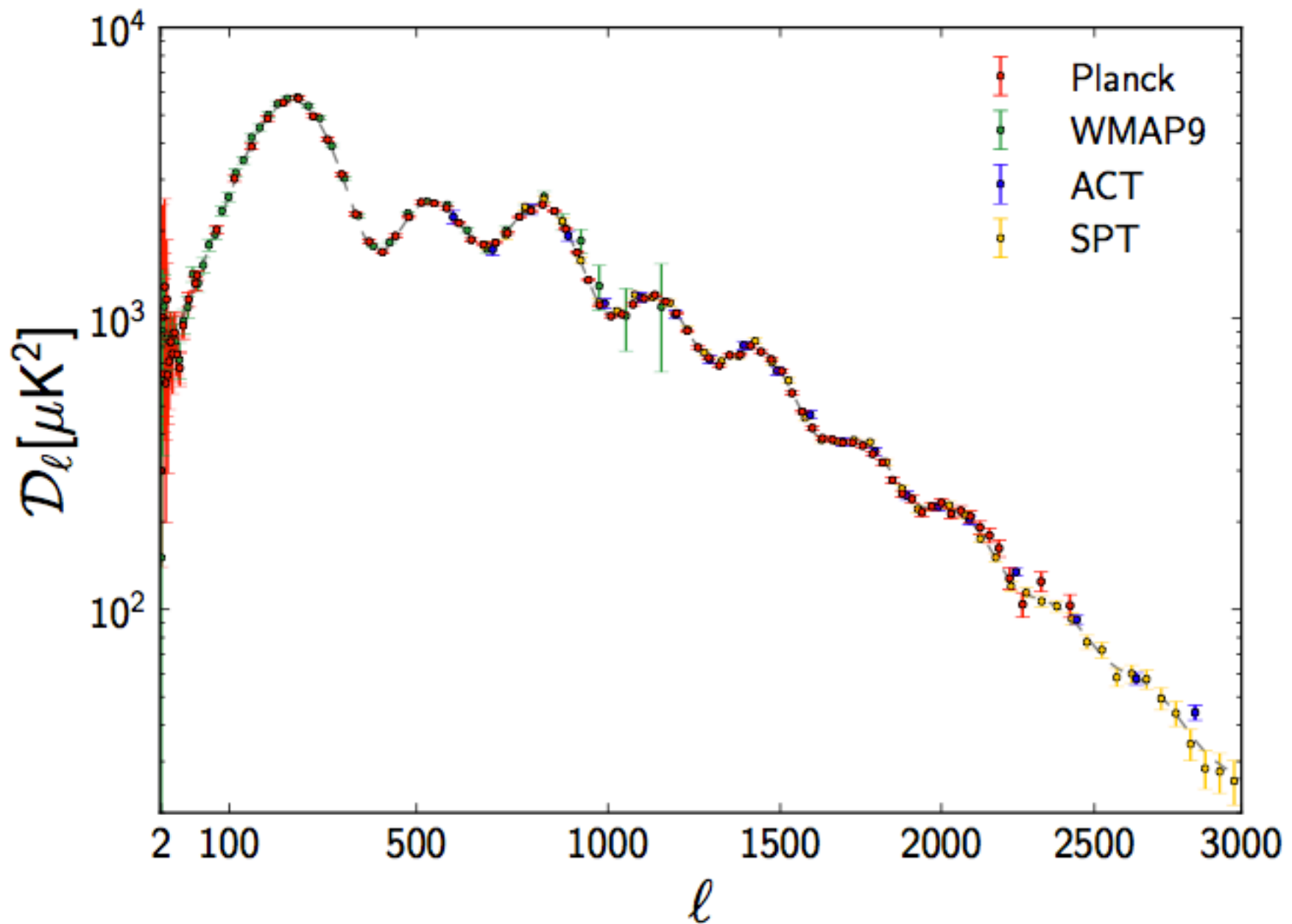


Reionization Optical Depth Tensor Modes

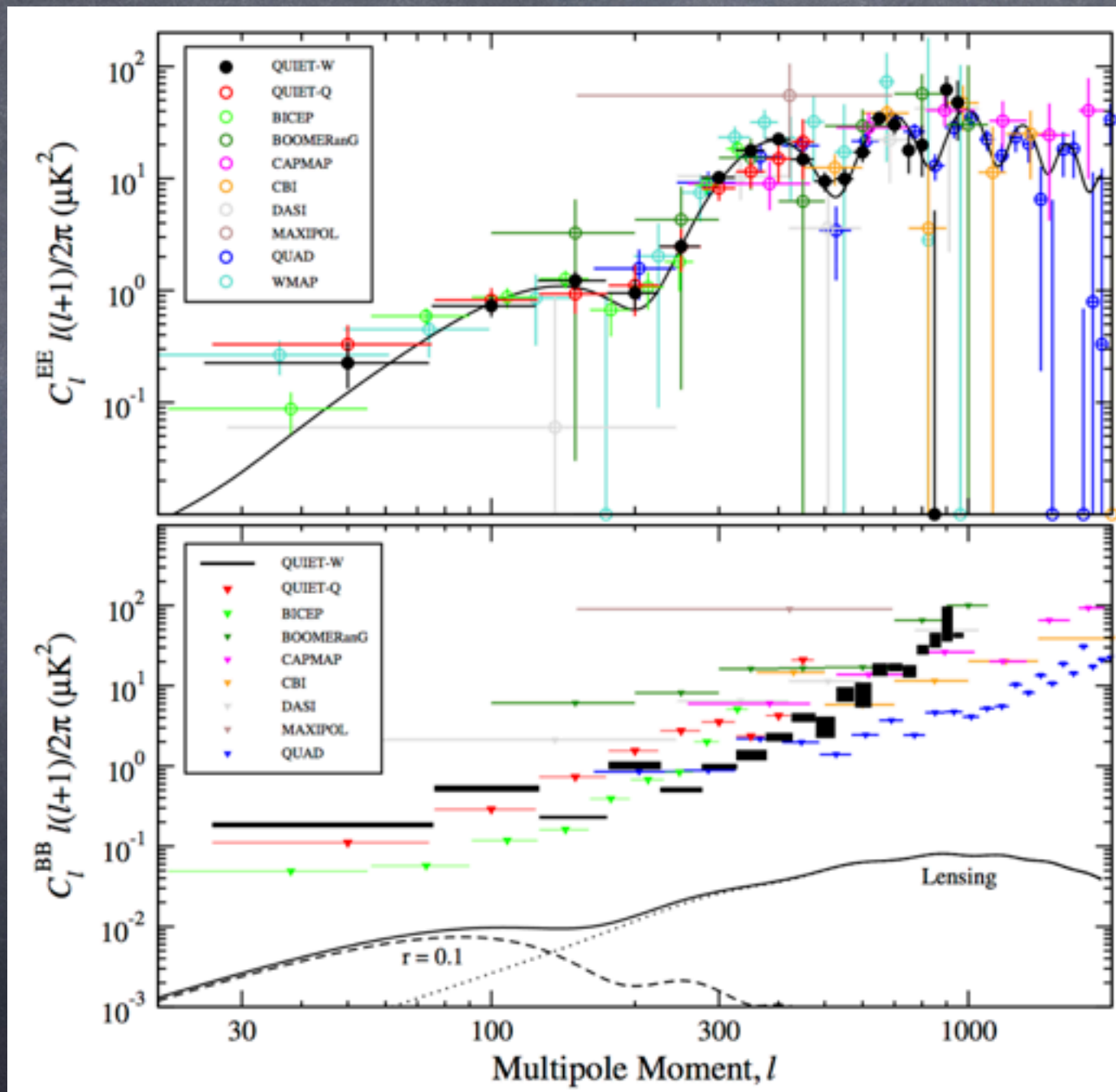


