ON NEXT-TO-EIKONAL CORRECTIONS TO THRESHOLD RESUMMATION FOR ELECTROWEAK ANNIHILATION CROSS SECTIONS

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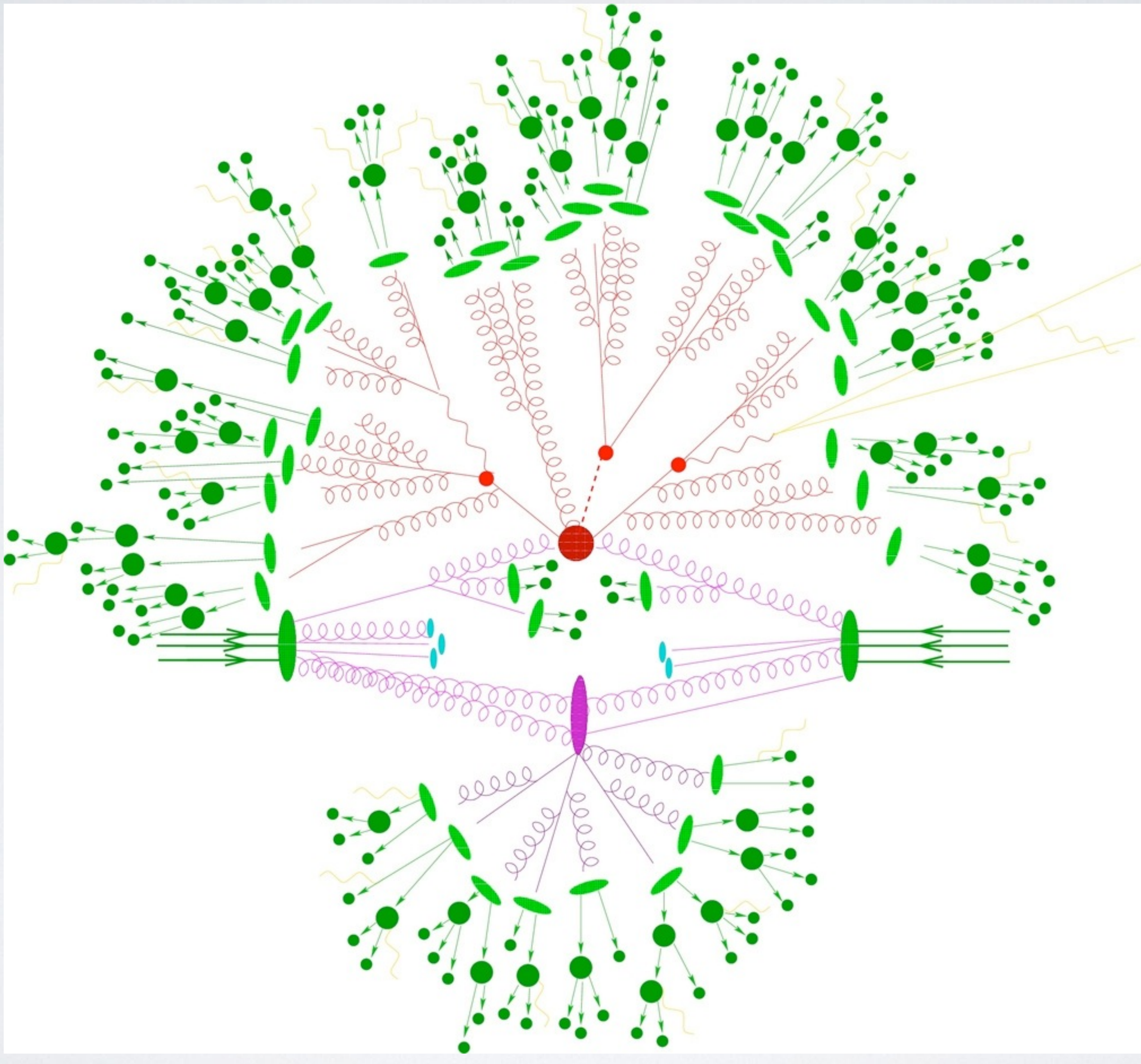
OUTLINE

- **Soft radiation at hadron colliders**
- **Soft radiation in Drell-Yan and electroweak annihilation**
- **Factorization at the eikonal level**
- **Factorization at the next-to-eikonal level.**

In collaboration with D. Bonocore, E. Laenen, L. Magnea, S. Melville, C. D. White.

Based on arXiv:0807.4412 (Phys. Lett. B 669 (2008) 173), arXiv:1010.1860 (JHEP 1101 (2011) 141) and in progress

