Hadronic Vacuum Polarization for g-2

Taku Izubuchi

Tom Blum, Hyung-Jin Kim , Eigo Shintani,
SM Theory

- QED, hadronic, EW contributions

QED (5-loop)  
Aoyama et al.  
PRL109,111808 (2012)

Hadronic vacuum polarization (HVP)

Hadronic light-by-light (Hlbl)  
[ T. Blum’s talk ]

Electroweak (EW)  
Knecht et al 02  
Czarnecki et al. 02
SM Theory prediction

- QED, EW, Hadronic contributions

\[
\begin{align*}
\alpha_{\mu}^{SM} & = (11.659 \pm 4.9) \times 10^{-10} \\
\alpha_{\mu}^{QED} & = (11.658 \pm 0.015) \times 10^{-10} \\
\alpha_{\mu}^{EW} & = (15.4 \pm 0.2) \times 10^{-10} \\
\alpha_{\mu}^{had,LOVP} & = (694.91 \pm 4.27) \times 10^{-10} \\
\alpha_{\mu}^{had,HVVP} & = (-9.84 \pm 0.07) \times 10^{-10} \\
\alpha_{\mu}^{had,LBL} & = (10.5 \pm 2.6) \times 10^{-10}
\end{align*}
\]

\[
\alpha_{\mu}^{EXP} - \alpha_{\mu}^{SM} = (26.1 \pm 8.0) \times 10^{-10}
\]

- Discrepancy between EXP and SM is larger than EW!
- Currently the dominant uncertainty comes from HVP, followed by HL̲bL
- Theoretical estimate of HL̲bL is really under control?
- LQCD ➞ the first principles’ estimate for the hadronic parts.
(g-2)$_{\mu}$ theory vs experiment


$\alpha_{\mu}^{\text{exp}} - \alpha_{\mu}^{\text{SM}} = (26.1 \pm 8.0) \cdot 10^{-10}$ [3.3$\sigma$] for $\alpha_{\mu}^{\text{HLxL}} = (10.5 \pm 2.6) \cdot 10^{-10}$

$\alpha_{\mu}^{\text{exp}} - \alpha_{\mu}^{\text{SM}} = (25.0 \pm 8.6) \cdot 10^{-10}$ [2.9$\sigma$] for $\alpha_{\mu}^{\text{HLxL}} = (11.6 \pm 4.0) \cdot 10^{-10}$

- ~ 3 $\sigma$ discrepancy ?
- SM prediction

$\rightarrow$ Hadronic uncertainties ?
Leading order of hadronic contribution (HVP)

- Hadronic vacuum polarization (HVP)
  \[ \nu^\mu \nu^\nu = (q^2 g_{\mu\nu} - q_\mu q_\nu) \Pi_V(q^2) \]

  quark’s EM current:
  \[ V_\mu = \sum_f Q_f \bar{f} \gamma_\mu f \]

- Optical Theorem
  \[ \text{Im} \Pi_V(s) = \frac{s}{4\pi \alpha} \sigma_{\text{tot}}(e^+ e^- \rightarrow X) \]

- Analyticity
  \[ \Pi_V(s) - \Pi_V(0) = \frac{k^2}{\pi} \int_{4m^2_{\pi}}^{\infty} ds \frac{\text{Im} \Pi_V(s)}{s(s - k^2 - i\epsilon)} \]
Leading order of hadronic contribution (HVP)

Hadronic vacuum polarization (HVP)

\[
\alpha^\text{had}_\mu = \int ds \quad \times \quad V \bullet \text{Had} \text{vac} \bullet V
\]

\[
= \frac{\alpha}{\pi^2} \int_{m^2_\pi}^{\infty} \frac{ds}{s} \text{Im}\Pi(s) K(s)
\]

\[
K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (s/m^2_\mu)(1-x)}
\]

\[
= \frac{1}{3} \left( \frac{\alpha}{\pi} \right)^2 \left[ \int_{m^2_\pi}^{s_{\text{cut}}} ds \frac{K(s)}{s} R^\text{data}_{\text{had}}(s) + \int_{s_{\text{cut}}}^{\infty} ds \frac{K(s)}{s} R^\text{pQCD}_{\text{had}}(s) \right]
\]