<http://background.uchicago.edu/~whu/animbut/anim4.html>

Results: Temperature

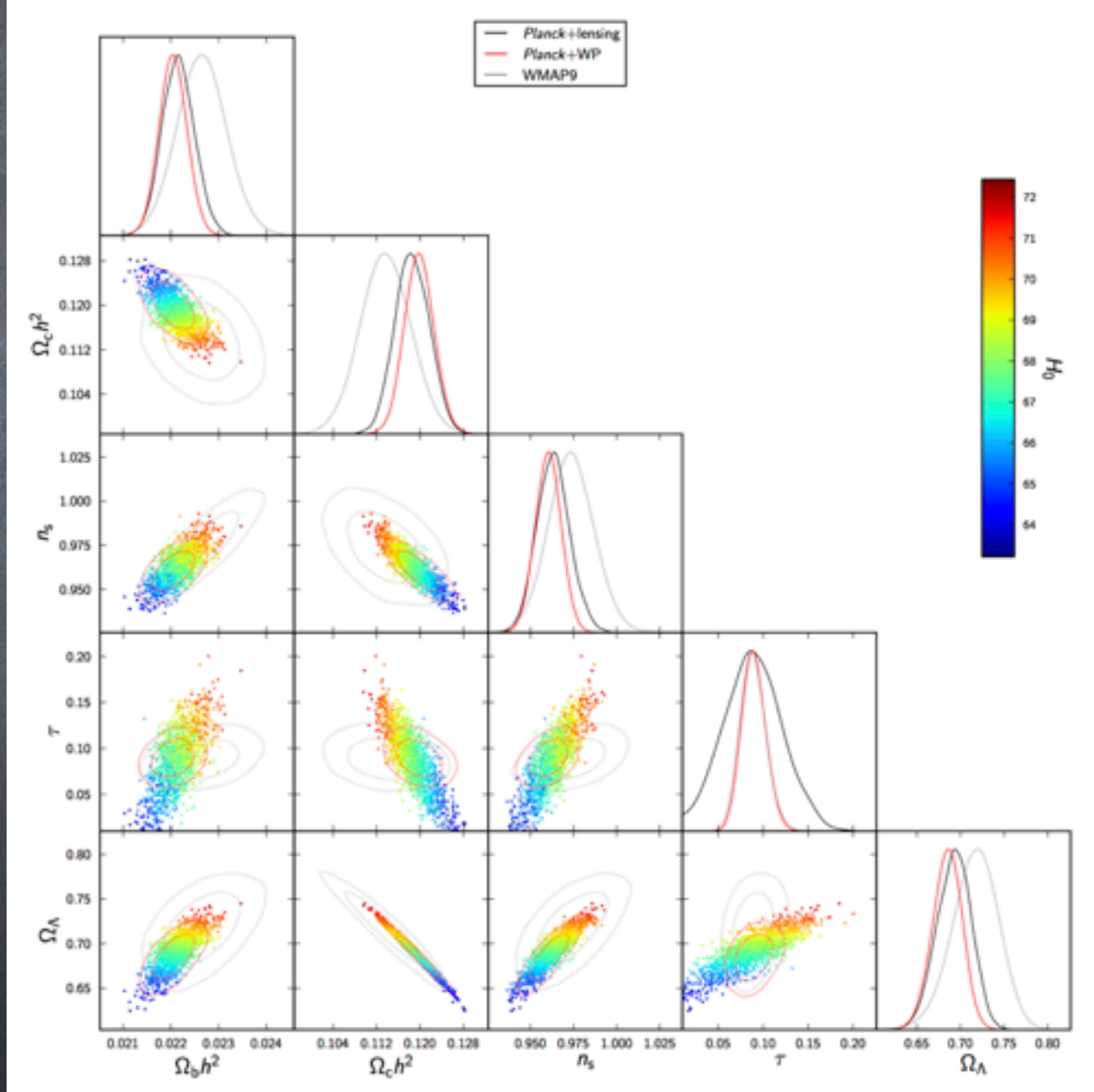


Results: Polarization



QUIET
2012