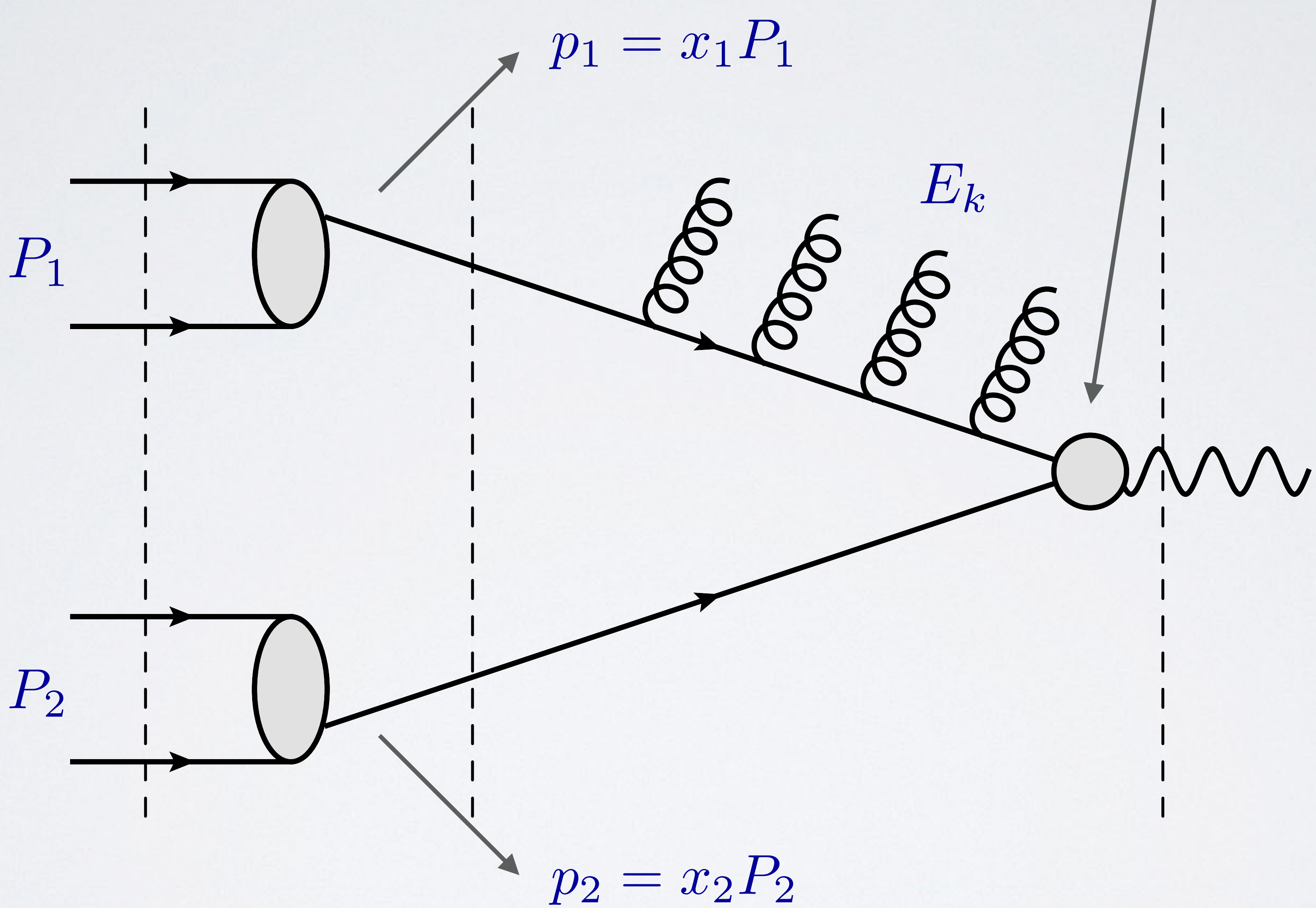
SOFT RADIATION AT HADRON COLLIDERS



SOFT RADIATION IN DRELL-YAN AND ELECTROWEAK ANNIHILATION

Multiple scale problem:

$$d\sigma = \sigma_0 \sum_{i,j} \int dx_1 \int dx_2 f_{i/N_1}(x_1,\mu) f_{j/N_2}(x_2,\mu) \hat{\sigma}_{ij}(x_1,x_2,s,M,\mu)$$

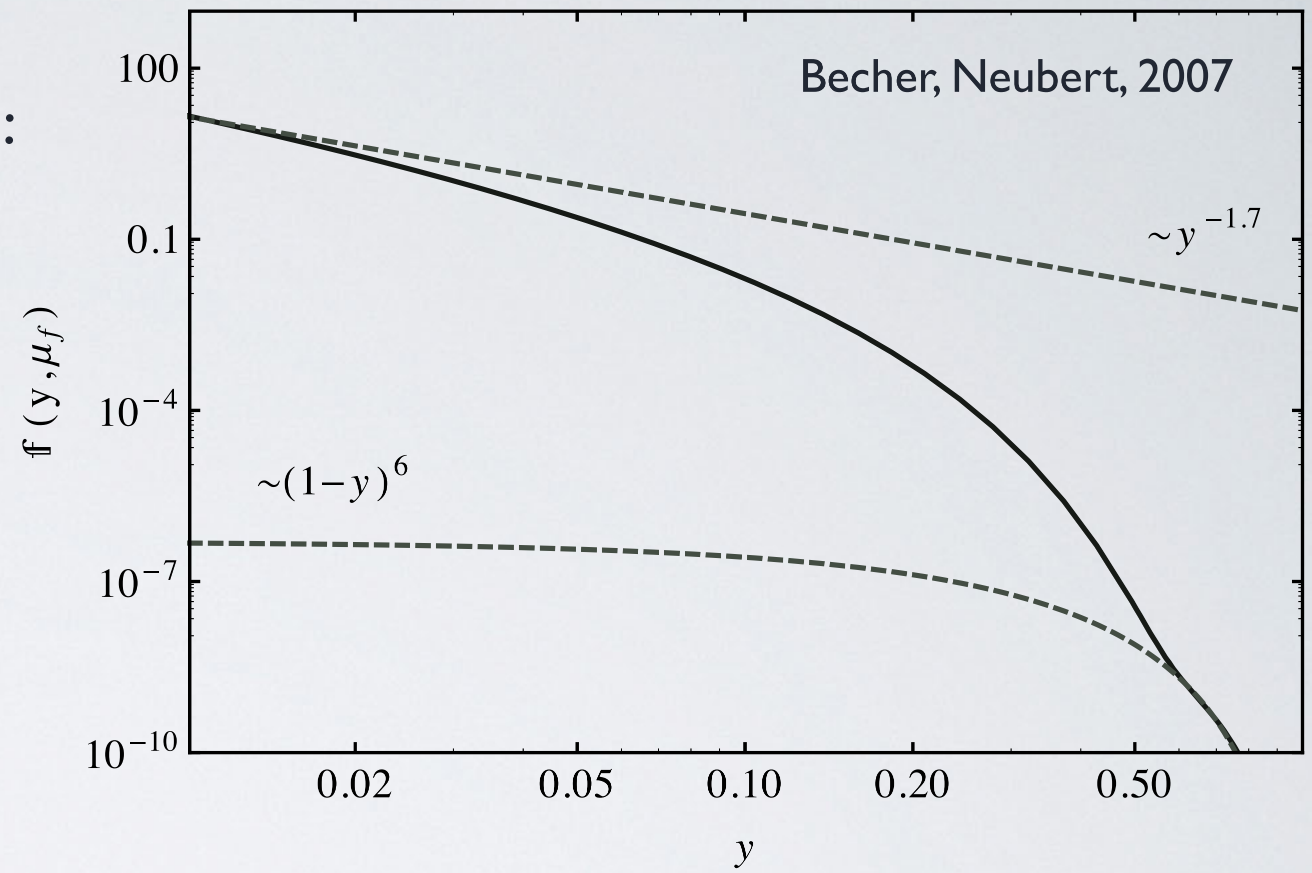


$$s = (P_1+P_2)^2 \quad > \quad \hat{s} = x_1 x_2 s \quad > \quad Q^2 = (p_3+p_4)^2 \quad \gg \quad E_k^2$$

$$\tau = \frac{Q^2}{s} \qquad z = \frac{Q^2}{\hat{s}} \qquad (z \geq \tau)$$

SOFT RADIATION IN DRELL-YAN AND ELECTROWEAK ANNIHILATION

- Does it matter? One can consider two regions:
  - threshold:
    $$\tau \to 1, \quad x_1 = x_2 = 1,$$
  - not relevant for phenomenology:
  - partonic threshold:
    $$z \to 1, \quad E_k \to 0.$$



