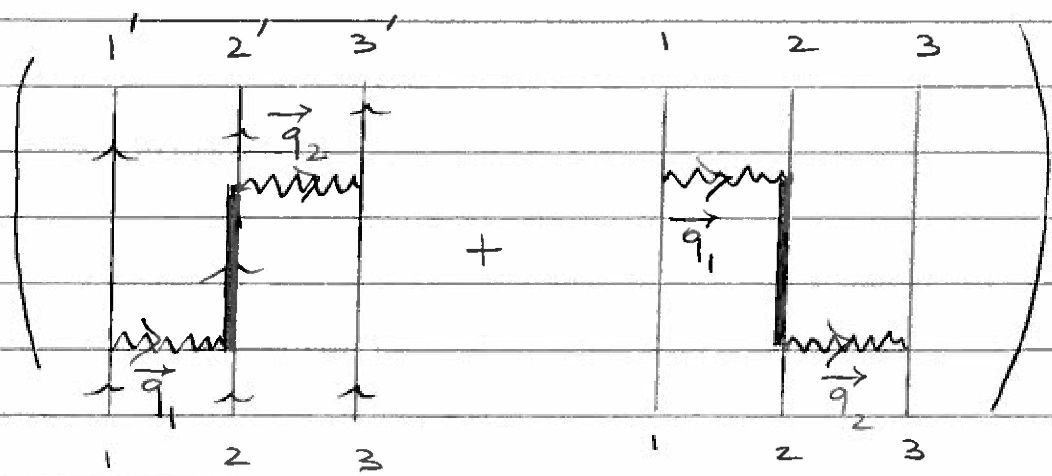
Two pion-exchange three-nucleon interaction

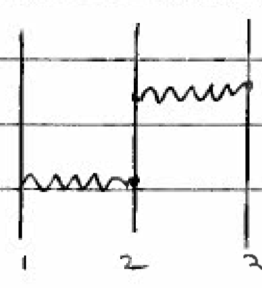
In nuclear physics three-nucleon interactions V_{ijk} play an important role, even though $\langle V_{ijk} \rangle \ll \langle v_{ij} \rangle$. This is because in a nucleus there is a large cancellation between $\langle T \rangle$ and $\langle v_{ij} \rangle$. It is an established fact that, in order to reproduce the observed spectra of light nuclei (up to ^{12}C) and the equilibrium density of symmetric nuclear matter, it is necessary to include three-nucleon potentials in the nuclear Hamiltonian.

The OPETP's generate three-nucleon interactions

$$V^{2\pi}_{123} = \sum_{cyc}$$



Three-body forces arise whenever one of the bodies is polarizable. In the nuclear case, the pion field from nucleon 1 "polarizes" the spins (and flavors) of the quarks in nucleon 2 (i.e., creates a Δ) and this changes the force between nucleons 2 and 3. Note that



$\simeq v_{23}\, v_{12}$ not a three-body force

A simple estimate of $V^{2\pi}$ is obtained in the static limit

$$V_{123}^{2\pi} \sim \sum_{cyc} \frac{\left[v^{\pi}_{NN\to\Delta N}(23) \right]^{\dagger} v^{\pi}_{NN\to N\Delta}(12) + \left[v^{\pi}_{NN\to N\Delta}(12) \right]^{\dagger} v^{\pi}_{NN\to\Delta N}(23)}{m_N - m_\Delta}$$

This potential is known as the Fujita-Miyazawa three-nucleon potential and is the leading contribution to V_{ijk}. It can be expressed as

$$V_{123}^{2\pi} = \sum_{cyc} \Big[A^{\Delta}_{2\pi} \{ X_{12}, X_{23} \} \{ \vec{\tau}_1 \cdot \vec{\tau}_2, \vec{\tau}_1 \cdot \vec{\tau}_2 \}$$

$$+ C^{\Delta}_{2\pi} [\ \ ,\ \ ] [\ \ ,\ \ ] \Big]$$

$$X_{ij} = T_\pi(r_{ij}) S_{ij} + Y_\pi(r_{ij}) \vec{\sigma}_i \cdot \vec{\sigma}_j$$

$$A^{\Delta}_{2\pi} = -\frac{2}{81} \frac{f^2_{\pi N\Delta}}{4\pi} \frac{f^2_{\pi NN}}{4\pi} \frac{m_\pi^2}{m_\Delta - m_N} \sim -0.044 \text{ MeV}$$