**HVP from experimental data**

- From experimental e+ e- total cross section $\sigma_{\text{total}}(e^+e^-)$ and dispersion relation

  \[ a_\mu^{\text{HVP}} = \frac{1}{4\pi^2} \int_{4m_\pi^2}^{\infty} ds K(s) \sigma_{\text{total}}(s) \]

  time like $q^2 = s \geq 4 m_\pi^2$

  \[ a_\mu^{\text{HVP,LO}} = (694.91 \pm 4.27) \times 10^{-10} \]

  \[ a_\mu^{\text{HVP,HO}} = (-9.84 \pm 0.07) \times 10^{-10} \]

[ ~ 0.6 % err ]
HVP from Lattice

\[ \int d^4 x \langle T \{ V_{\mu}^{em}(x)V_{\nu}^{em}(0) \} \rangle e^{iQx} = (Q^2 \delta_{\mu\nu} - Q_{\mu}Q_{\nu})\Pi_V(Q^2) \]

\[ a_{\mu}^{\text{had}} = \frac{\alpha}{\pi^2} \int_{m^2}^{\infty} \frac{ds}{s} \text{Im}\Pi(s)K(s) = \left( \frac{\alpha}{\pi} \right)^2 \int_0^{\infty} dQ^2 f(Q^2) 4\pi^2 \left[ \Pi_V(0) - \Pi_V(Q^2) \right] \]


- Interpolation and extrapolation to \( Q^2 = 0 \) ⇒ systematic error
- Large statistical error

\[ Q^2_{\text{cut}} \approx 1 \text{ GeV}^2 \]

pQCD: well describe lattice data

Aubin et al. (2007), (2012)
Challenges in HVP on lattice

- Chiral extrapolation (unphysically heavy quark mass) → We now have \( \text{Mpi} \sim 135 \text{ MeV} \) QCD ensemble, so no problem for the next calculation!
- Need more data in small momentum \( q^2 \) region
  \[ p = \frac{2\pi}{L} \times n \]  (need larger \( L \), larger \( \text{Vol} \))
  → exploring various ideas
  - twieak boundary conditions
    - (partially) Minkowskian/Time-like momentum
    - simply going to larger Volume

- Statistical error
  → A new class of error reduction technique

-Disconnected diagram / Higher Order
- Discretization error
- Isospin breaking effects .....
**parameterize q2 dependence**

- $\pi(q^2)$ at small $q^2 \sim m^2$ region, which dominates the integral of HVP, is statistically noisier and sparse (small number of $q^2$ variation).

- By fitting $\pi(q^2)$ for $q^2 < O(1)$ GeV$^2$
  - extract $\pi(0)$ to subtract, $\pi(q^2) - \pi(0)$
  - perform the integration of HVP using the fit function
  \[
  \int_0^{Q_c^2} dQ^2 f(Q^2) \times \hat{\Pi}(Q^2) \rightarrow \int_0^1 dt f(Q^2) \times \hat{\Pi}(Q^2) \times \frac{Q^2}{t^2} \quad \text{where} \quad t = \frac{1}{1 + \log \frac{Q_c^2}{Q^2}}
  \]
Fit functions

- **Vector Meson Dominance**
  \[
  \Pi_V^{\text{tree}}(Q^2) = \frac{2}{3} \frac{f_V^2}{Q^2 + m_V^2}
  \]

- **Multi point Pade fit** [2012, Aubin et al.]
  \[
  \Pi(Q^2) = \Pi(0) - Q^2 \left( a_0 + \sum_{n=1}^{[P/2]} \frac{a_n}{b_n + Q^2} \right)
  \]
  Conditions: \( a_n > 0, \quad b_n > 4 m_\pi^2 \)
solid: correlated fit ($q^2 \leq 0.6$ GeV$^2$), dash: uncorrelated fit ($q^2 \leq 1$ GeV$^2$)

Pade approximation converges, results stable.

