Hadronic Vacuum Polarization for g-2

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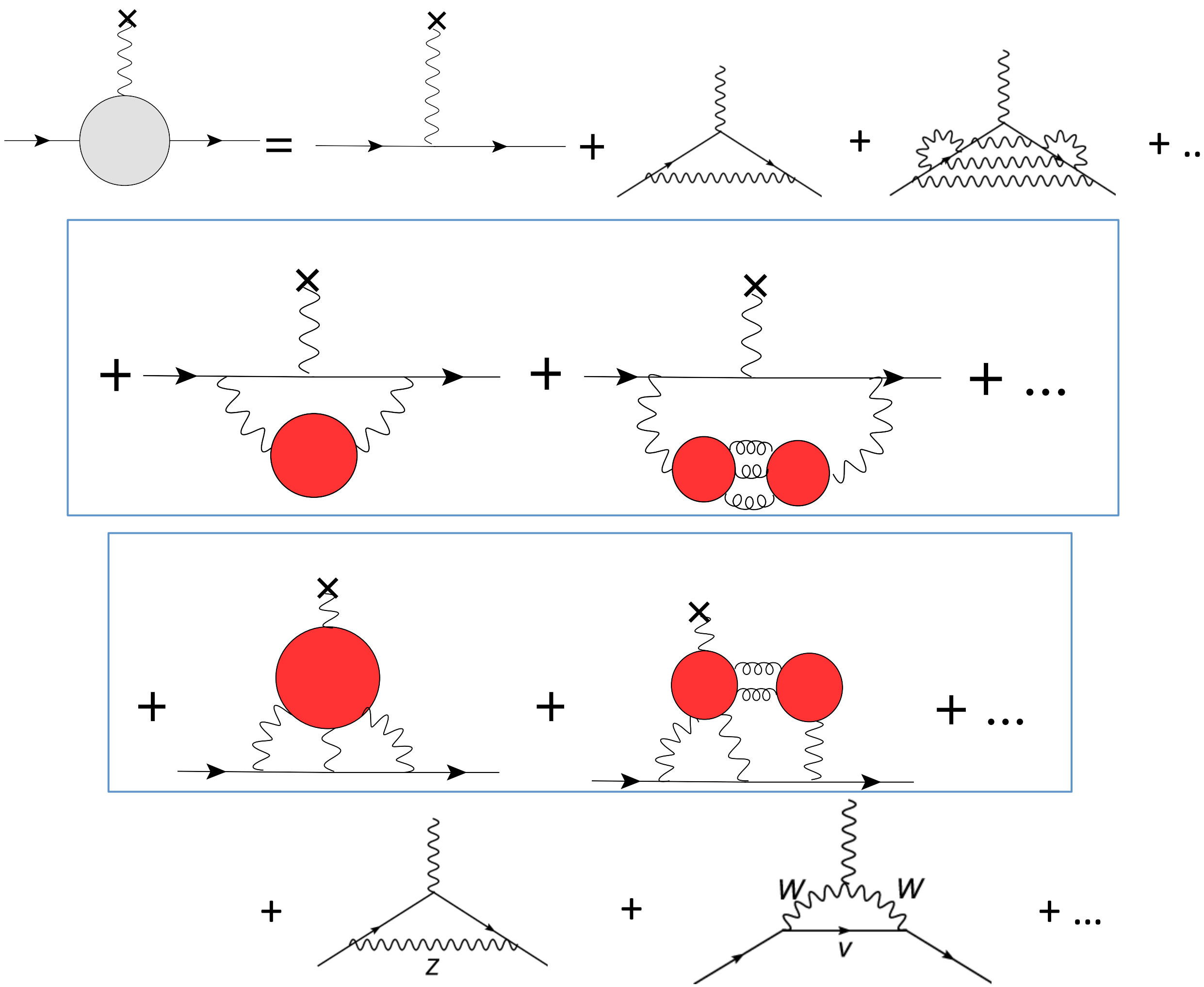
Fermilab, Lattice QCD Meets Experiment, Batavia, March 8, 2013

SM Theory

- QED, hadronic, EW contributions



QED (5-loop)
Aoyama et al.
PRL109,111808 (2012)

Hadronic vacuum polarization (HVP)

Hadronic light-by-light (Hlbl)
[T. Blum's talk]

Electroweak (EW)
Knecht et al 02
Czarnecki et al. 02

SM Theory prediction

- QED, EW, Hadronic contributions

K. Hagiwara et al. , J. Phys. G: Nucl. Part. Phys. 38 (2011) 085003

$$a_\mu^{\rm SM} = (11\ 659\ 182.8 \pm 4.9) \times 10^{-10}$$

$$\begin{aligned}
a_\mu^{\rm QED} &= (11\ 658\ 471.808 \pm 0.015) \times 10^{-10} \\
a_\mu^{\rm EW} &= (15.4 \pm 0.2) \times 10^{-10} \\
\boxed{a_\mu^{\rm had,LOVP} = (694.91 \pm 4.27) \times 10^{-10}} \\
a_\mu^{\rm had,HOVP} &= (-9.84 \pm 0.07) \times 10^{-10} \\
a_\mu^{\rm had,lbl} &= (10.5 \pm 2.6) \times 10^{-10}
\end{aligned}$$

$$a_\mu^{\rm EXP} - a_\mu^{\rm SM} = (26.1 \pm 8.0) \times 10^{-10}$$

- Discrepancy between EXP and SM is larger than EW!
- Currently the dominant uncertainty comes from HVP, followed by HLbL
- Theoretical estimate of HLbL is really under control?
- LQCD ➡ the first principles' estimate for the hadronic parts.

$(g\text{-}2)_\mu$ theory vs experiment

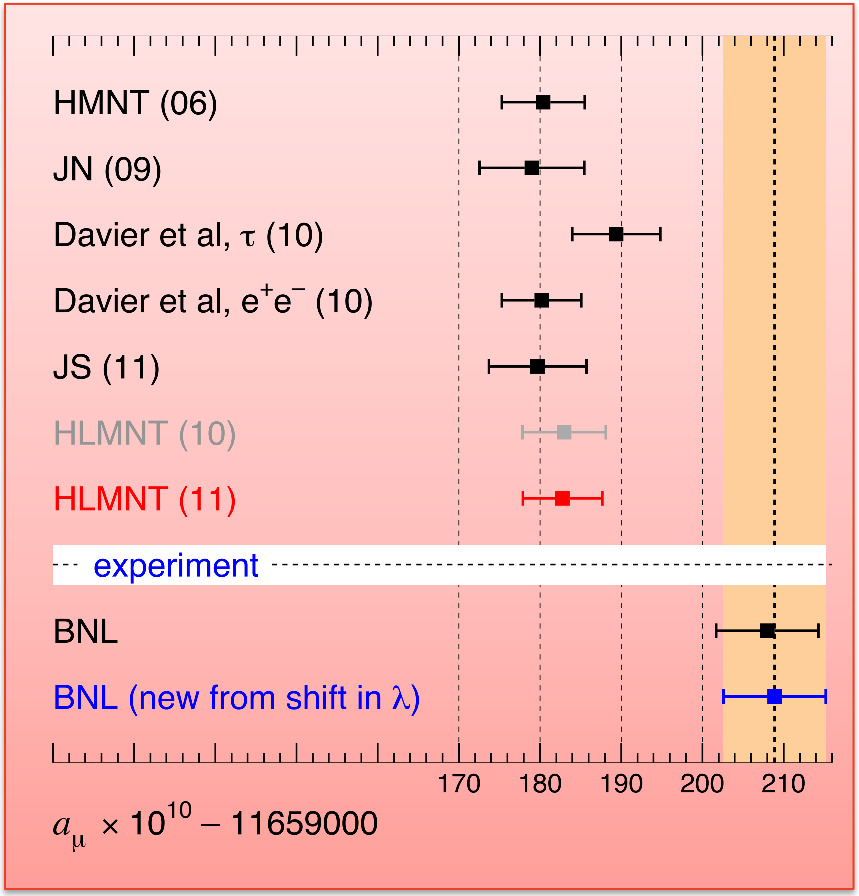
[K. Hagiwara et al., J. Phys. G 38, 085003 (2011)]

$$a_\mu^{\rm exp} - a_\mu^{\rm SM} = (26.1 \pm 8.0) \cdot 10^{-10} \ \ [3.3\sigma] \quad \text{for } a_\mu^{\rm HLxL} = (10.5 \pm 2.6) \cdot 10^{-10}$$

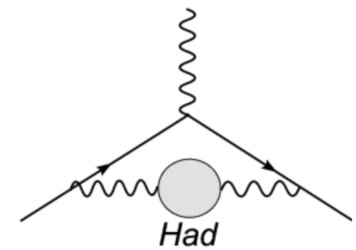
$$(a_\mu^{\rm exp} - a_\mu^{\rm SM} = (25.0 \pm 8.6) \cdot 10^{-10} \ \ [2.9\sigma] \quad \text{for } a_\mu^{\rm HLxL} = (11.6 \pm 4.0) \cdot 10^{-10})$$

- ~ 3 σ discrepancy ?
- SM prediction

  → Hadronic uncertainties ?