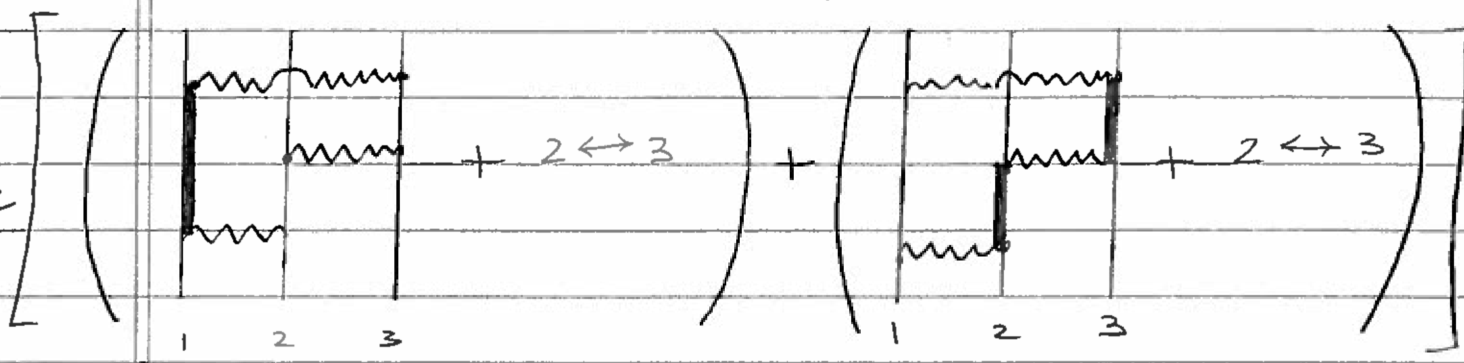
$$C^{\Delta}_{2\pi} = \frac{1}{4} A^{\Delta}_{2\pi}.$$

In fact the strengths $A_{2\pi}$ and $C_{2\pi}$ are obtained by fitting the binding energies of light nuclei.

Tree pion exchange three-nucleon interaction

There are strong indications from calculations of energy spectra in nuclei with A > 4 that three pion-exchange three-nucleon interactions are not negligible. Among the many possibilities, two contributions have been considered in detail with the OPETP technique

$$\sum_{cyc}$$



After some tedious algebra, the $V_{123}^{3\pi}$ above can be written as

$$V_{123}^{3\pi} = A_{3\pi}\, O_{123}^{3\pi}$$

$$O_{123}^{3\pi} = \frac{50}{3} S_\sigma^I S_\tau^I + \frac{26}{3} A_\sigma^I A_\tau^I$$

$$S_\tau^I = 2 + \frac{2}{3}(\vec{\tau}_i \cdot \vec{\tau}_j + \vec{\tau}_j \cdot \vec{\tau}_k + \vec{\tau}_k \cdot \vec{\tau}_i) = 4P^{T=3/2}$$

$$A_\tau^I = \vec{\tau}_i \cdot (\vec{\tau}_j \times \vec{\tau}_k)$$

Finally, the two and three pion-exchange potentials are found to be mostly attractive in light nuclei calculations. In addition $V^{2\pi}$ and $V^{3\pi}$, a repulsive term is included

$$V^R$$

A realistic nuclear Hamiltonian

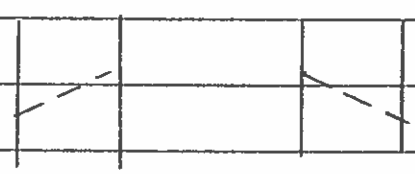
The "basic model" Hamiltonian reads

$$H = \sum_i \frac{\vec{p}_i^{\,2}}{2m_i} + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk}, \qquad V_{ijk} \sim V^{2\pi} + V^{3\pi} + V^{R}$$

v_{ij} has long and intermediate range ($r \gtrsim 2$ fm and $1\,\text{fm} \lesssim r \lesssim 2$ fm, respectively) from

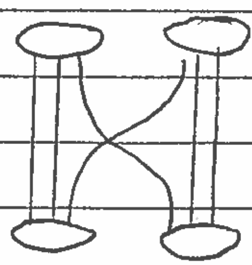
i) γ-exchange
ii) π-exchange
iii) 2π-exchange (leading process via Δ excitation)