Twisted boundary condition

- On a torus, the action must be single-valued, while fields do not have to be.
- Impose the **twisted boundary condition** on quark fields.
  \[ q(x+L) = q(x)\exp(i\theta) \]
  \[ \rightarrow \quad p = (2\pi n + \theta)/L \]
  (\( \theta \):arbitrary input)
- \( q^2 \) can be arbitrary small.
- Breaking isospin, Vector ward identity is broken, could be exactly subtracted [Aubin et al 2012]
- Noise in small \( q^2 \)
Exploring time-like mom

[ Eigo Shintani, Hyung-Jin Kim & TI ]

- To reduce systematic error
  Transformation to time-like momentum using analytical continuation


- Domain-wall fermion (RBC/UKQCD) in $N_f = 2+1$
  - $24^3\times64$ ($a^{-1} = 1.73$ GeV), $32^3\times64$ ($a^{-1} = 2.25$ GeV): $m_\pi = 300$--400 MeV
  - Good chiral property and scaling behavior.

  Remark: precise determination of $\alpha_s$ with pQCD in high $Q^2$.

Time-like momentum

\[ Q_4 = i\omega \]

\[ \int d^4x \langle T \{ V_{\mu}^{\text{em}}(x)V_{\nu}^{\text{em}}(0) \} \rangle e^{i\mathbf{q}\cdot\mathbf{x}} = \Pi_{\mu\nu}(\mathbf{q},\omega) = (q^2 g_{\mu\nu} - q_\mu q_\nu) \Pi_V(q^2) \]

\[ q = (\omega, \mathbf{q}), \quad g_{\mu\nu} = \text{diag}(1, -1, -1, -1), \quad q^2 = \omega^2 - q^2 = -Q^2 \]

- \( \omega \) is “photon energy” which can be controlled by hand.
- Temporal integral from \(-\infty < t < \infty\) (Laplace transformation)

\[ \Pi_{\mu\nu}(\mathbf{q},\omega) = \int_0^\infty dt \sum_{\mathbf{\bar{x}}} e^{-\omega t - i\mathbf{\xi}\cdot\mathbf{q}} \langle V_{\mu}(\mathbf{\bar{x}}, t)V_{\nu}(0) \rangle_c + \int_{-\infty}^0 dt \sum_{\mathbf{\bar{x}}} e^{-\omega t - i\mathbf{\xi}\cdot\mathbf{q}} \langle V_{\nu}(0)V_{\mu}(\mathbf{\bar{x}}, t) \rangle_c \]

\( \rho \) state or \( \pi\pi \) state

Resonance poles

\[ 0 \quad -q_1^2 \quad -q_2^2 \quad -q^2 \]
Time-like momentum

Modeling large time behavior

To perform the infinite temporal integral, we need to model 2pt at large time

\[ \sum_{\vec{x}} e^{i\vec{q}\cdot\vec{x}} \langle V_\mu(x)V_\nu(0) \rangle \simeq g_V e^{-E_V t} \]  
(asympotic state dominance at \( t \geq t_{cut} \))

\[ \int_0^{t_{cut}} dt e^{-\omega t} \sum_{\vec{x}} e^{i\vec{q}\cdot\vec{x}} \langle V_\mu(x)V_\nu(0) \rangle \simeq \sum_{t=0}^{t_{cut}} C_{VV}(\vec{q}, \omega; t) \]  
(numerical integral with lattice data from \( 0 \leq t \leq t_{cut} \))