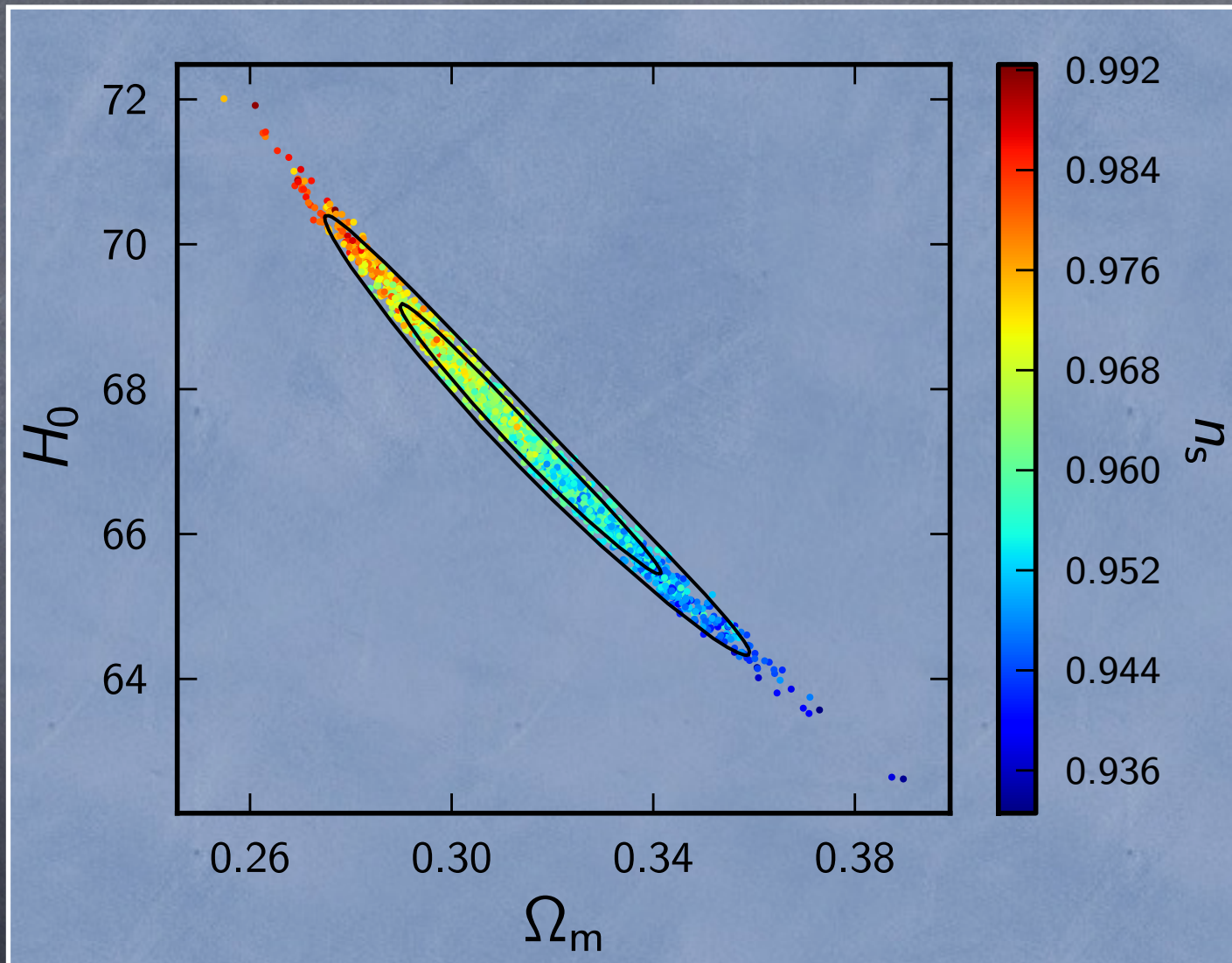
Results: Parameters



PLANCK
2013

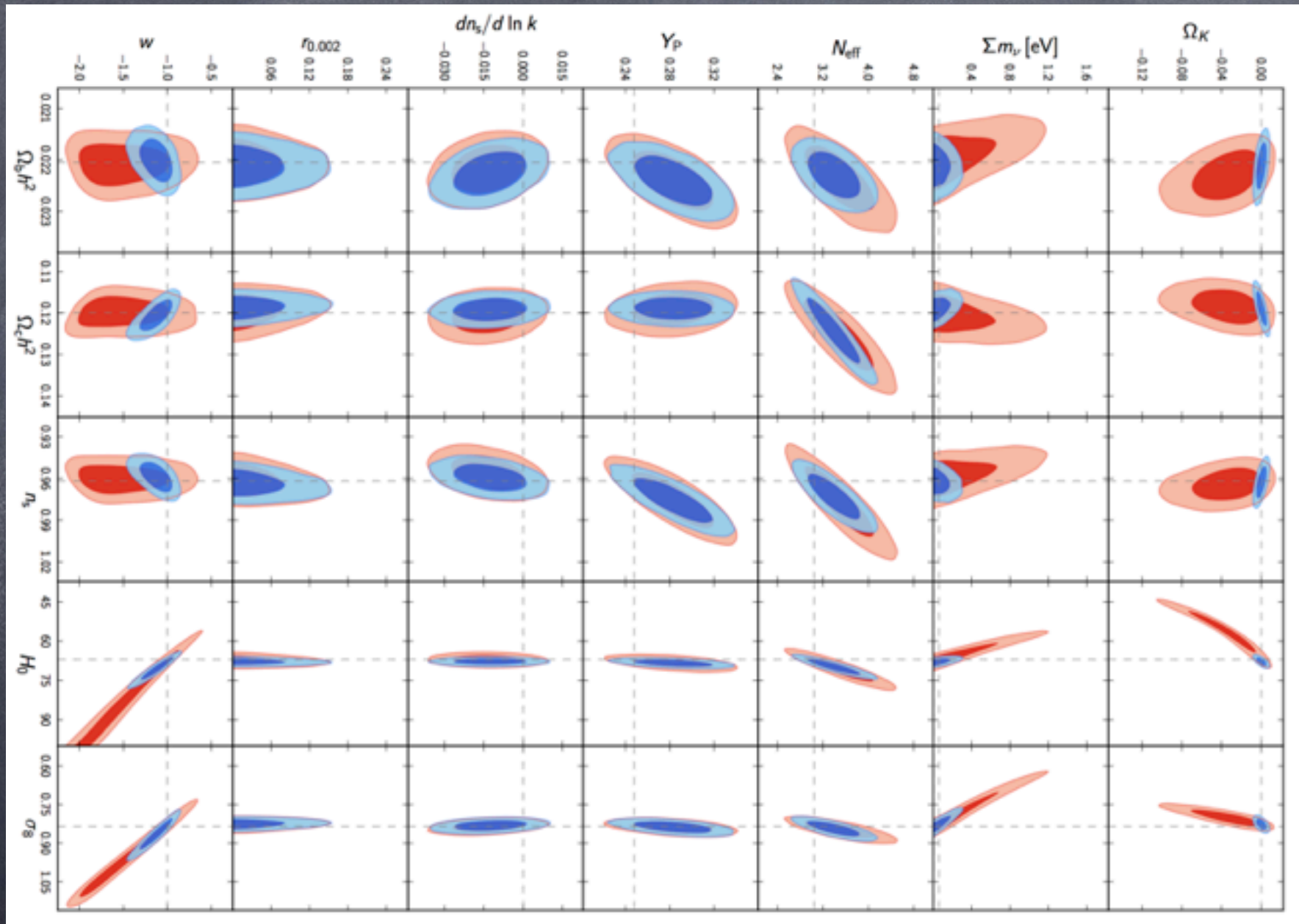
WP=WMAP
Polarization