There are additional contributions to v_{ij}, but these have short range ($r \lesssim 1$ fm) and are theoretically not well known. Mechanisms influencing this short-range component include: i) heavy meson exchange



	J^π	I	mass (GeV)
η	0^-	0	0.55
ρ	1^-	1	0.77
ω	1^-	0	0.78

ii) quark exchange



The operator structure of this short-range part can be constrained by symmetry requirements (invariance under isospin rotations, hermiticity $v^\dagger = v$, translational invariance, Galilean invariance, invariance under space inversion and time reversal, rotational invariance). One can show that

$$v = v_0 + v_\tau \, \vec{\tau}_1 \cdot \vec{\tau}_2$$

and each v_a, $a = 0$ or τ, can be written as

$$v_a = v_{a,1}(r^2, p^2, L^2) + \vec{L}\cdot\vec{S}\, v_{a,2}(\ ,\ ,\ )$$
$$+ \vec{\sigma}_1\cdot\vec{\sigma}_2 \, v_{a,3}(\ ,\ ,\ ) + \left\{ S_{12}, v_{a,4}(\ ,\ ,\ ) \right\}$$
$$+ \left\{ P_{12}, v_{a,5}(\ ,\ ,\ ) \right\} + \frac{1}{2}(\vec{\sigma}_1\cdot\vec{L}\,\vec{\sigma}_2\cdot\vec{L} + \vec{\sigma}_2\cdot\vec{L}\,\vec{\sigma}_1\cdot\vec{L})\, v_{a,6}(\ ,\ ,\ )$$

where

$$P_{12} = 3\,\vec{\sigma}_1\cdot\vec{p}\;\vec{\sigma}_2\cdot\vec{p} - p^2\,\vec{\sigma}_1\cdot\vec{\sigma}_2 .$$

In the AV18: $v_{a,1} = v^{(1)}_{a,1}(r) + v^{(2)}_{a,1}(r)\, L^2$ and similarly for $v_{a,3}$; $v_{a,2}$, $v_{a,4}$, and $v_{a,6}$ are taken to depend on r alone; the term with P_{12} is neglected.

Finally, in order to fit pp and np data precisely it is necessary to include isospin symmetry breaking terms induced by the strong interactions (in addition to the isospin symmetry breaking terms due to electromagnetic interactions). An important contribution comes from the $\pi^{\pm}$ and π^{0} mass difference, and leads to the OPEP

$$v^{\pi}(\vec{r}) = v^{\pi}_{IS}(\vec{r})\,\vec{\tau}_1\cdot\vec{\tau}_2 + v^{\pi}_{IT}\,T_{12}\quad 3\tau_{1z}\tau_{2z} - \vec{\tau}_1\cdot\vec{\tau}_2$$

$$v_{IS}(\vec{r}) = \frac{1}{3}\left[v_0(\vec{r}) + 2v_+(\vec{r})\right]$$

$$v_{IT}(\vec{r}) = \frac{1}{3}\left[v_0(\vec{r}) - v_+(\vec{r})\right]$$

Note that $T_{12}\,|T=0\rangle = 0$ (so v^{π}_{IT} does not have any effect on the deuteron); however,

$$T_{12}\,|T=1, M_T=\pm1\rangle = 2\,|T=1, M_T=\pm1\rangle$$

$$T_{12}\,|T=1, M_T=0\rangle = -4\,|T=1, M_T=0\rangle$$

and therefore v^{π}_{IT} affects differently pp or nn pairs from $T=1$ np pairs.

Electromagnetic current of nucleons and nuclei

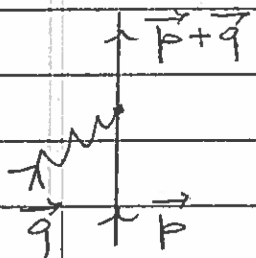
The interaction of an external EM field with a nucleus is given by

$$H_{EM} = e\int d\vec{x}\, A^{\mu}(\vec{x})\, j_{\mu}(\vec{x}) \qquad \text{current density}$$

$$= e\int d\vec{x}\left[\phi^{0}(\vec{x})\,\underbrace{\rho_c(\vec{x})}_{\text{charge density}} - \vec{A}(\vec{x})\cdot\vec{j}(\vec{x})\right]$$