Leading order of hadronic contribution (HVP)



- Hadronic vacuum polarization (HVP)

V_μ  V_ν $= (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi_V(q^2)$

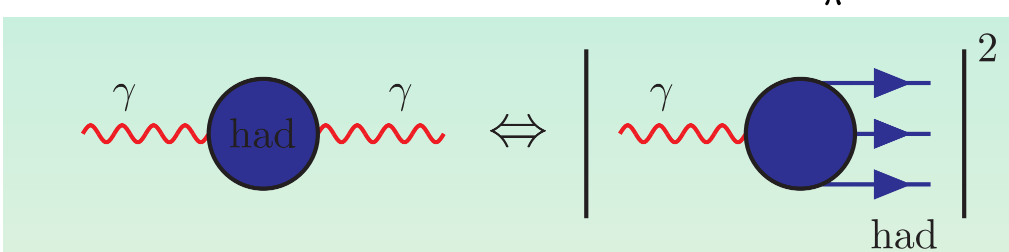
quark's EM current : $V_\mu = \sum_f Q_f \bar{f}\gamma_\mu f$

- Optical Theorem

$$\mathrm{Im}\Pi_V(s) = \frac{s}{4\pi\alpha}\sigma_{\mathrm{tot}}(e^+e^- \to X)$$

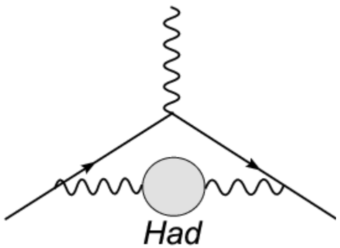
- Analycity

$$\Pi_V(s) - \Pi_V(0) = \frac{k^2}{\pi}\int_{4m_\pi^2}^{\infty} ds \frac{\mathrm{Im}\Pi_V(s)}{s(s-k^2-i\epsilon)}$$

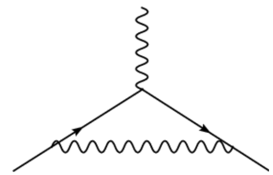


Leading order of hadronic contribution (HVP)

- Hadronic vacuum polarization (HVP)



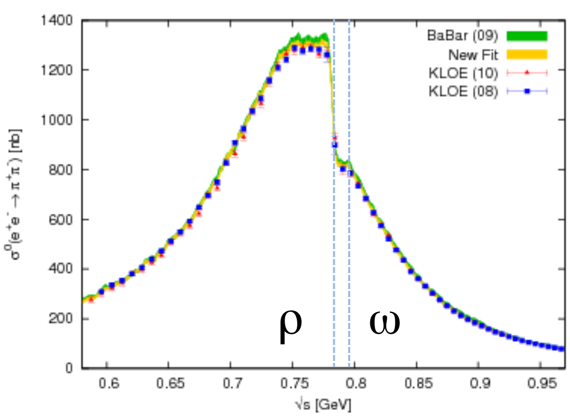
$$a_\mu^{\text{had}} = \int ds$$

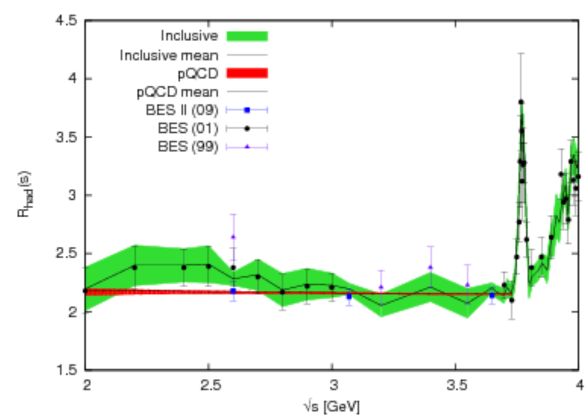


×

$$= \frac{\alpha}{\pi^2} \int_{m_\pi^2}^{\infty} \frac{ds}{s} \mathrm{Im}\Pi(s) K(s) \qquad K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (s/m_\mu^2)(1-x)}$$

$$= \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \left[\int_{m_\pi^2}^{s_{\text{cut}}} ds \frac{K(s)}{s} R_{\text{had}}^{\text{data}}(s) + \int_{s_{\text{cut}}}^{\infty} ds \frac{K(s)}{s} R_{\text{had}}^{\text{pQCD}}(s) \right]$$



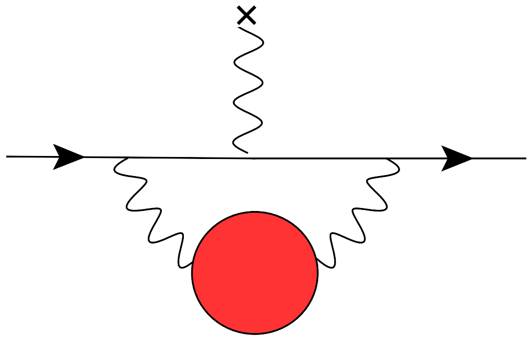


Hagiwara, et al.
J.Phys. G38,085003
(2011)

HVP from experimental data

- From experimental e+ e- total cross section σ_{total}(e+e-) and dispersion relation

$$a_\mu^{\rm HVP} = \frac{1}{4\pi^2}\int_{4m_\pi^2}^{\infty} ds K(s)\sigma_{\rm total}(s)$$

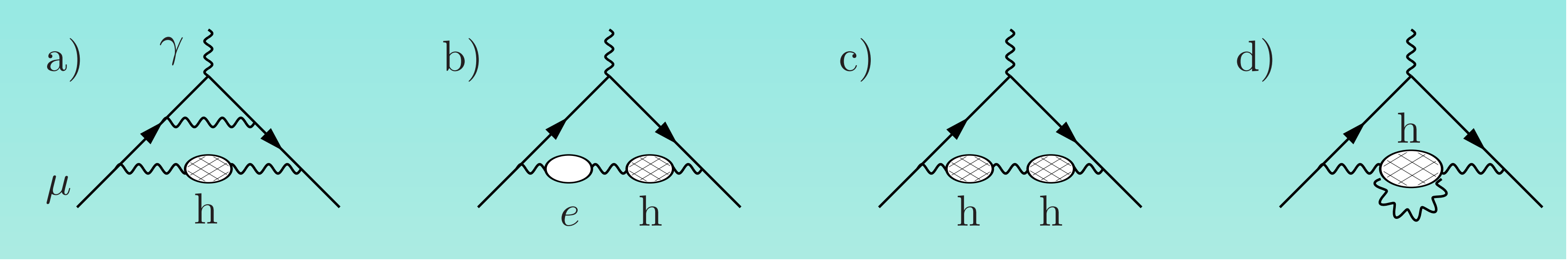


time like $q^2 = s >= 4\, m_\pi^2$

$$a_\mu^{\rm HVP,LO} = (694.91 \pm 4.27) \times 10^{-10}$$

$$a_\mu^{\rm HVP,HO} = (-9.84 \pm 0.07) \times 10^{-10}$$

[~ 0.6 % err]



[T. Blum PRL91 (2003) 052001]