Longitudinal part will be

\[ \Pi_{long}(\vec{q}, \omega) = \frac{g_V}{E_V + \omega} e^{-(E_V + \omega)t_{cut}} + \frac{g_V}{E_V - \omega} e^{-(E_V - \omega)t_{cut}} + \sum_{t=0}^{t_{cut}} 2F(t) \cosh \omega t \]

Finally we consider the particular momentum \( q_\mu \neq 0, q_\nu \neq 0 = 0 \)

\[ \Pi_{long}(\vec{q}, \omega) = -\omega^2 \Pi_V(q^2), \quad q^2 = \omega^2 - q_\mu^2 \]
Demonstrating that finite-size effects for the average with a weight of 1 in various momentum modes, we can average them to obtain the final result. Here we show discrepancies for finite-size effects using the conventional approach. We therefore conclude that applying a cut in the Euclidean times induces indeed a finite-size effect.

- tm-Wilson quark (maximal twist)
- pion mass: 290 MeV - 650 MeV
- a = 0.08 fm, 0.06 fm


- larger stat. error than conventional method
HVP with time-like momentum

- $t_{\text{cut}} = 9 \ (24^3), 10 \ (32^3)$
- Fitting range at large $t$ $[8,13] \ (24^3), [10,15] \ (32^3)$

- Similar behavior with results obtained in Euclid momentum
- Slight discrepancy from HVP in space-like momentum, especially for light mass.

More carefully systematic study is necessary!
Covariant Approximation Averaging (CAA) 
a new class of Error reduction techniques

\[ \mathcal{O} = \mathcal{O}^{\text{appx}} + \mathcal{O}^{\text{rest}} \]

\[ \mathcal{O}^{\text{imp}} = \mathcal{O}^{\text{rest}} + \frac{1}{N_G} \sum_{g \in G} \mathcal{O}^{\text{appx}}, g \]

Expensive: infrequently measured

Cheap: frequently measured

- \( \mathcal{O}^{\text{imp}} \) has smaller error
- \( \mathcal{O}^{\text{appx}} \) need to be cheap & not to be too accurate

\( N_G \) suppresses the bulk part of noise cheaply

\[ \text{Residue (}\alpha\text{)} \quad L^2 \text{ Force} \quad \alpha/(\beta+0.125) \quad \text{CG iterations} \]

- [Blum, TI, Shintani PRD 88 (2013) 094503]
Examples of Covariant Approximations (contd.)

- **All Mode Averaging (AMA)**

  Sloppy CG or Polynomial approximations

  \[ O^{(\text{appx})} = O[S_l] , \]

  \[ S_l = \sum_\lambda v_\lambda f(\lambda) v_\lambda^\dagger , \]

  \[ f(\lambda) = \begin{cases} \frac{1}{\lambda} , & |\lambda| < \lambda_{\text{cut}} \\ P_n(\lambda) , & |\lambda| > \lambda_{\text{cut}} \end{cases} \]

  \[ P_n(\lambda) \approx \frac{1}{\lambda} \]

If quark mass is heavy, e.g. ~ strange, low mode isolation may be unnecessary.
AMA at work

- **Target**: $V=32^3 \times 64 = (4.6\text{fm})^3 \times 9.6\text{fm}$, $L_\text{s}=32$ Shamir-DWF, $a^{-1}=1.37$ GeV, $\text{M}_{\pi}=170$ MeV

- Use $L_\text{s}=16$ Mobius as the approximation
  [Brower, Neff, Orginos, arXiv:1206.5214]

- Quark propagator cost on SandyBridge 1024 cores (XSEDE gordon@SDSC)
  - non-deflated CG, $r(\text{stop})=1\times10^{-8}$: $\sim9,800$ iteration, 5.7 hours / prop
  - Implicitly restarting Lanczos of Chebyshev polynomials of even-odd prec operator for 1000 eigenvectors
    [Neff et al. PRD64, 114509 (2001)]: 12 hours
  - deflated CG with 1000 eigenvectors: $\sim700$ iteration, 20 min /prop
  - deflation+sloppy CG, $r(\text{stop})=5\times10^{-3}$: $\sim125$ iteration, 3.2 min /prop