The EM field is quantized, and $A(\vec{x})$ either creates or destroys photons

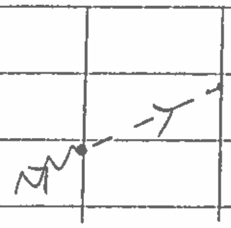
A selection of processes: 1-body



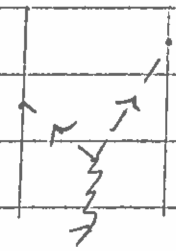
$$j_1^{(i)},\; i=1,\dots,A$$

Only one nucleon absorbs the momentum of the photon via one-body current. However, that nucleon can later share the absorbed momentum with other nucleons via v_{ij} or V_{ijk}. These interactions determine the final state of the nucleus, and are not a part of the current operator. They are called final state interactions in approaches based on perturbation theory. Interactions that take place before the absorption of the photon are called initial state interactions. Non-perturbative approaches use eigenstates of the nuclear Hamiltonian, and treat only the interaction of the external photon (real or virtual) as a weak perturbation. The nuclear eigenstates contain all effects of nuclear forces, including those of the EM interactions between nucleons in the nucleus.

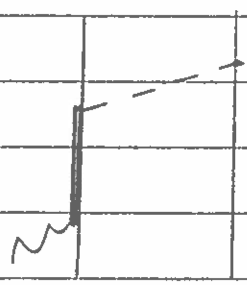
2-body:



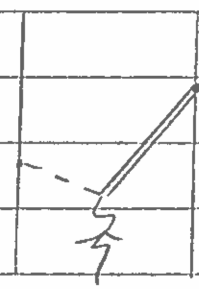
photomeson current



meson current

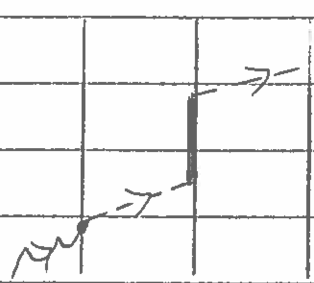


nucleon excitation current

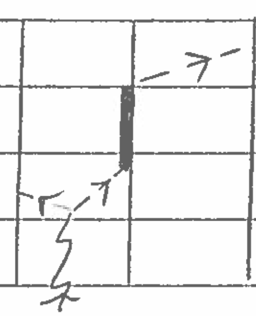


meson excitation current

3-body:



photomeson current



meson current

$$j = \sum_i j_1^{(i)} + \sum_{i<j} j_2^{(ij)} + \sum_{i<j<k} j_3^{(ijk)} + \dots$$

Non-relativistic one-body current

The photon gives its momentum $\vec{q}$ to the nucleon that absorbs it. Hence it is convenient to consider

$$j(\vec{q}) = \int d\vec{x}\, e^{i\vec{q}\cdot\vec{x}}\, j(\vec{x})$$

Including the finite size of the nucleon, one finds

$$\rho_{c,1}(\vec{q}) = \sum_i e^{i\vec{q}\cdot\vec{r}_i} \frac{1}{2}\left[G_E^p(q)\frac{1+\tau_{z,i}}{2} + G_E^n(q)\frac{1-\tau_{z,i}}{2}\right]$$

$$= \sum_i e^{i\vec{q}\cdot\vec{r}_i} \frac{1}{2}\left[G_E^S(q) + G_E^V(q)\tau_{z,i}\right],$$

$$G_E^S(0) = 1 = G_E^V(0)$$

$$\vec{j}_1(\vec{q}) = \vec{j}_{c,1}(\vec{q}) + \vec{j}_{m,1}(\vec{q})$$

with

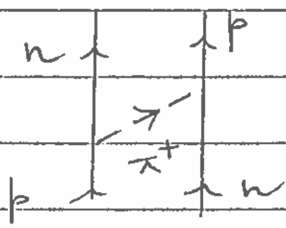
$$\vec{j}_{c,1}(\vec{q}) = \sum_i \frac{1}{2m_N}\left\{e^{i\vec{q}\cdot\vec{r}_i}, \vec{p}_i\right\} \frac{1}{2}\left[G_E^S(q) + G_E^V(q)\tau_{z,i}\right]$$

$$\vec{j}_{m,1}(\vec{q}) = \sum_i e^{i\vec{q}\cdot\vec{r}_i} \frac{i\,\vec{\sigma}_i\times\vec{q}}{2m_N} \frac{1}{2}\left[G_M^S(q) + G_M^V(q)\tau_{z,i}\right]$$

$$G_M^S(0) = \mu_p + \mu_n \simeq 0.88\,\mu_N \qquad G_M^V(0) = \mu_p - \mu_n \sim 4.71\,\mu_N$$