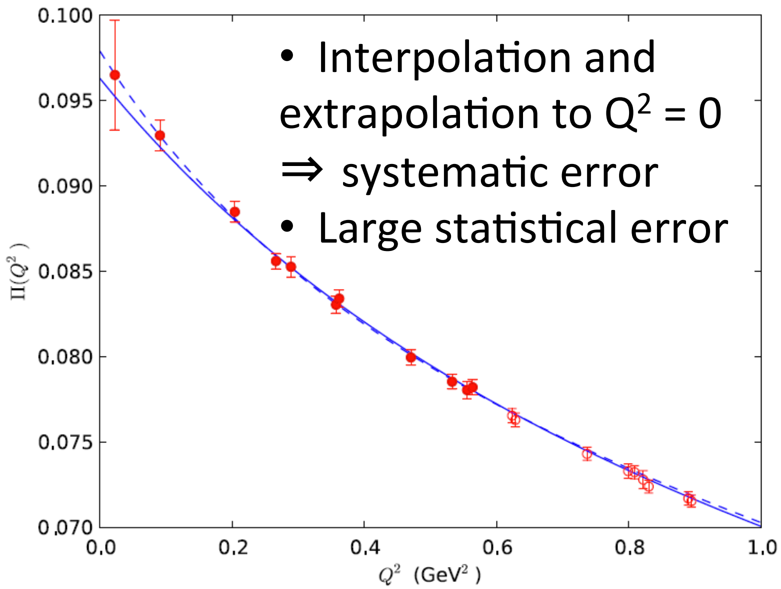
HVP from Lattice

$Q^2 = -q^2 > 0$ (Euclidean/space-like momentum)

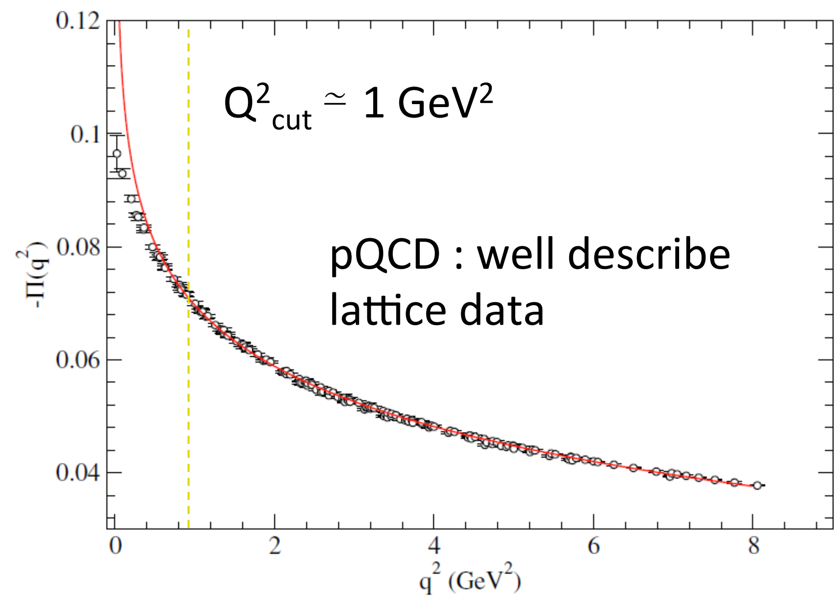
$$\int d^4x \langle T\{V_\mu^{\rm em}(x)V_\nu^{\rm em}(0)\}\rangle e^{iQx} = (Q^2\delta_{\mu\nu} - Q_\mu Q_\nu)\Pi_V(Q^2)$$

$$a_\mu^{\rm had} = \frac{\alpha}{\pi^2}\int_{m_\pi^2}^{\infty}\frac{ds}{s}{\rm Im}\Pi(s)K(s) = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dQ^2 f(Q^2)4\pi^2\left[\Pi_V(0) - \Pi_V(Q^2)\right]$$

Aubin, Blum, Phys. Rev.D75,114502 (2007), Feng, et al., Phys.Rev.Lett. 107, 081802 (2011), Bolye et al., Phys.Rev. D85,074504(2012), Della Morte, et al., JHEP 1203,055(2012),Aubin et al.,Phys.Rev.D86, 054509(2012)



- Interpolation and extrapolation to $Q^2 = 0$ ⇒ systematic error
- Large statistical error



$Q^2_{\rm cut} \simeq 1$ GeV^2

pQCD : well describe lattice data

Aubin et al. (2007), (2012)

Challenges in HVP on lattice

- Chiral extrapolation (unphysically heavy quark mass) → We now have Mpi ~ 135 MeV QCD ensemble, so no problem for the next calculation !
- Need more data in small momentum q2 region

  p = 2π/L x n (need larger L , larger Vol)

  → exploring various ideas

  twieak boundary conditions

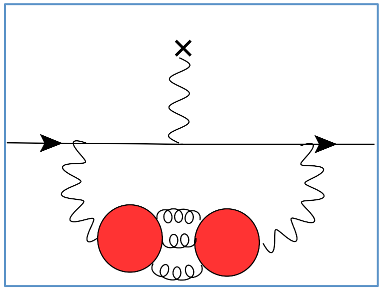
  (partially) Minkowskian/Time-like momentum

  simply going to larger Volume

- Statistical error

  → A new class of error reduction technique

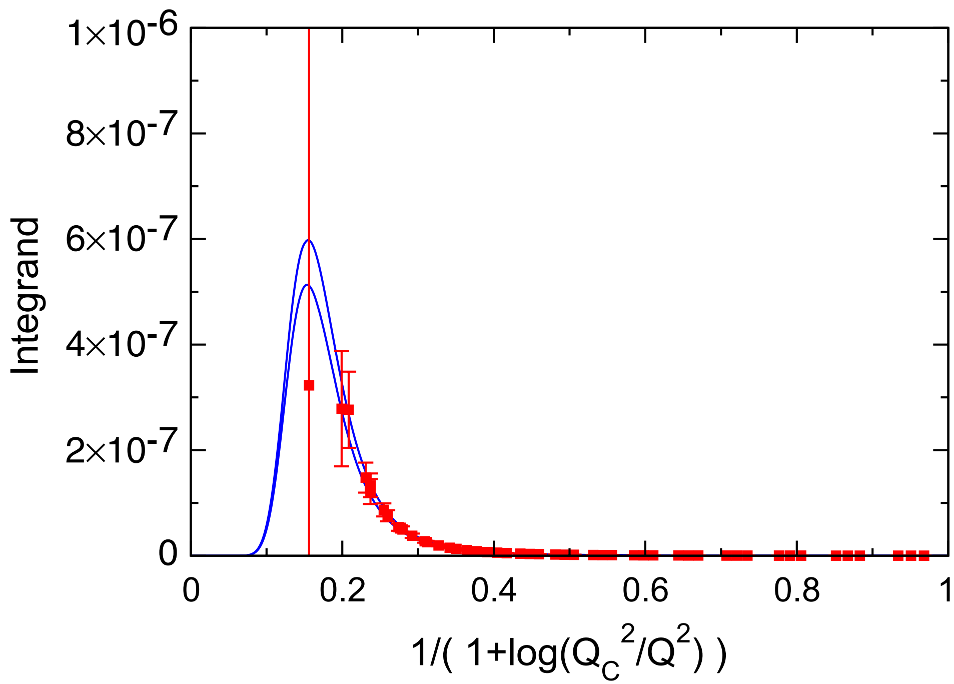
- Disconnected diagram / Higher Order
- Discretization error
- Isospin breaking effects …..



parameterize q2 dependence

- $\pi(q2)$ at small $q^2 \sim m_\mu^2$ region, which dominates the integral of HVP, is statistically noisier and sparse (small number of q2 variation).
- By fitting $\pi(q2)$ for q2 < O(1) GeV^2
  - extract $\pi(0)$ to subtract, $\pi(q2)-\pi(0)$
  - perform the integration of HVP using the fit function

$$\int_0^{Q_C^2} dQ^2 f(Q^2) \times \hat{\Pi}(Q^2) \to \int_0^1 dt f(Q^2) \times \hat{\Pi}(Q^2) \times \frac{Q^2}{t^2} \qquad \text{where} \quad t = \frac{1}{1 + \log\frac{Q_C^2}{Q^2}}$$