- **Multiplicative Cost reduction for General hadrons** could combine with $\{\text{EigCG} \mid \text{AMG}\}$ and Distillation:
  $x1.2$ (Mobius) $x$ $14$ (deflation) $x$ $7$ (sloppy CG) $\sim x 110$
AMA at work
[ M. Lin ]

- $F_1(Q^2) : \text{tsep} = 9 \ a \sim 1.3 \ \text{fm}$
  1 forward + 2 (up and down) seq-props, contraction cost is $\sim 15\%$ of sloppy propagator

- Error bar
  $x \ 2 - 2.7 \sim \sqrt{4400/600}$

- Total cost reduction upto
  $(430/160) \times (4400/600) \sim x19.7$

- Note this is still sub-optimal, 4 exact source and without deflation. (would be x30 for 2 exact sources)

- non-deflated CG, 150 config x 4 sources = 600 measurements :
  $5.7 \times 3 \times 4 \times 150 \ \text{config} = 10K \ \text{hours, 430 days}$

- AMA : 39 config, 4 exact solves / config (perhaps overkill) , $N_G=112$ sloppy solves
  $\Rightarrow 39 \times 112 = 4400 \ \text{AMA measurements}$ :
  $(5.7 \times 3 \times 4 + 12 + 0.06 \times 3 \times 112) \times 39 \ \text{config} = 3.9 \ K \ \text{hours, 160 days}$
  4-exact (68%) + Lanczos (12%) + sloppy CG (20%)
Improving HVP statistics using AMA

- Staggered Fermion (MILC Asqtad, $M_{\pi} = 300$ MeV)
  2.6 -- 20 times smaller error with same cost

- $\Pi(Q^2)$
  - $q_{\text{min}}^2 = 1.5 \, m(\mu)$
  - Now getting to all stat error $< 2%$
Two lattice spacings 
\(a = 0.11, 0.088 \text{fm},\)
\(M_{\pi} = 0.28-0.33 \text{ GeV}\)
All stat err < 0.7%
\(q_{\text{min}} = 2 \, m(\mu)\)

Applied [2,1] Pade,
can’t fit with
\(b_1 \geq 4 \, M_{\pi}^2\) bound

<table>
<thead>
<tr>
<th>Lattice</th>
<th>(m_u)</th>
<th>(\Pi_V(0))</th>
<th>(a_0) (GeV(^{-2}))</th>
<th>(a_1)</th>
<th>(b_1) (GeV(^2))</th>
<th>(\chi^2/\text{dof})</th>
<th>(4m_{\pi}^2) GeV(^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>24(^3)x64</td>
<td>0.005</td>
<td>0.1752(2)</td>
<td>0.0325(2)</td>
<td>0.0407(1)</td>
<td>0.139(1)</td>
<td>2.7(4)</td>
<td>0.44</td>
</tr>
<tr>
<td></td>
<td>0.01</td>
<td>0.1603(2)</td>
<td>0.0219(3)</td>
<td>0.0434(4)</td>
<td>0.408(7)</td>
<td>0.4(1)</td>
<td>0.71</td>
</tr>
<tr>
<td>32(^3)x64</td>
<td>0.004</td>
<td>0.197(2)</td>
<td>0.026(3)</td>
<td>0.052(3)</td>
<td>0.227(37)</td>
<td>0.08(7)</td>
<td>0.31</td>
</tr>
<tr>
<td></td>
<td>0.006</td>
<td>0.190(3)</td>
<td>0.027(7)</td>
<td>0.043(11)</td>
<td>0.253(25)</td>
<td>0.4(5)</td>
<td>0.44</td>
</tr>
</tbody>
</table>
Subtraction Strategy: Derivative of Twisting Angle
[Divitiis et al. PLB 718(2012) 589]

\[ p_i = \left( \frac{2\pi n}{L} + \theta_i \right) \]