- It is possible to prove that the partonic threshold is dynamically enhanced, because of the convolution with PDFs:

$$\frac{d\sigma}{dQ^2} \sim \int_\tau^1 \frac{dz}{z}\, \hat{\sigma}_{q\bar{q}}(z, Q, \mu)\, \mathcal{L}_{q\bar{q}}\left(\frac{\tau}{z}, \mu\right),$$

- where the PDFs are organised into the luminosity function

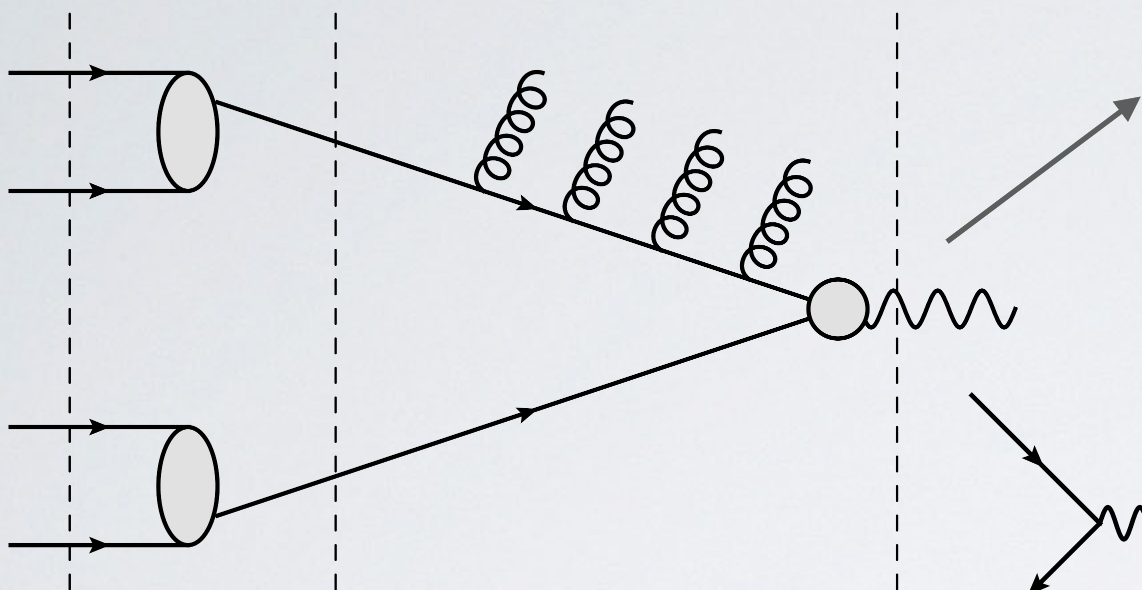
$$\mathcal{L}_{q\bar{q}}(y, \mu) = \sum_q e_q^2 \int_y^1 \frac{dx}{x} \left(f_{q/N_1}(x, \mu) f_{q/N_2}(\frac{y}{x}, \mu) + q \leftrightarrow \bar{q} \right).$$

- Enhancement of the z → 1 region already for $\tau \gtrsim 0.3$. It must be analysed for each process.

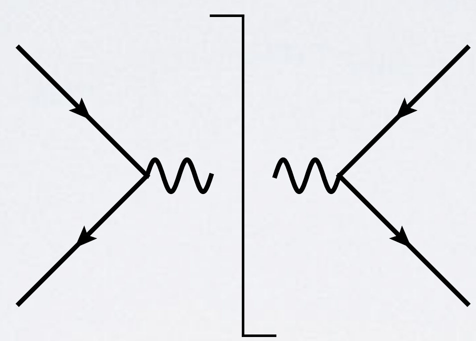
Bonvini, Forte, Ridolfi, 2010

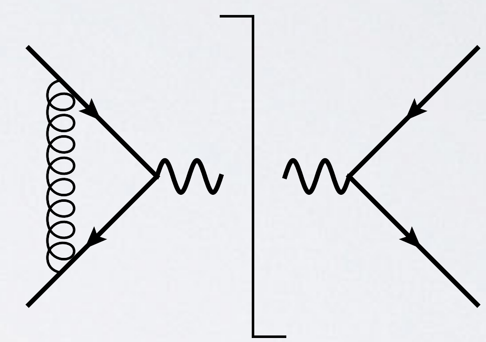
SOFT RADIATION IN DRELL-YAN AND ELECTROWEAK ANNIHILATION

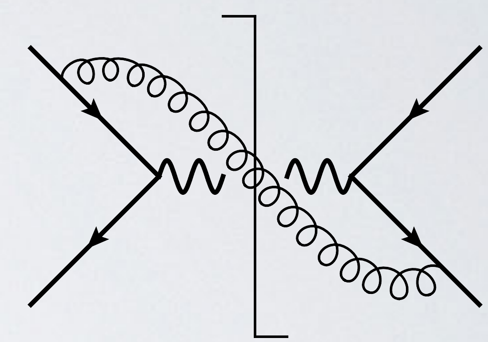
- Why needs special treatment? Have a closer look at the partonic cross section:



$$\hat{\sigma}_{ij} = \hat{\sigma}_{ij}^{(0)} + \frac{\alpha_s}{4\pi}\hat{\sigma}_{ij}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 \hat{\sigma}_{ij}^{(2)} + \dots$$







$$\propto \int d^d k \qquad\qquad \propto \int d_{\mathrm{PS}} k$$

- Virtual and real emission have infrared divergences, which cancel in the sum, leaving large (Sudakov) logarithms, which spoil reliability of the perturbative expansion. E.g.