P. Boyle et al, PRD85, 074504(2012)

Fit functions

- Vector Meson Dominance

$$\Pi_V^{tree}(Q^2) = \frac{2}{3}\frac{f_V^2}{Q^2 + m_V^2}$$

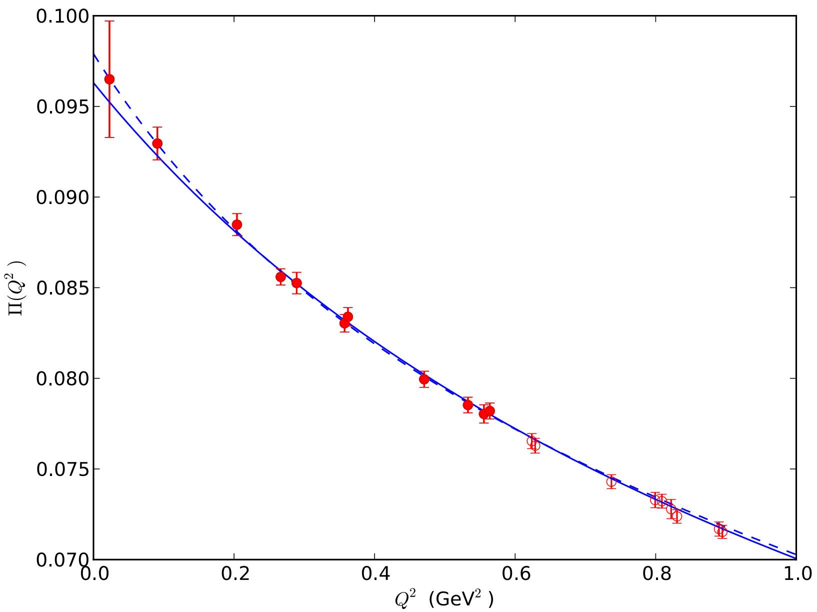
- Multi point Pade fit   [2012, Aubin et al.]

$$\Pi(Q^2) = \Pi(0) - Q^2\left(a_0 + \sum_{n=1}^{[P/2]}\frac{a_n}{b_n + Q^2}\right)$$

Conditions : $a_n > 0$, $b_n > 4\, m_\pi^2$

Pade fit results

- solid: correlated fit (q2 <=0.6 GeV2) ,
  dash : uncorrelated fit (q2 <= 1 GeV2)

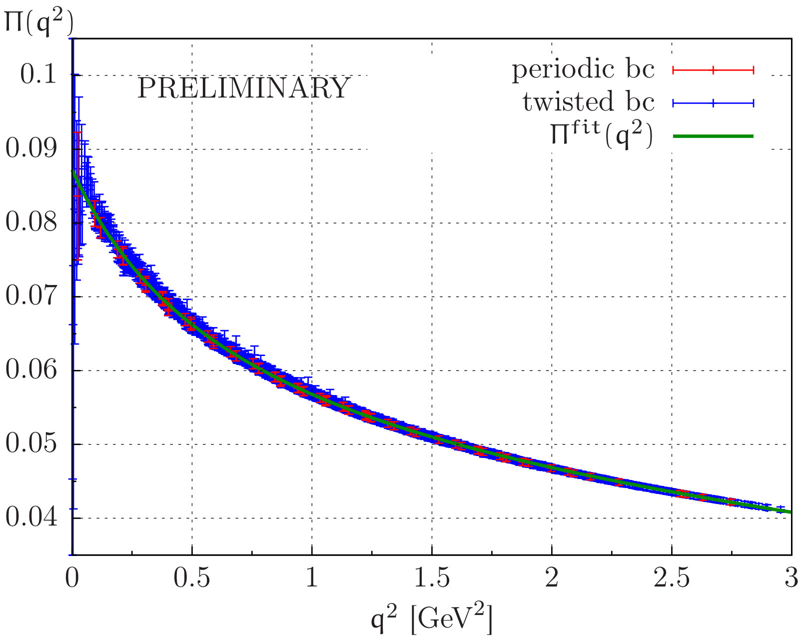


	χ^2/dof	$10^{10} a_\mu^{\mathrm{HLO},Q^2\le 1}$	$\Pi(0)$	a_i	b_i	a_0
VMD	38.6/18	646(8)	0.1222(6)	0.0595(8)	0.64 (fixed)	–
[0, 1]	14.3/17	550(20)	0.1203(7)	0.0646(16)	0.83(5)	–
[1, 1]	13.9/16	572(41)	0.1206(8)	0.052(16)	0.68(20)	0.005(7)
[1, 2]	13.9/15	572(37)	0.1206(8)	0.052(14)	0.68(19)	–
				1(6)	$0.3(1.0) \times 10^3$	
[2, 2]	13.9/14	572(38)	0.1206(8)	0.052(14)	0.68(18)	0.003(27)
				1(31)	$0.4(6.0) \times 10^3$	

[Aubin et al. Phys. Rev. D 86 (2012) 054509]

- Pade approximation converges, results stable.

Twisted boundary condition



B. Jaeger [Mainz group] @ Lattice 2012

- On a torus, the action must be single-valued, while fields do not have to be.
- Impose the twisted boundary condition on quark fields.

$$q(x+L) = q(x)\exp(i\theta) \quad \rightarrow \quad p = (2\pi n + \theta)/L$$

(θ :arbitrary input)

- q^2 can be arbitrary small.
- Breaking isospin, Vector ward identity is broken, could be exactly subtracted [Aubin et al 2012]
- Noise in small q2

Exploring time-like mom
[Eigo Shintani, Hyung-Jin Kim & TI]

- To reduce systematic error

  Transformation to time-like momentum using analytical continuation

  Ji and Jung, Phys.Rev.Lett. 86, 208(2001); Dudek et al. Phys.Rev.Lett. 97 (2006) 172001; Feng, et al.(JLQCD),Phys.Rev.Lett.109,182001(2012)

- Domain-wall fermion (RBC/UKQCD) in N_f = 2+1
  - $24^3 \times 64$ (a^{-1} = 1.73 GeV), $32^3 \times 64$ (a^{-1} = 2.25 GeV): m_π = 300--400 MeV
  - Good chiral property and scaling behavior.

  Remark: precise determination of α_s with pQCD in high Q^2.

Shintani, et al.(JLQCD), Phys. Rev. D79, 074510 (2009); Shintani, et al.(JLQCD), Phys. Rev. D82, 074505 (2010)

Time-like momentum

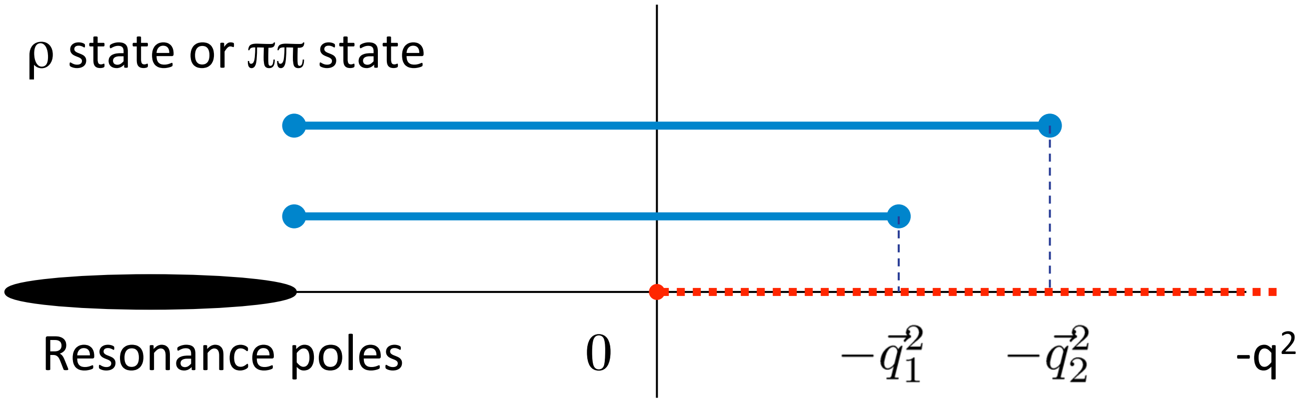
- $Q_4 = i\omega$

$$\int d^4x \langle T\{V^{\rm em}_\mu(x) V^{\rm em}_\nu(0)\}\rangle e^{iqx} = \Pi_{\mu\nu}(\vec{q},\omega) = (q^2 g_{\mu\nu} - q_\mu q_\nu)\Pi_V(q^2)$$

$$q = (\omega, \vec{q}), \quad g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1), \quad q^2 = \omega^2 - \vec{q}^2 = -Q^2$$

  - ω is "photon energy" which can be controlled by hand.
  - Temporal integral from $-\infty < t < \infty$ (Laplace transformation)

$$\Pi_{\mu\nu}(\vec{q},\omega) = \int_0^\infty dt \sum_{\vec{x}} e^{-\omega t - i\vec{x}\vec{q}} \langle V_\mu(\vec{x},t) V_\nu(0)\rangle_c + \int_{-\infty}^0 dt \sum_{\vec{x}} e^{-\omega t - i\vec{x}\vec{q}} \langle V_\nu(0) V_\mu(\vec{x},t)\rangle_c$$