\[
C^{\mu\nu}(p) = \frac{1}{(TL^3)^2} \sum_{x,y} e^{ip(y-x+\hat{v}/2-\hat{\mu}/2)} \langle V_{em}^\mu(x) V_{em}^\nu(y) \rangle \\
= (\delta^{\mu\nu} \hat{p}^2 - \hat{p}^\mu \hat{p}^\nu) \Pi(p^2),
\]

\[
\Pi(0) = -\frac{\partial^2 \hat{C}^{12}(p)}{\partial p_1 \partial p_2} \bigg|_{p^2=0}
\]

\[
\begin{align*}
\frac{\partial^2 \hat{C}^{12}(p)}{\partial p_1 \partial p_2} & = \frac{1}{2} - \frac{1}{2} \times 2 - \frac{1}{4} = 0.5
\end{align*}
\]

\[
\begin{align*}
\Pi(0, 24^3 \times 48) & \quad \Pi(p^2 > 0, 24^3 \times 48) \\
\Pi(0, 32^3 \times 64) & \quad \Pi(p^2 > 0, 32^3 \times 64)
\end{align*}
\]
Use of Time-Moments [ HPQCD ]

- Compute Time-moments of 2pt [P. Lepage’s talk]

\[
G_{2n} \equiv a^4 \sum_t \sum_{\vec{x}} t^{2n} Z_V^2 \langle j^i(\vec{x}, t) j^i(0) \rangle
= (-1)^n \left. \frac{\partial^{2n}}{\partial q^{2n}} q^2 \hat{\Pi}(q^2) \right|_{q^2=0}.
\]

\[
\hat{\Pi}(q^2) = \sum_{j=1}^{\infty} q^{2j} \Pi_j
= (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!}.
\]

- Subtraction by taking derivatives, use local currents
- Padé approximation, determined from \( \Pi_j \), for high q2 integration

\[ a_s^\mu = 53.41(59) \times 10^{-10}. \]
[ 1.1% ~ lattice spacing error ]

\[ a_c^\mu = 14.42(39) \times 10^{-10}. \]
[ 2.7% ~ Z_V error ]
Recent results

roughly

5–10 % error

---

**Table:**

<table>
<thead>
<tr>
<th>$a_\mu$</th>
<th>$N_f$</th>
<th>errors</th>
<th>action</th>
<th>group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$713(15)$</td>
<td>$2+1$</td>
<td>stat.</td>
<td>Asqtad</td>
<td>Aubin, Blum (2006)</td>
</tr>
<tr>
<td>$748(21)$</td>
<td>$2+1$</td>
<td>stat.</td>
<td>Asqtad</td>
<td>Aubin, Blum (2006)</td>
</tr>
<tr>
<td>$641(33)(32)$</td>
<td>$2+1$</td>
<td>stat., sys.</td>
<td>DWF</td>
<td>UKQCD (2011)</td>
</tr>
<tr>
<td>$572(16)$</td>
<td>$2$</td>
<td>stat.</td>
<td>TM</td>
<td>ETMC (2011)</td>
</tr>
<tr>
<td>$618(64)$</td>
<td>$2+1^1$</td>
<td>stat., sys.</td>
<td>Wilson</td>
<td>Mainz (2011)</td>
</tr>
</tbody>
</table>

[BMW Collaboration, 1311.4446]
HVP Summary and future prospects

- Lattice HVP issues
  - Parameterize / Fit low Q2
    - Model Independent Pade, Time-moments of HVP (error from parameterization dependence)
  - More precise data at low Q2
    - Twisted B.C., Derivatives of twist angle, Time-like momentum, or simply large volume
  - Discretization error, Quark Mass dependence
    - Nf=2+1, 2+1+1, Physical quark mass calculations are running
  - Statistical error
    - All Mode Averaging (AMA) helps to reduction of statistical error.
  - Disconnected quark loop
  - EM Isospin / wall