Photopion and pion currents

The dominant two-body current occurs when a proton (neutron) i at $\vec{r}_i$ changes into a neutron (proton) j at $\vec{r}_j$ by a charge-exchange interaction



It is necessary if the current carried by the protons is to be conserved.

To generate the currents due to pion exchange (charged pions!), use minimal substitution

$$\vec{\nabla}\phi_{\pm} \longrightarrow \left[\vec{\nabla} \pm ie\vec{A}\right]\phi_{\pm}$$

and in the $H_{\pi NN}$ interaction this leads to the $\gamma\pi N$ interaction

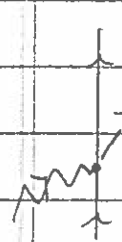
$$H_{\gamma\pi N} = -i\frac{f_{\pi NN}}{m_\pi}\vec{\sigma}\cdot\vec{A}(\vec{r})\left[\phi_+(\vec{r})\tau_- - \phi_-(\vec{r})\tau_+\right]$$

$$= \frac{f_{\pi NN}}{m_\pi}\vec{\sigma}\cdot\vec{A}(\vec{r})\;\varepsilon_{abz}\tau_a\phi_b(\vec{r}) ,$$

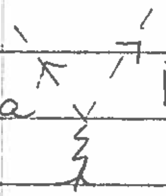
while in the pion Hamiltonian it leads to the $\gamma\pi\pi$ interaction

$$H_{\gamma\pi\pi} = -\varepsilon_{abz}\int d\vec{r}\,\vec{A}(\vec{r})\cdot\left[\vec{\nabla}\phi_a(\vec{r})\right]\phi_b(\vec{r}) .$$

The vertices associated with $H_{\gamma\pi N}$ and $H_{\gamma\pi\pi}$ are given by

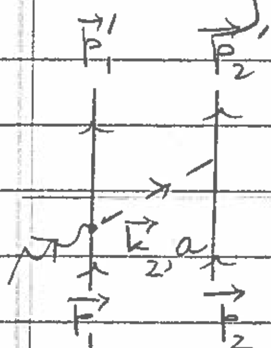


$$-\frac{f_{\pi NN}}{m_\pi}\,\vec{\sigma}\cdot\vec{A}\,\frac{1}{\sqrt{2\omega_k}}\,\varepsilon_{abz}\,\tau_b$$

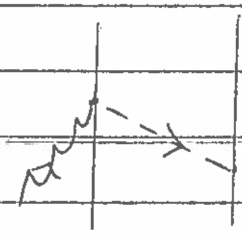


$$i\,\varepsilon_{abz}\,\frac{1}{\sqrt{4\omega_{k_i}\omega_{k_j}}}\,\vec{A}\cdot(\vec{k}_i-\vec{k}_j)$$

The two-body photopion current follows from



(a)



(b)

$$E_i = E_1 + E_2 + \omega_q$$

$$E_f = E_1' + E_2'$$

$$(a) = \left(-\frac{f_{\pi NN}}{m_\pi}\,\vec{\sigma}_1\cdot\vec{A}\,\frac{1}{\sqrt{2\omega_{k_2}}}\,\varepsilon_{abz}\,\tau_{b,1}\right)\frac{1}{E_i - E_1' - \omega_{k_2} - E_2}$$

$$\times\left(-i\frac{f_{\pi NN}}{m_\pi}\,\frac{1}{\sqrt{2\omega_k}}\,\vec{\sigma}_2\cdot\vec{k}_2\,\tau_{a,2}\right)$$

static limit

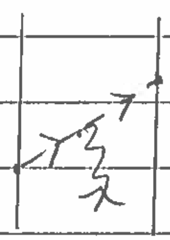
$$\sim\ i\,\frac{f^2_{\pi NN}}{m_\pi^2}\,\frac{1}{2\omega_k^2}\,(\vec{\tau}_1\times\vec{\tau}_2)_z\,\vec{\sigma}_1\cdot\vec{A}\,\vec{\sigma}_2\cdot\vec{k}_2$$