Time-like momentum

- Modeling large time behavior

To perform the infinite temporal integral, we need to model 2pt at large time

$$\sum_{\vec{x}} e^{i\vec{q}\vec{x}} \langle V_\mu(x) V_\nu(0) \rangle \simeq g_v e^{-E_V t}$$

(asymptotic state dominance at t $\geqq$ t_{cut})

$$\int_0^{t_{cut}} dt e^{-\omega t} \sum_{\vec{x}} e^{i\vec{q}\vec{x}} \langle V_\mu(x) V_\nu(0) \rangle \simeq \sum_{t=0}^{t_{cut}} C_{VV}(\vec{q}, \omega; t)$$

(numerical integral with lattice data from 0 $\leqq$ t $\leqq$ t_{cut})

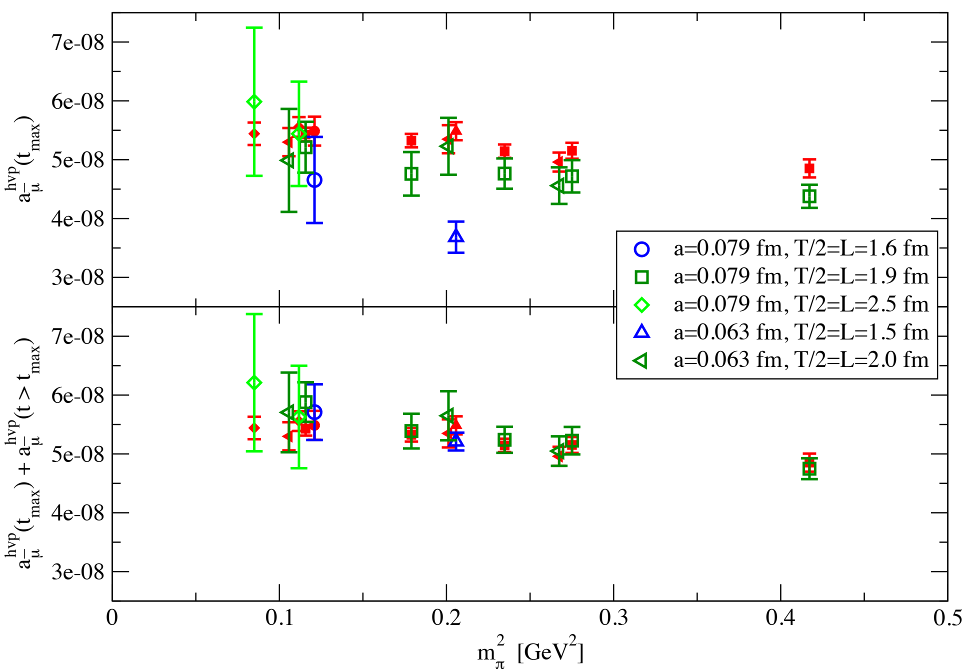
Longitudinal part will be

$$\Pi_{\rm long}(\vec{q}, \omega) = \frac{g_V}{E_V + \omega} e^{-(E_V + \omega) t_{\rm cut}} + \frac{g_V}{E_V - \omega} e^{-(E_V - \omega) t_{\rm cut}} + \sum_{t=0}^{t_{\rm cut}} 2F(t) \cosh \omega t$$

Finally we consider the particular momentum $q_\mu \neq 0, \; q_{j\neq\mu} = 0$

$$\Pi_{\rm long}(\vec{q}, \omega) = -\omega^2 \Pi_V(q^2), \quad q^2 = \omega^2 - q_\mu^2$$

Time-like momentum result



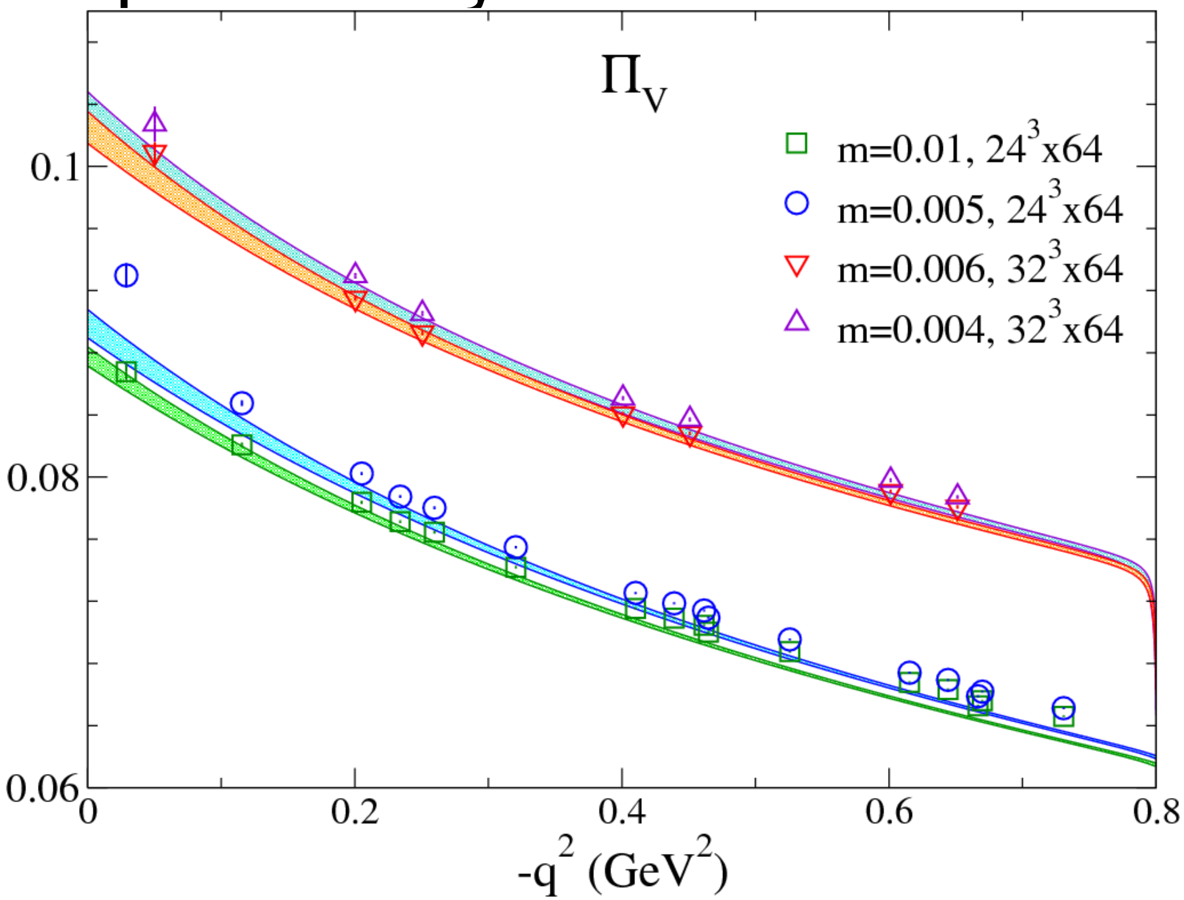
- tm-Wilson quark (maximal twist)
- pion mass : 290 MeV – 650 MeV
- a = 0.08 fm, 0.06 fm

[Xu Feng et al. [ETMC+JLQCD] Phys.Rev. D88 (2013) 034505]

- larger stat. error than conventional method

HVP with time-like momentum

- preliminary



t_{cut} = 9 (24^3), 10 (32^3)
Fitting range at large t
[8,13] (24^3), [10,15] (32^3)

- Similar behavior with results obtained in Euclid momentum
- Slight discrepancy from HVP in space-like momentum, especially for light mass.

More carefully systematic study is necessary !