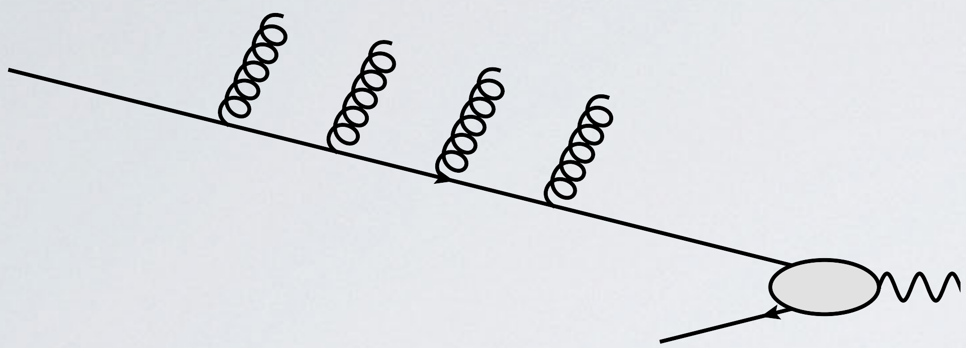
$$\hat{\sigma}^{(1)} \sim \delta(1-z)\left[\frac{3}{2}\log\left(\frac{Q^2}{\mu^2}\right) + \frac{2\pi^2}{3} - 4\right] + \frac{2}{1-z}\log\left(\frac{q^2(1-z)^2}{\mu^2 z}\right)\Bigg|_+$$

- In general one has

$$\hat{\sigma}^{(n)} \sim \sum_{m=0}^{2n-1}\left[a_{nm}\frac{\log^m(1-z)}{1-z}\Bigg|_+ + b_{nm}\log^m(1-z) + \mathcal{O}(1-z)\right]$$

SOFT RADIATION IN DRELL-YAN AND ELECTROWEAK ANNIHILATION

- How do we deal with soft gluons? Key ideas are factorisation and exponentiation:



$$\mathcal{M}_n(z_1,\ldots,z_n) \overset{\text{soft}}{\sim} \frac{1}{n!}\prod_{i=1}^{n}\mathcal{M}_1(z_i)$$

  - Factorization: physics occurring at well separate scales do not "talk"
  - Exponentiation: at leading order parton does not recoil: soft interaction give just a phase.

- In Mellin space, $$\sigma(N,Q^2) = \int_0^1 d\tau\, \tau^{N-1}\sigma(\tau,Q^2) = \hat{\sigma}(N,Q^2)\,\mathcal{L}(N)$$

- the log of the amplitude can be written as:

$$\ln[\hat{\sigma}(N,\alpha_s)] = \mathcal{F}_{DY}\left(\alpha_s(Q^2)\right) + \int_0^1 dz\, z^{N-1}\left\{\frac{1}{1-z}D\left[\alpha_s\left((1-z)^2Q^2\right)\right] + \int_{Q^2}^{(1-z)^2Q^2}\frac{dq^2}{q^2}\frac{2}{1-z}A\left[\alpha_s\left(q^2\right)\right]\right\}_+ .$$

- it takes into account
  - running of α_s;
  - soft and collinear gluon radiation.

Catani, Trentadue, 1989; Sterman, 1987

- In N space correctly reproduces terms a_{nn} below:

$$\hat{\sigma}(N) = \sum_{i=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n\left[\sum_{m=0}^{2n}a_{nm}\ln^m\left(Ne^{\gamma_E}\right) + \sum_{m=0}^{2n-1}b_{nm}\frac{\ln^m\left(Ne^{\gamma_E}\right)}{N}\right] + \mathcal{O}\left(\frac{\ln^p N}{N^2}\right)$$

SOFT RADIATION IN DRELL-YAN AND ELECTROWEAK ANNIHILATION

- What about the subleading b_{nn} terms?

Laenen, Magnea, Stavenga, 2008

- a simple ansatz succeeds in reproducing correctly some of the b_{nn} terms:

Modification of running

Modification of the phase space

Inclusion of the next-to-leading order in z of the Altarelli-Parisi Kernel

$$\ln[\hat{\sigma}(N,\alpha_s)] = \mathcal{F}_{DY}\left(\alpha_s(Q^2)\right)$$
$$+\int_0^1 dz\, z^{N-1}\left\{\frac{1}{1-z}D\left[\alpha_s\left(\frac{(1-z)^2}{z}Q^2\right)\right]\right.$$
$$\left.+\int_{Q^2}^{\frac{(1-z)^2}{z}Q^2}\frac{dq^2}{q^2}2\left(\frac{z}{1-z}A\left[\alpha_s\left(q^2\right)\right]+C_\gamma\ln(1-z)+\bar{D}_\gamma\right)\right\}_+ .$$

- indicating that exponentiation occurs at least for some of the next-to-leading terms in the soft gluon expansion, namely, some of the next-to-eikonal terms.
- The ansatz can be understood noting that the singular terms arise from integration of the real emission diagrams over the transverse momentum of the gluon, which is better described by the modifications above.
- Additional terms C and D follow from Dokshitzer, Marchesini, Salam (2006), (Attempt to put on the same ground evolution of PDF and fragmentation functions).