Diagram (b) gives an identical contribution (in the static limit) and

$$\vec{j}_2^{\,PM,\pi}(\vec k_1,\vec k_2) = i\frac{f_{\pi NN}^2}{m_\pi^2}(\vec\tau_1\times\vec\tau_2)_z\,\frac{\vec\sigma_1\cdot\vec\sigma_2\cdot\vec k_2}{k_2^2+m_\pi^2} + (1\rightleftarrows 2)$$

$$\vec q = \vec k_1 + \vec k_2 .$$

The two-body pion current is obtained by considering the six time ordered diagrams



$+ \cdots$

In the static limit, one obtains

$$\vec{j}_2^{\,MC,\pi}(\vec k_1,\vec k_2) = i\frac{f_{\pi NN}^2}{m_\pi^2}(\vec\tau_1\times\vec\tau_2)_z(\vec k_1-\vec k_2)\,\frac{\vec\sigma_1\cdot\vec k_1}{k_1^2+m_\pi^2}\,\frac{\vec\sigma_2\cdot\vec k_2}{k_2^2+m_\pi^2}$$

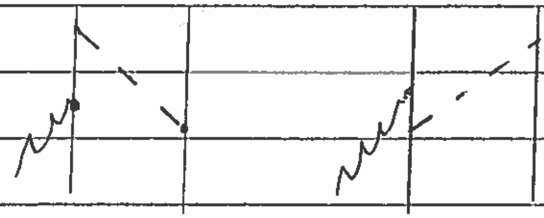
There is a one pion exchange coupling to the nucleon via the scalar potential $\phi^0(\vec r)$

$$H'_{\gamma\pi N} = -\frac{1}{2m_N}\frac{f_{\pi NN}}{m_\pi}\left[\vec\phi(\vec r)\cdot\vec\tau + \phi_z(\vec r)\right]\vec\sigma\cdot\vec\nabla\phi^0(\vec r)$$

It represents a relativistic correction, and vanishes in the static limit $m_N\to\infty$. It leads to a corresponding two-body charge operator

$$\rho_{c,2}^{PM,\pi}(\vec{k}_1,\vec{k}_2) = \frac{1}{2m_N}\frac{f_{\pi NN}^2}{m_\pi^2}\left(\vec{\tau}_1\cdot\vec{\tau}_2+\tau_{z,2}\right)\frac{\vec{\sigma}_1\cdot\vec{q}\;\vec{\sigma}_2\cdot\vec{k}_2}{k_2^2+m_\pi^2} + (1\rightleftarrows 2)$$

The derivation of the two-body charge operator due to the direct coupling of the pion field to the scalar potential is more complicated, since one needs to consider non-static corrections of the six time ordered diagrams as well as of



The operator is given in the class notes.

There is a lot of evidence for the importance of $\vec{j}_2^{PM,\pi}$ and $\vec{j}_2^{MC,\pi}$ in nuclear electromagnetic observables. The classic example is the 10% discrepancy between experiment and theory in the np radiative capture at thermal neutron energies, when the cross section is calculated with $\vec{j}_1$. But other more striking examples are the isovector magnetic moments and form factors of nuclei, nd and n^{3}He radiative captures at threshold ($\vec{j}_2$ gives ~90% of n^{3}He cross section), quasielastic (e,e') scattering, ...

There is also significant evidence for the importance of $\rho_{c,2}^{PM,\pi}$ and $\rho_{c,2}^{MC,\pi}$ mostly from charge form factors of few-nucleon systems.

Charge conservation

The total charge operator is given by

$$\int d\vec{x}\, \rho_c(\vec{x}) = \rho_c(\vec{q}=0)$$

In a nucleus with Z protons

$$\langle A^Z | \int d\vec{x}\, \rho_c(\vec{x}) | A^Z \rangle = Z \quad (\text{in units of } e).$$

Note

$$\rho_{c,1}(\vec{q}=0) = \sum_i \frac{1+\tau_{z,i}}{2}$$

and the requirement above is trivially satisfied by $\rho_{c,1}$. However, it implies

$$\langle A^Z | \rho_{c,2}(\vec{q}=0) + \cdots | A^Z \rangle = 0$$

The two-body pion-exchange charge operators comply with the above.