[Blum, TI, Shintani PRD 88 (2013) 094503]

Covariant Approximation Averaging (CAA) a new class of Error reduction techniques

Original

$$\mathcal{O} = \mathcal{O}^{(\mathrm{appx})} + \mathcal{O}^{(\mathrm{rest})}$$

Lattice Symmetry

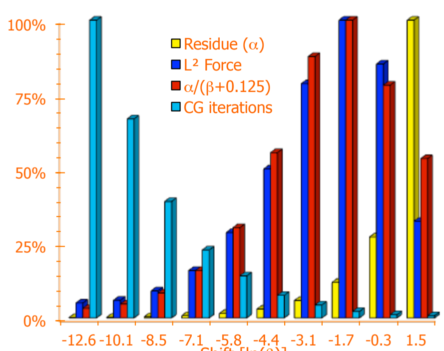
unbiased imporved

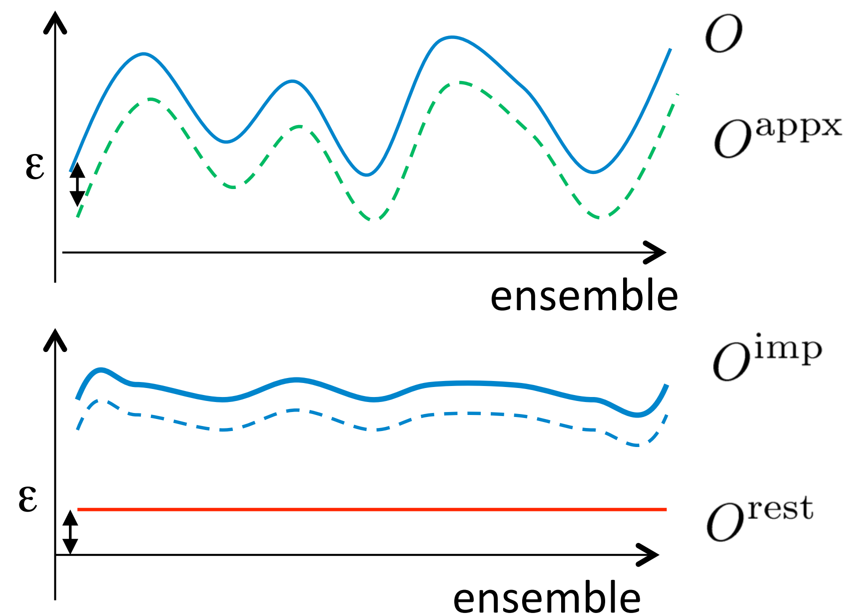
$$\mathcal{O}^{(\mathrm{imp})} = \mathcal{O}^{(\mathrm{rest})} + \frac{1}{N_G} \sum_{g \in G} \mathcal{O}^{(\mathrm{appx}),g}$$

Expensive : infrequently measured

Cheap : frequently measured

- $O^{(\mathrm{imp})}$ has smaller error
  $O^{(\mathrm{appx})}$ need to be cheap & not to be too accurate
  N_G suppresses the bulk part of noise cheaply





Examples of Covariant Approximations (contd.)

- All Mode Averaging **AMA**

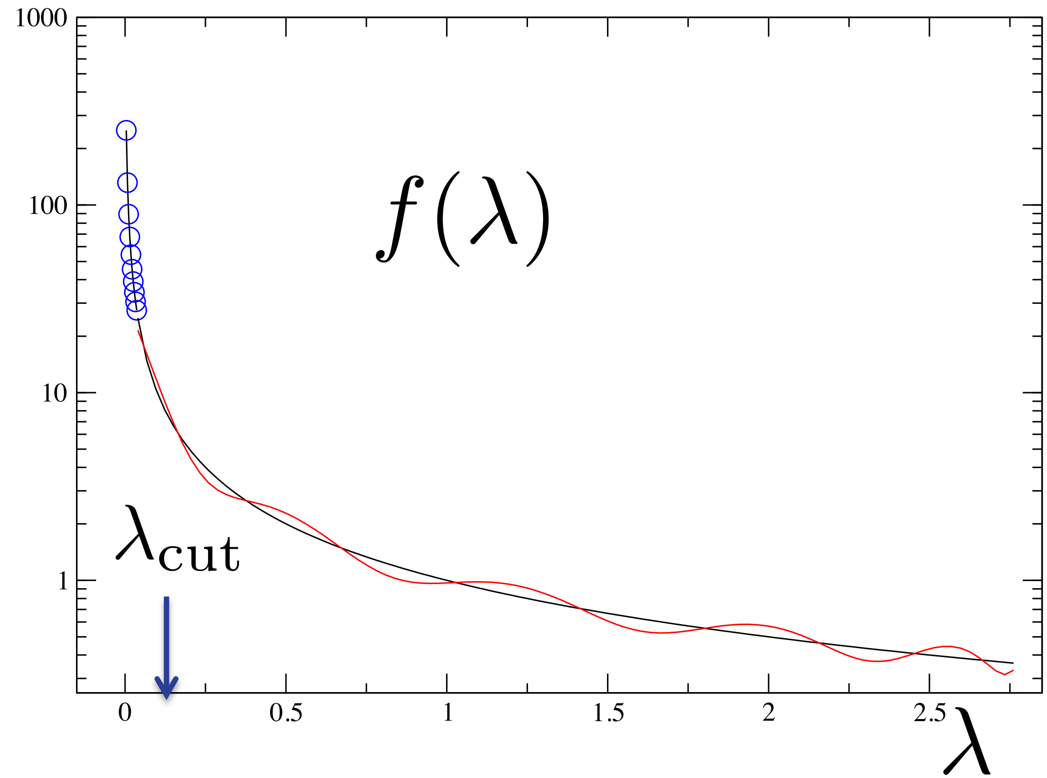
  Sloppy CG or Polynomial approximations

$$\mathcal{O}^{(\mathrm{appx})} = \mathcal{O}[S_l],$$

$$S_l = \sum_\lambda v_\lambda f(\lambda) v_\lambda^\dagger,$$

$$f(\lambda) = \begin{cases} \frac{1}{\lambda}, & |\lambda| < \lambda_{\mathrm{cut}} \\ P_n(\lambda) & |\lambda| > \lambda_{\mathrm{cut}} \end{cases}$$

$$P_n(\lambda) \approx \frac{1}{\lambda}$$



accuracy control :
- low mode part : # of eig-mode
- mid-high mode : degree of poly.

If quark mass is heavy, e.g. ~ strange, low mode isolation may be unneccesary

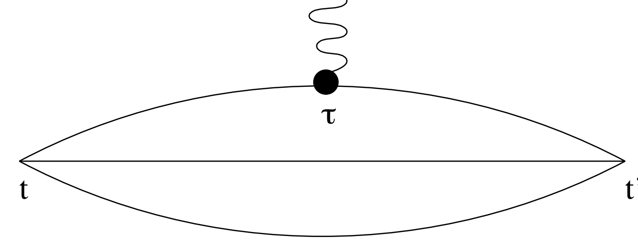
AMA at work

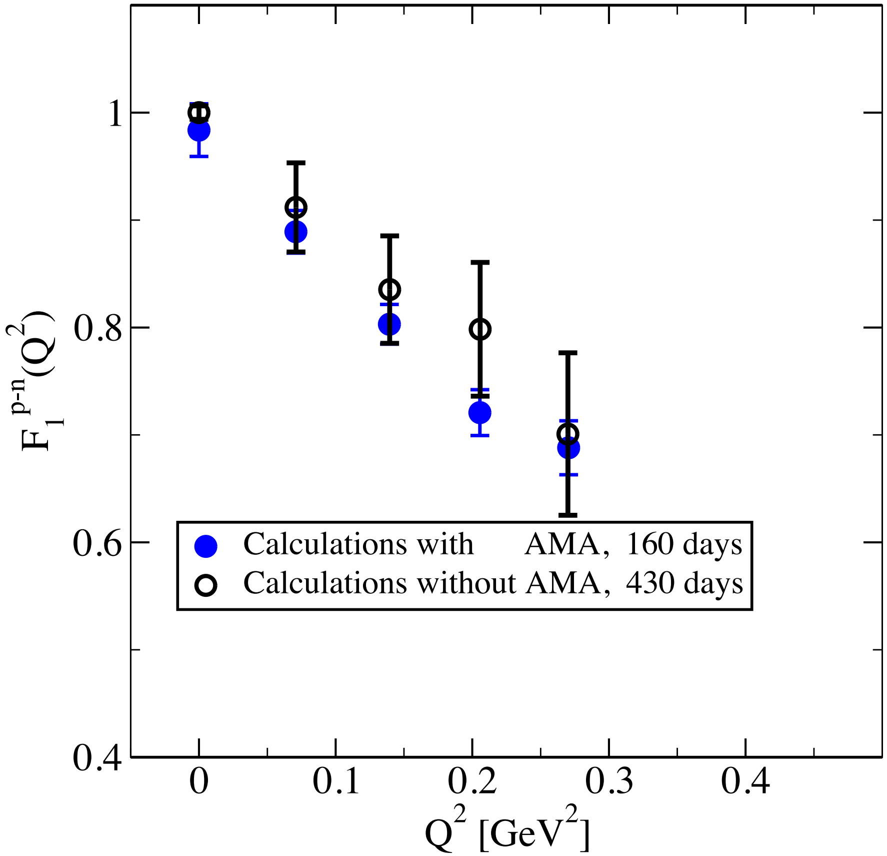
- Target : $V=32^3 \times 64 = (4.6\text{fm})^3 \times 9.6\text{fm}$, Ls=32 Shamir-DWF, a^{-1}=1.37 GeV, Mpi = 170 MeV
- Use Ls=16 Mobius as the approximation
  [Brower, Neff, Orginos, arXiv:1206.5214]