Parameter Degeneracy



PLANCK
2013

Results: Other Parameters

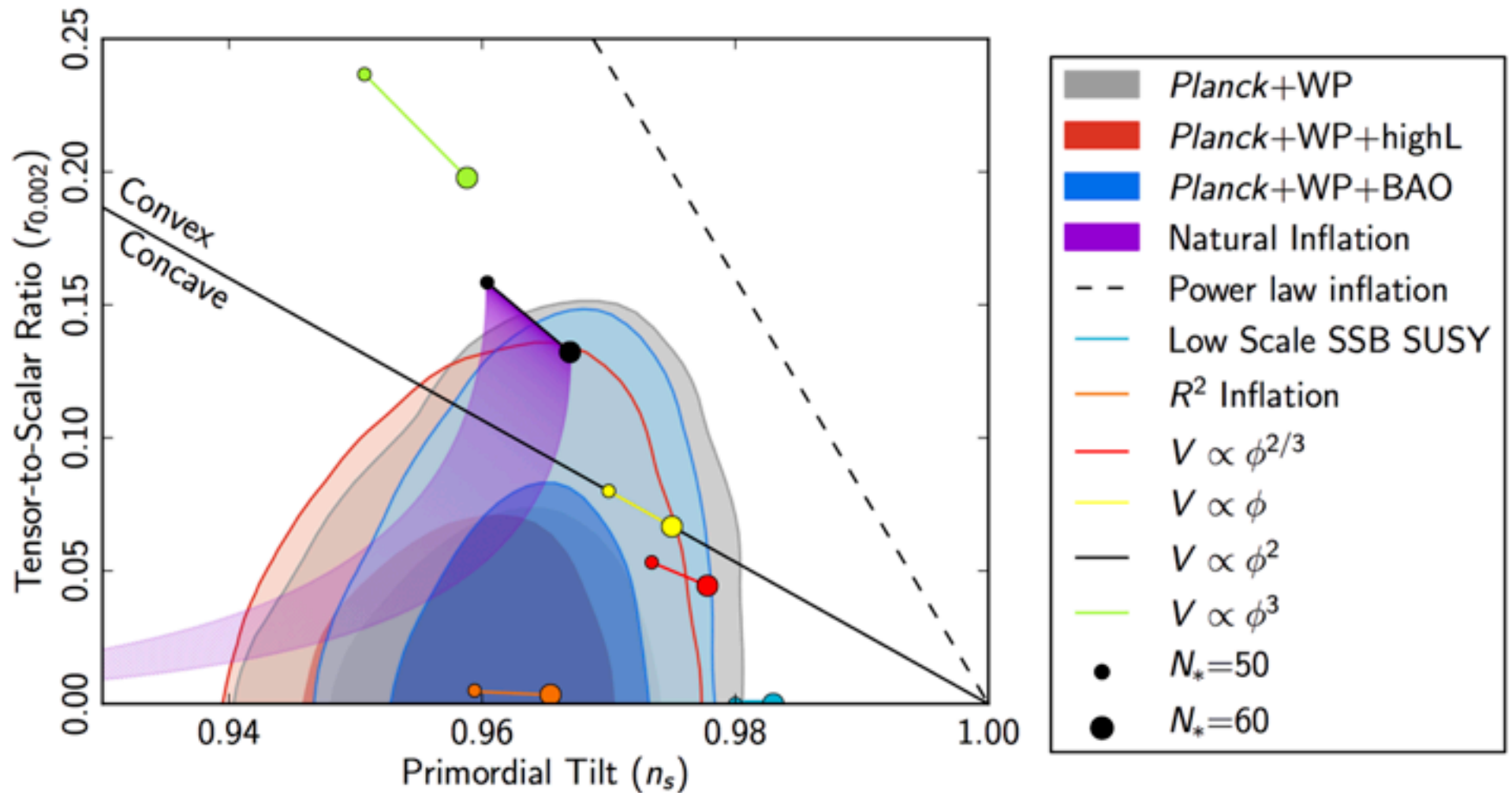


PLANCK 2013

PLANCK+WP

PLANCK+WP+BAO

Constraints on Inflation



PLANCK 2013

CONCLUSIONS

- THE CMBR IS A FAIRLY SIMPLE AND CLEAN AND EASY TO UNDERSTAND SYSTEM ALLOWING VERY PRECISE MEASUREMENTS OF ITS PROPERTIES
- BECAUSE OF THIS THE CMBR HAS AND WILL CONTINUE TO PROVIDE SOME OF THE BEST CONSTRAINTS ON COSMOLOGICAL PARAMETERS