SOFT RADIATION IN DRELL-YAN AND DIS

- Compare coefficients of the logs obtained with this procedure with exact result at NNLO:
- for Drell-Yan

Laenen, Magnea, Stavenga, 2008

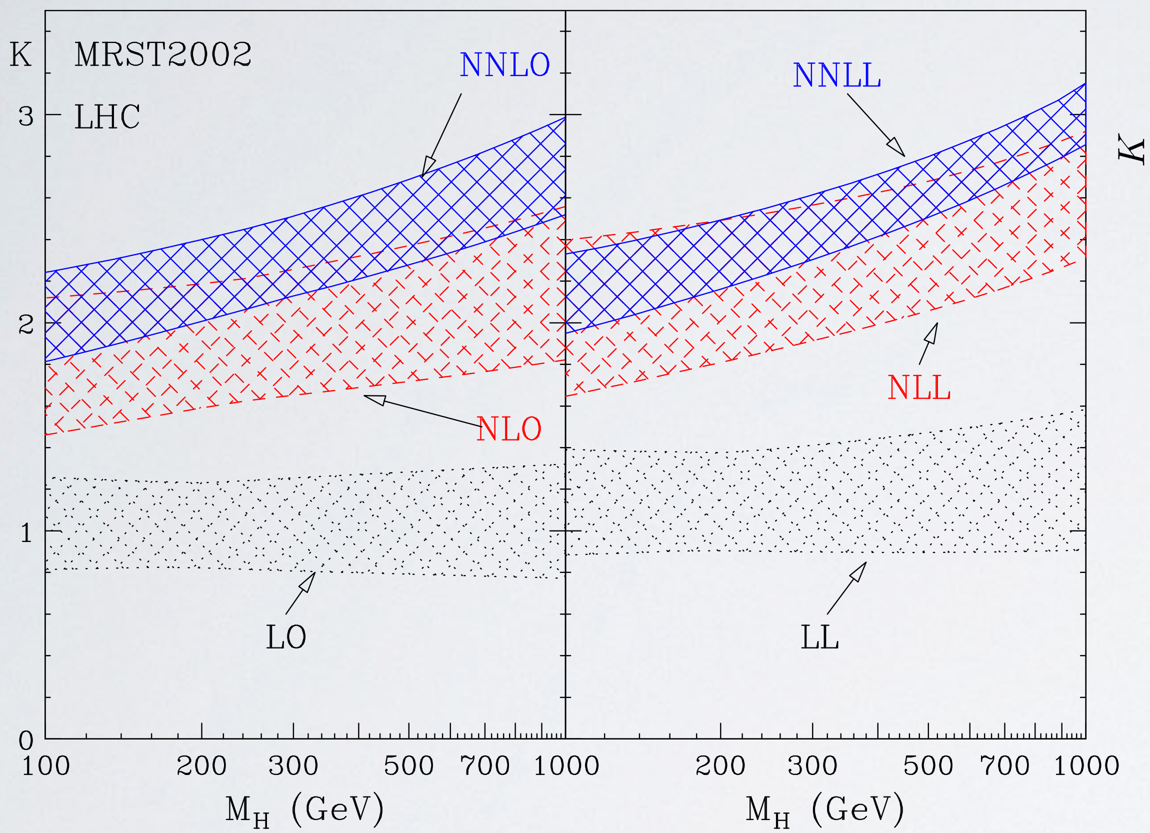
	C_F^2		$C_A C_F$		$n_f C_F$	
b_{23}	4	4	0	0	0	0
b_{22}	$\frac{7}{2}$	4	$\frac{11}{6}$	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
b_{21}	$8\zeta_2 - \frac{43}{4}$	$8\zeta_2 - 11$	$-\zeta_2 + \frac{239}{36}$	$-\zeta_2 + \frac{133}{18}$	$-\frac{11}{9}$	$-\frac{11}{9}$
b_{20}	$-\frac{1}{2}\zeta_2 - \frac{3}{4}$	$4\zeta_2$	$-\frac{7}{4}\zeta_3 + \frac{275}{216}$	$\frac{7}{4}\zeta_3 + \frac{11}{3}\zeta_2 - \frac{101}{54}$	$-\frac{19}{27}$	$-\frac{2}{3}\zeta_2 + \frac{7}{27}$

- and deep inelastic scattering:

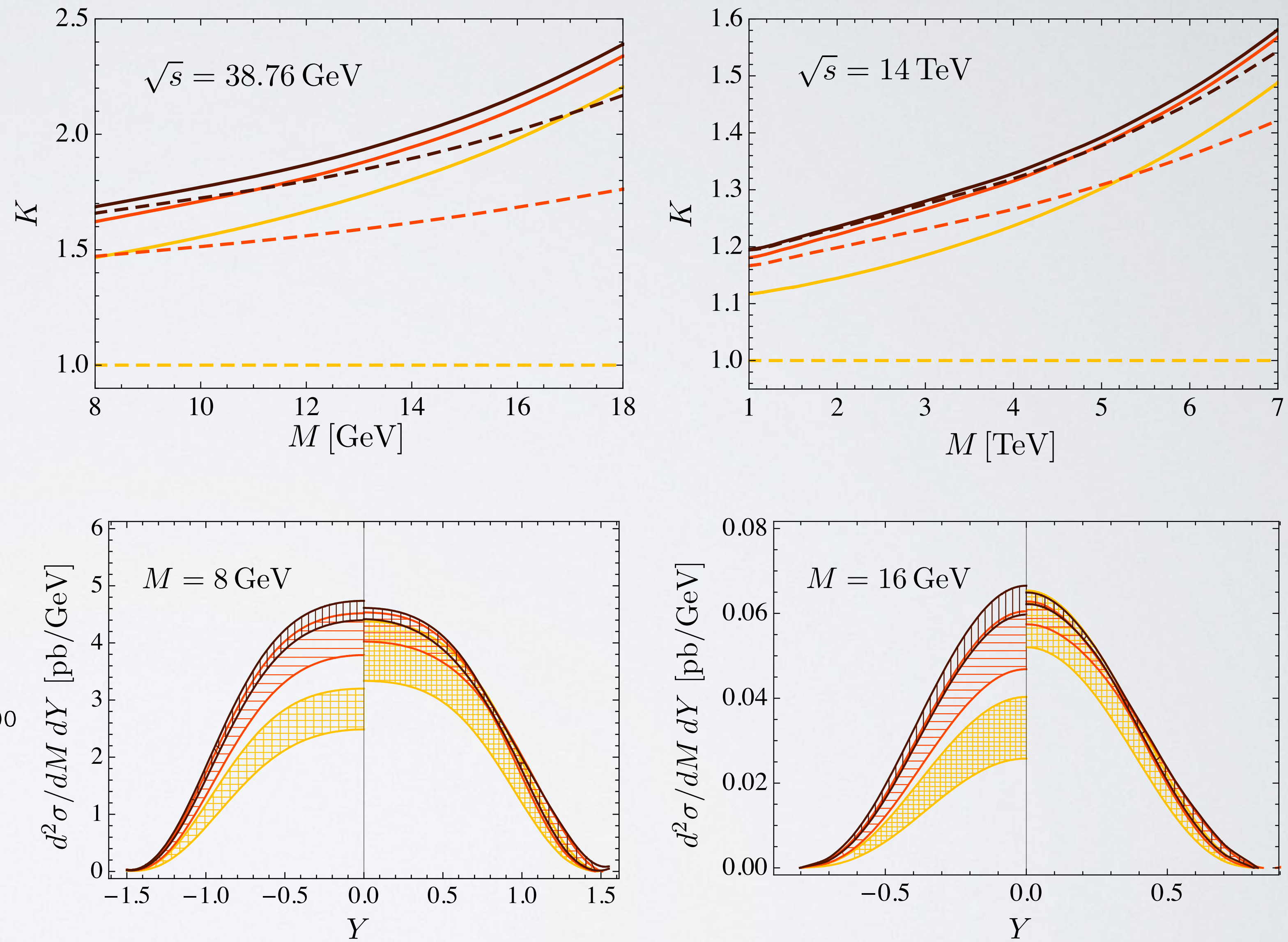
	C_F^2		$C_A C_F$		$n_f C_F$	
d_{23}	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0
d_{22}	$\frac{39}{16}$	$\frac{55}{16}$	$\frac{11}{48}$	$\frac{11}{48}$	$-\frac{1}{24}$	$-\frac{1}{24}$
d_{21}	$\frac{7}{4}\zeta_2 - \frac{49}{32}$	$-\frac{1}{4}\zeta_2 - \frac{105}{32}$	$-\frac{5}{4}\zeta_2 + \frac{1333}{288}$	$-\frac{1}{4}\zeta_2 + \frac{565}{288}$	$-\frac{107}{144}$	$-\frac{47}{144}$
d_{20}	$\frac{15}{4}\zeta_3 - \frac{47}{16}\zeta_2 - \frac{431}{64}$	$-\frac{3}{4}\zeta_3 + \frac{53}{16}\zeta_2 - \frac{21}{64}$	$-\frac{11}{4}\zeta_3 + \frac{13}{48}\zeta_2 - \frac{17579}{1728}$	$\frac{5}{4}\zeta_3 + \frac{7}{16}\zeta_2 - \frac{953}{1728}$	$\frac{1}{24}\zeta_2 - \frac{1699}{864}$	$-\frac{1}{8}\zeta_2 + \frac{73}{864}$