- quark propagator cost on SandyBridge 1024 cores (XSEDE gordon@SDSC)
  - non-deflated CG, r(stop)=1e-8 : ~9,800 iteration, 5.7 hours / prop
  - Implicitly restarting Lanczos of Chebyshev polynomials of even-odd prec operator for 1000 eigenvectors
    [Neff et al. PRD64, 114509 (2001)] : 12 hours
  - deflated CG with 1000 eigenvectors : ~700 iteration, 20 min /prop
  - deflation+sloppy CG, r(stop)=5e-3 : ~125 iteration, 3.2 min /prop
- Multiplicative Cost reduction for General hadrons could combine with {EigCG | AMG} and Distillation:
  x1.2 (Mobius) x 14 (deflation) x 7 (sloppy CG) ~ x 110

AMA at work

[M. Lin]





- $F_1(Q^2)$: tsep = 9 a ~ 1.3 fm

  1 forward + 2 (up and down) seq-props, contraction cost is ~15% of sloppy propagator

- Error bar
  x 2 – 2.7 ~ sqrt(4400/600)
- Total cost reduction upto
  (430 / 160) * (4400/600)
  ~ x19.7

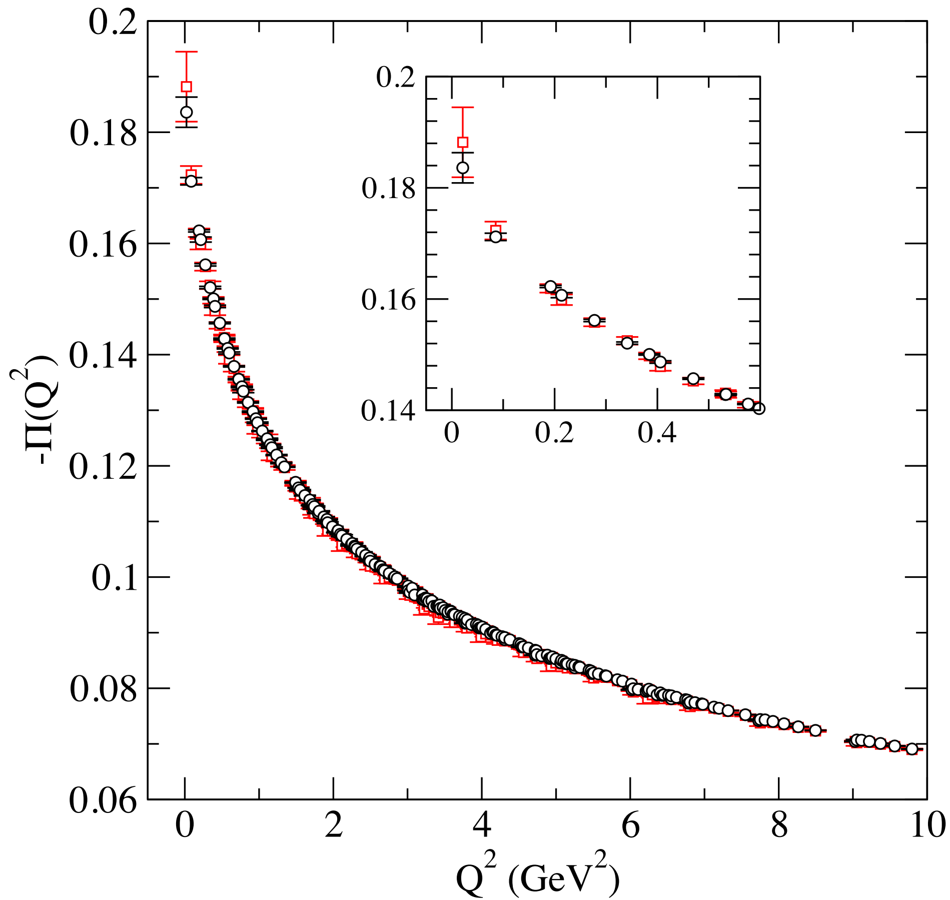
- Note this is still sub-optimal, 4 exact source and without deflation. (would be x30 for 2 exact sources)

- non-deflated CG, 150 config x 4 sources = 600 measurements :
  5.7 * 3 * 4 * 150 config = 10K hours, 430 days

- AMA : 39 config, 4 exact solves / config (perhaps overkill) , N_G=112 sloppy solves
  => 39 x 112 = 4400 AMA measurements :
  (5.7 * 3 * 4 + 12 + 0.06 * 3 * 112) * 39 config = 3.9 K hours, 160 days
  4-exact (68%) + Lanczos (12%) + sloppy CG (20%)

Improving HVP statistics using AMA

- Staggered Fermion (MILC Asqtad, Mpi=300 MeV) 2.6 -- 20 times smaller error with same cost

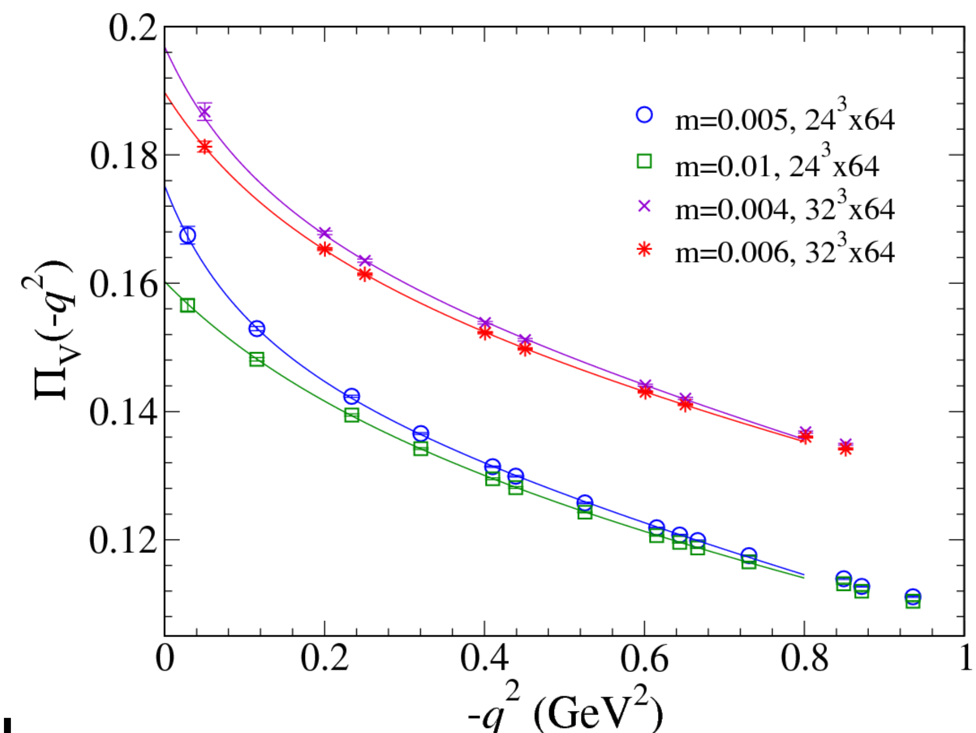


Now getting to
all stat error < 2%

qmin^2 = 1.5 m(mu)

RBC/UKQCD DWF AMA Results

- Two lattice spacings
  a = 0.11, 0.088fm,
  Mpi=0.28-0.33 GeV
  All stat err < 0.7%
  q_min = 2 m(mu)
- Applied [2,1] Pade,
  can't fit with
  b1 >= 4 Mpi^2 bound