Locally charge conservation demands

$$\vec{\nabla}\cdot \vec{j}(\vec{x}) + \frac{\partial}{\partial t}\rho_c(\vec{x}) = 0.$$

The time derivative of the Schrödinger picture operator $\rho_c(\vec{x})$ is defined as

$$\frac{\partial}{\partial t}\rho_c(\vec{x}) = i\left[H, \rho_c(\vec{x})\right]$$

In terms of Fourier transforms, one has

$$\vec{q}\cdot\vec{J}(\vec{q}) = \left[H, \rho_c(\vec{q}) \right]$$

To lowest order $1/m_N$, this implies

$$\vec{q}\cdot\vec{J}_1(\vec{q};i) = \left[\frac{p_i^2}{2m_N}, \rho_{c,1}(\vec{q};i) \right]$$

$$\vec{q}\cdot\left[\vec{J}^{PM,\pi}(\vec{q},ij) + \vec{J}^{MC,\pi}(\vec{q},ij) \right] = \left[v^{\pi}_{ij}, \rho_{c,1}(\vec{q};i) + \rho_{c,1}(\vec{q};j) \right]$$

These relations are easily shown to be satisfied (see class notes). Note that the relation above implies that $G_E^V(q)$ should be used in both PM, π and MC, π currents, at least in their longitudinal components.

The nucleon excitation current

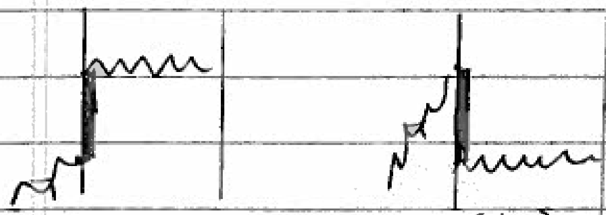
The $N \rightleftarrows \Delta$ transition currents are believed to be the leading currents due to the excitation of nucleons by photons. In a quark model picture, the photon can flip the spin of one of the quarks by a magnetic dipole interaction and convert the spin $1/2$ nucleon to the spin $3/2$ Δ. The $\gamma N\Delta$ interaction reads   from γN data at resonance $\mu_{\gamma N\Delta} \simeq 3\mu_N$

$$H_{\gamma N\Delta} = -\frac{\mu_{\gamma N\Delta}}{2m_N}\Big[\underbrace{T_z \vec{S}\times\vec{\nabla}}_{N\to\Delta} + \underbrace{T_z^{\dagger}\, \vec{S}^{\dagger}\times\vec{\nabla}}_{\Delta\to N} \Big]\cdot\vec{A}(\vec{r}).$$

and the $N \to \Delta$ current follows as

$$\vec{j}_{N\to\Delta}(q;i) = i\,\frac{\mu_{\gamma N\Delta}}{2m_N}\,T_{z,i}\,\vec{S}_i \times \vec{q}\; e^{i\vec{q}\cdot\vec{r}_i}$$

The two-body Δ-excitation currents can be estimated in perturbation theory by using the OPETP



(a) (b)

$$\vec{j}_2^{\,NE,\Delta}(\vec{q};ij) = \left[v^{(ij)}_{NN\to\Delta N}\right]^{\dagger} \frac{1}{m_N - m_\Delta}\, \vec{j}_{N\to\Delta}(\vec{q};i) \qquad (a)$$

$$+\, \vec{j}_{\Delta\to N}(\vec{q};i)\, \frac{1}{m_N - m_\Delta}\, v^{(ij)}_{NN\to\Delta N} \qquad (b)$$

and

$$\vec{j}_{\Delta\to N}(q;i) = i\,\frac{\mu_{\gamma N\Delta}}{2m_N}\,T^{\dagger}_{z,i}\,\vec{S}^{\dagger}_i \times \vec{q}\; e^{i\vec{q}\cdot\vec{r}_i}$$

More sophisticated treatments of Δ excitation in nuclei go beyond perturbation theory and are briefly discussed in the class notes.

Note that the current $\vec{j}_2^{\,NE,\Delta}$ is transverse, and drops out of the current conservation relation.