- Many logs are found, but not all: are they non-factorizing logs? Is there a way to systematise the resummation for 1/N logs?

EFFECTS OF RESUMMATION

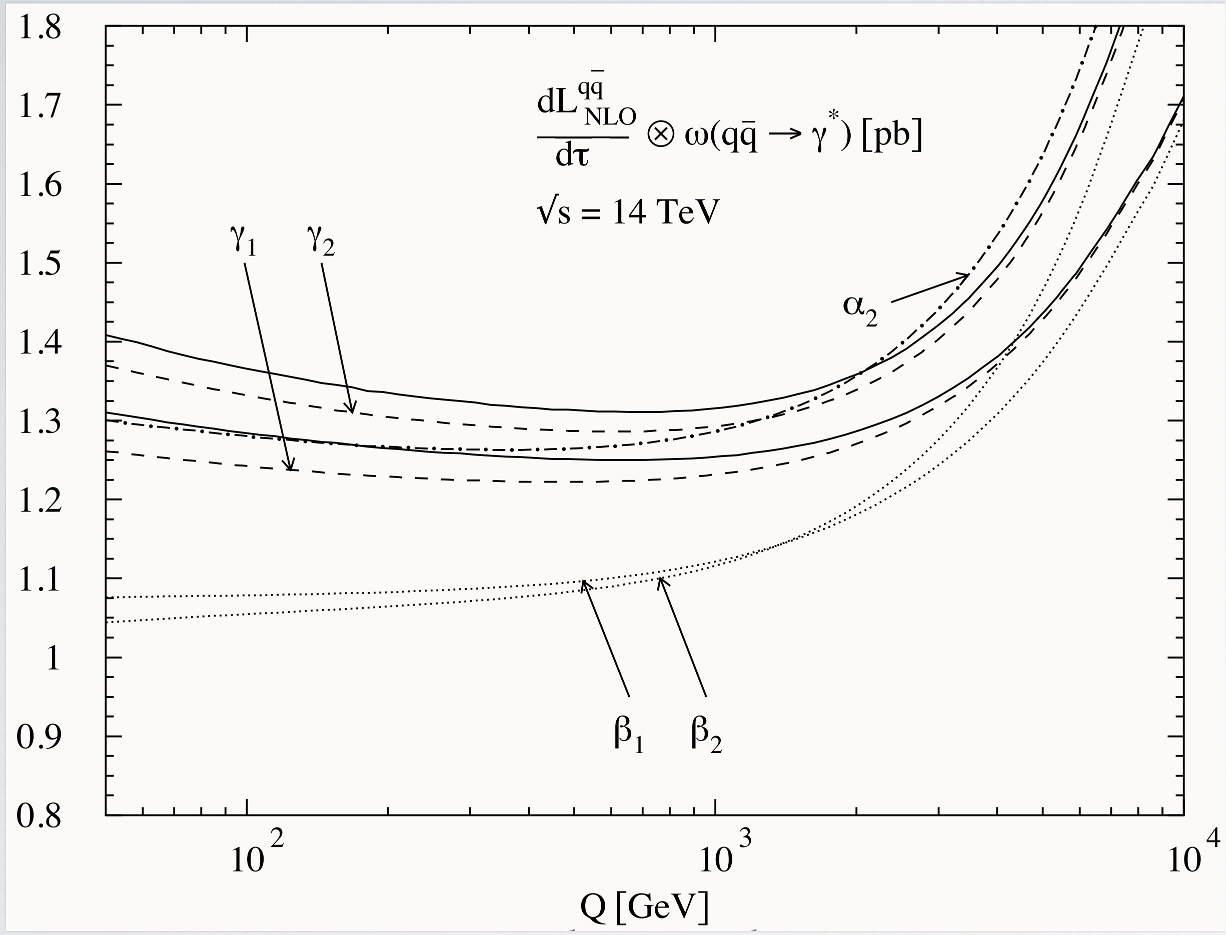


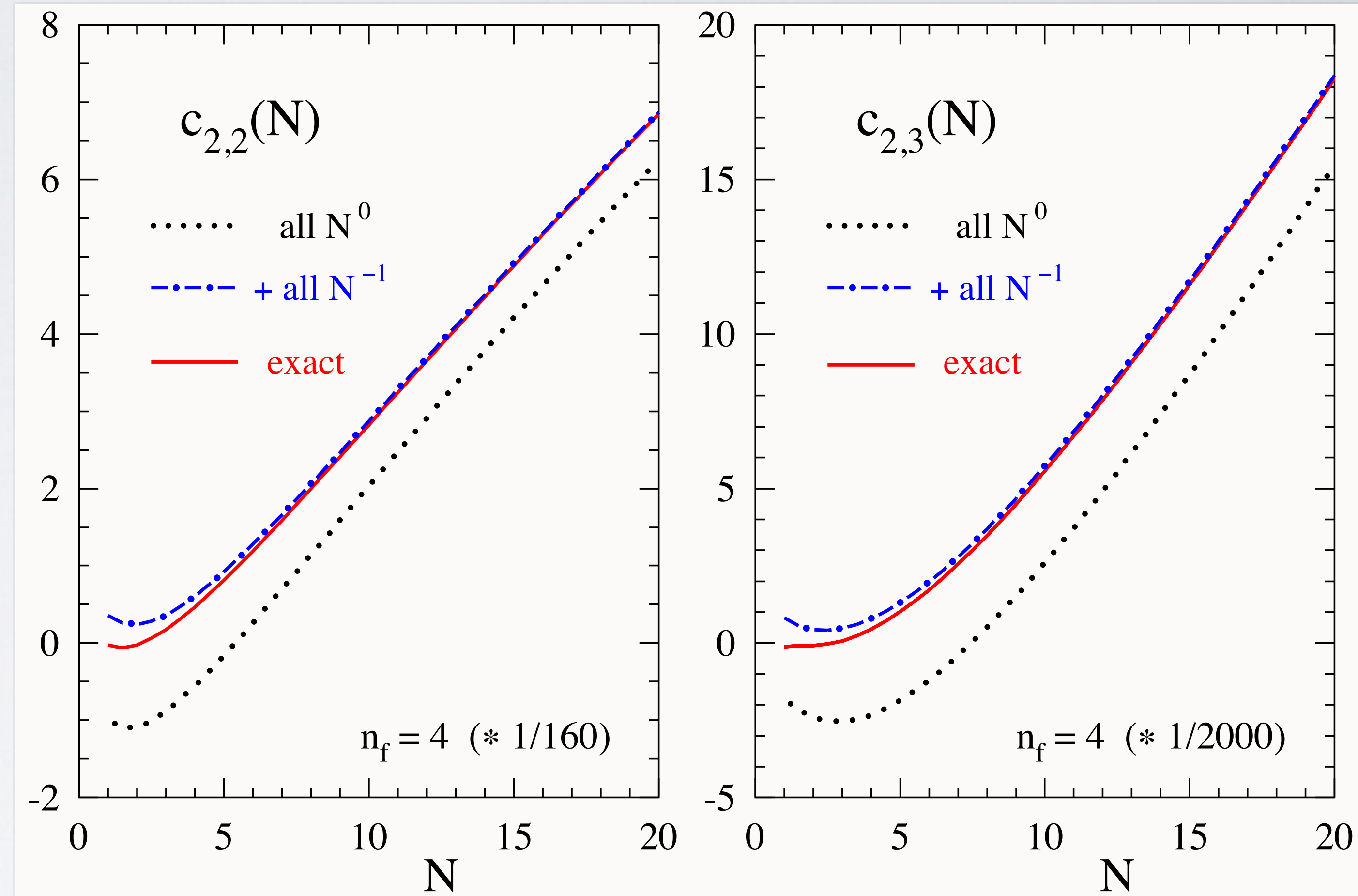
Catani, De Florian,
Grazzini, Nason, 2003