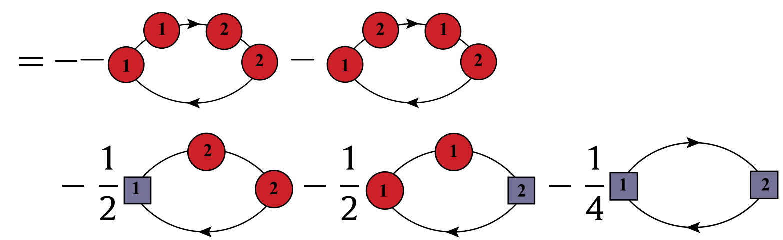
Lattice	m_u	$\Pi_V(0)$	a_0 (GeV^{-2})	a_1	b_1 (GeV^2)	χ^2/dof	$4m_\pi^2$ GeV^2
$24^3\times64$	0.005	0.1752(2)	0.0325(2)	0.0407(1)	0.139(1)	2.7(4)	0.44
	0.01	0.1603(2)	0.0219(3)	0.0434(4)	0.408(7)	0.4(1)	0.71
$32^3\times64$	0.004	0.197(2)	0.026(3)	0.052(3)	0.227(37)	0.08(7)	0.31
	0.006	0.190(3)	0.027(7)	0.043(11)	0.253(25)	0.4(5)	0.44

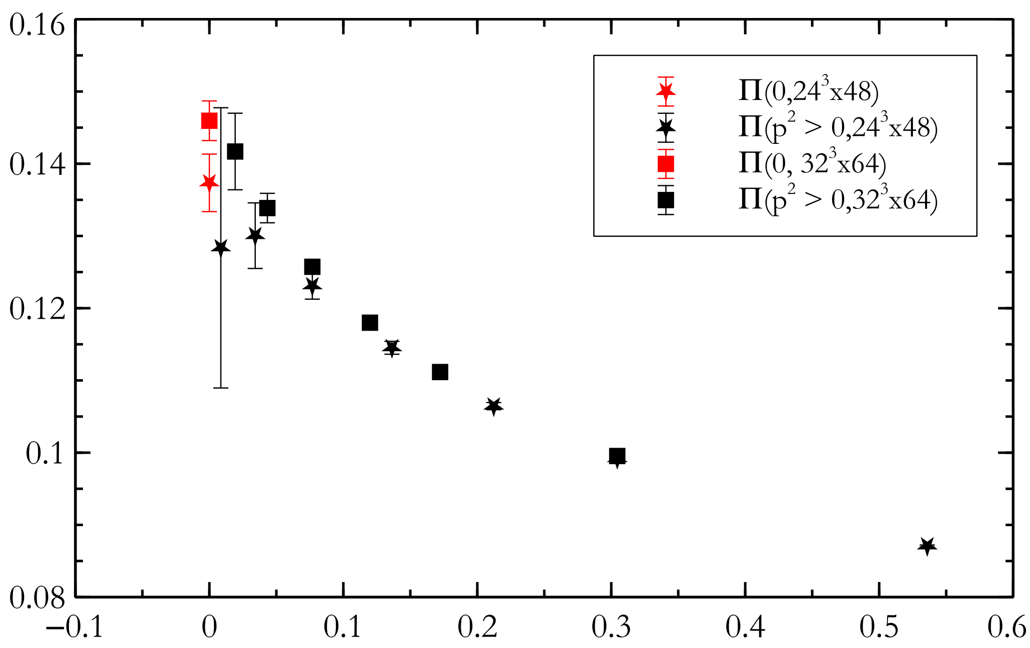
Subtraction Strategy: Derivative of Twisting Angle [Divitiis et al. PLB 718(2012) 589]

- $p_i = (2\pi n + \theta_i)/L$

$$C^{\mu\nu}(p) = \frac{1}{(TL^3)^2}\sum_{x,y} e^{ip(y-x+\hat{\nu}/2-\hat{\mu}/2)}\langle V^{\mu}_{em}(x)V^{\nu}_{em}(y)\rangle$$
$$= \left(\delta^{\mu\nu}\hat{p}^2 - \hat{p}^{\mu}\hat{p}^{\nu}\right)\Pi(p^2),$$

$$\Pi(0) = -\left.\frac{\partial^2 \hat{C}^{12}(p)}{\partial p_1 \partial p_2}\right|_{p^2=0}$$





Use of Time-Moments [HPQCD]

- Compute Time-moments of 2pt [P. Lepage's talk]

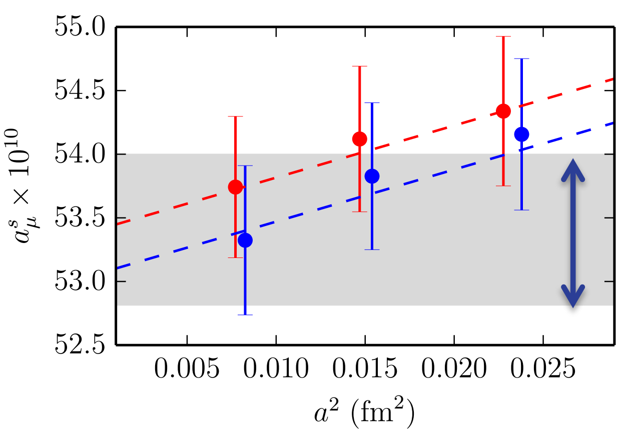
$$G_{2n} \equiv a^4 \sum_t \sum_{\vec{x}} t^{2n} Z_V^2 \langle j^i(\vec{x},t) j^i(0) \rangle$$

$$= (-1)^n \frac{\partial^{2n}}{\partial q^{2n}} q^2 \hat{\Pi}(q^2) \Big|_{q^2=0} .$$

$$\hat{\Pi}(q^2) = \sum_{j=1}^{\infty} q^{2j} \Pi_j$$

$$\Pi_j = (-1)^{j+1} \frac{G_{2j+2}}{(2j+2)!} .$$

- subtraction by taking derivatives, use local currents
- Pade approximation, <u>determined</u> from Πj , for high q2 integration



1.1 %

Strange

$$a_\mu^s = 53.41(59) \times 10^{-10}.$$

[1.1% ~ lattice spacing error]

charm

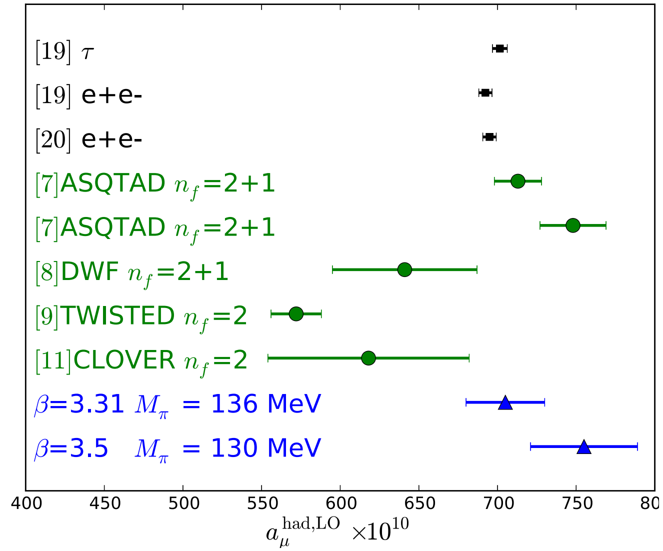
$$a_\mu^c = 14.42(39) \times 10^{-10}.$$

[2.7% ~ Z_V error]

Recent results

roughly

5—10 % error



[BMW Collaboration, 1311.4446]

a_μ	N_f	errors	action	group
713(15)	2+1	stat.	Asqtad	Aubin, Blum (2006)
748(21)	2+1	stat.	Asqtad	Aubin, Blum (2006)
641(33)(32)	2+1	stat., sys.	DWF	UKQCD (2011)
572(16)	2	stat.	TM	ETMC (2011)
618(64)	2+1[1]	stat., sys.	Wilson	Mainz (2011)

HVP Summary and future prospects

- Lattice HVP issues

  - Parameterize / Fit low Q2

    → Model Independent Pade, Time-moments of HVP

    (error from parameterization dependence)
  - More precise data at low Q2

    → Twisted B.C. , Derivatives of twist angle,

    Time-like momentum, or simply large volume
  - Discretization error, Quark Mass dependence

    → Nf=2+1, 2+1+1, Physical quark mass calculations are running
  - Statistical error

    → All Mode Averaging (AMA) helps to reduction of statistical error.
  - Disconnected quark loop
  - EM Isospin / wall