Becher, Neubert, 2007

EFFECTS OF RESUMMATION

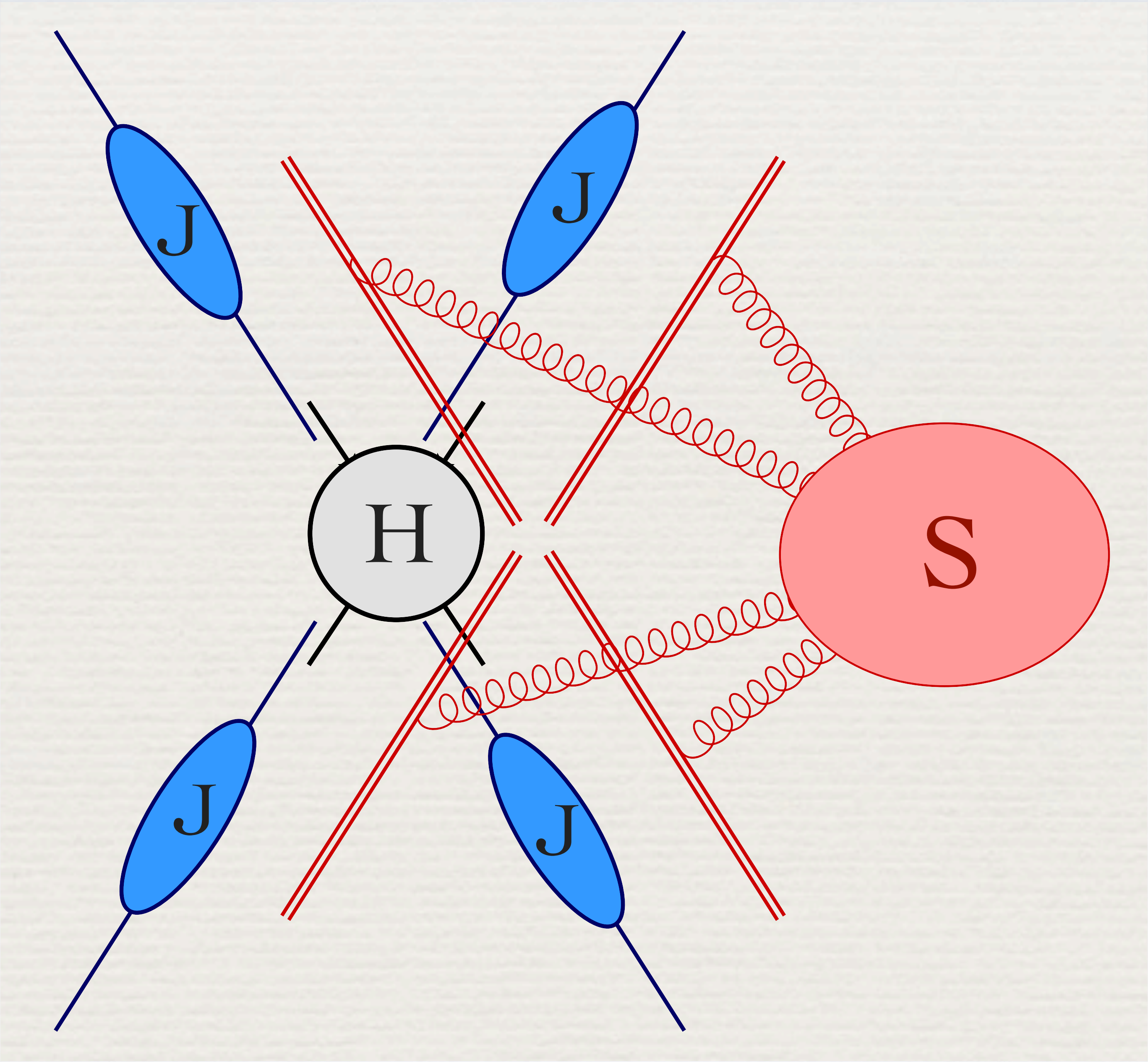




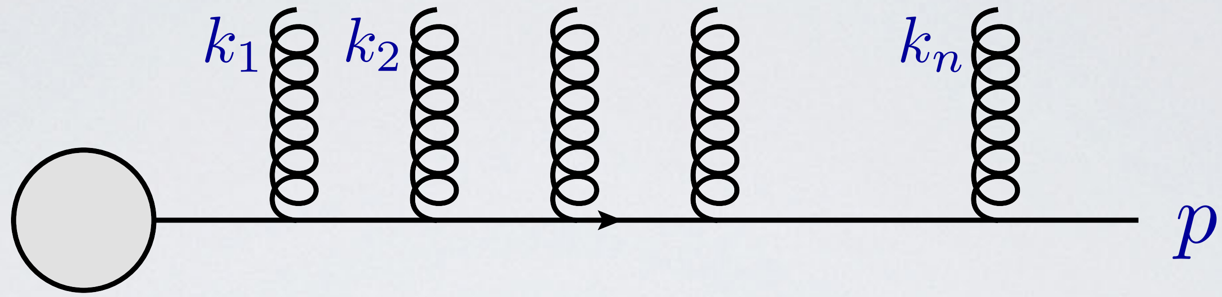
Kraemer, Laenen, Spira, 1996;
Schemes γ contains sub-leading logs

Moch, Vogt, 2009

FACTORIZATION AT THE EIKONAL LEVEL



SOFT RADIATION AT THE EIKONAL LEVEL



Laenen, Magnea, Stavenga, White, 2010

- Consider the emission of n (abelian) gluons from a fermion line:

$$\mathcal{M}^{\mu_1\ldots\mu_n}(p,k_i) = \mathcal{M}_0(p)\,\frac{\not{p}+\not{K}_1}{(p+K_1)^2}\gamma^{\mu_1}\ldots\frac{\not{p}+\not{K}_n}{(p+K_n)^2}\gamma^{\mu_n}u(p)\,, \qquad K_i = \sum_{m=i}^{n} k_m$$

- Consider one of the propagators: when k is soft expand

$$\frac{\not{p}+\not{K}_i}{(p+K_1)^2} = \underbrace{\frac{\not{p}}{2p\cdot K_i}}_{\mathrm{E}} + \underbrace{\frac{\not{K}_i}{2p\cdot K_i} - \frac{K_i^2\,\not{p}}{(2p\cdot K_i)^2}}_{\mathrm{NE}} + \ldots$$

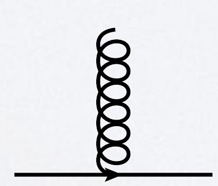
- Consider leading order (eikonal):

$$E^{\mu_1\ldots\mu_n}(p,k_i) = \frac{1}{n!}\,p^{\mu_1}\ldots p^{\mu_n}\sum_{\pi}\frac{1}{p\cdot k_{\pi_1}}\,\frac{1}{p\cdot(k_{\pi_1}+k_{\pi_2})}\cdots\frac{1}{p\cdot(k_{\pi_1}+\ldots+k_{\pi_n})}\,,$$

- Eikonal identity gives:

$$\sum_{\pi}\frac{1}{p\cdot k_{\pi_1}}\,\frac{1}{p\cdot(k_{\pi_1}+k_{\pi_2})}\cdots\frac{1}{p\cdot(k_{\pi_1}+\ldots k_{\pi_n})} = \prod_i \frac{1}{p\cdot k_i}\,.$$

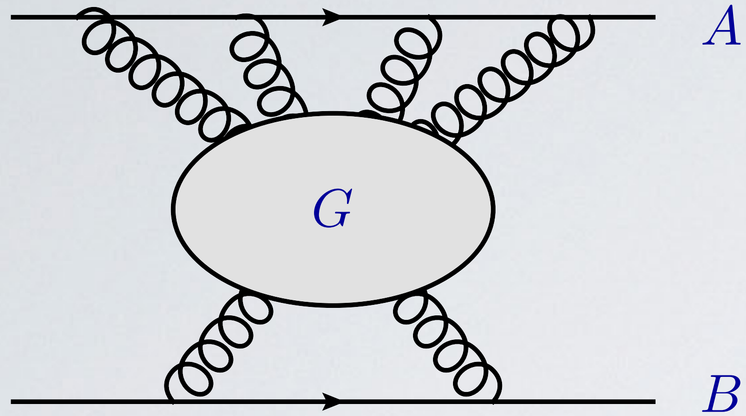
- This is equivalent to an effective Feynman rule for soft gluon emission:



$$= \frac{p^{\mu}}{p\cdot k}$$

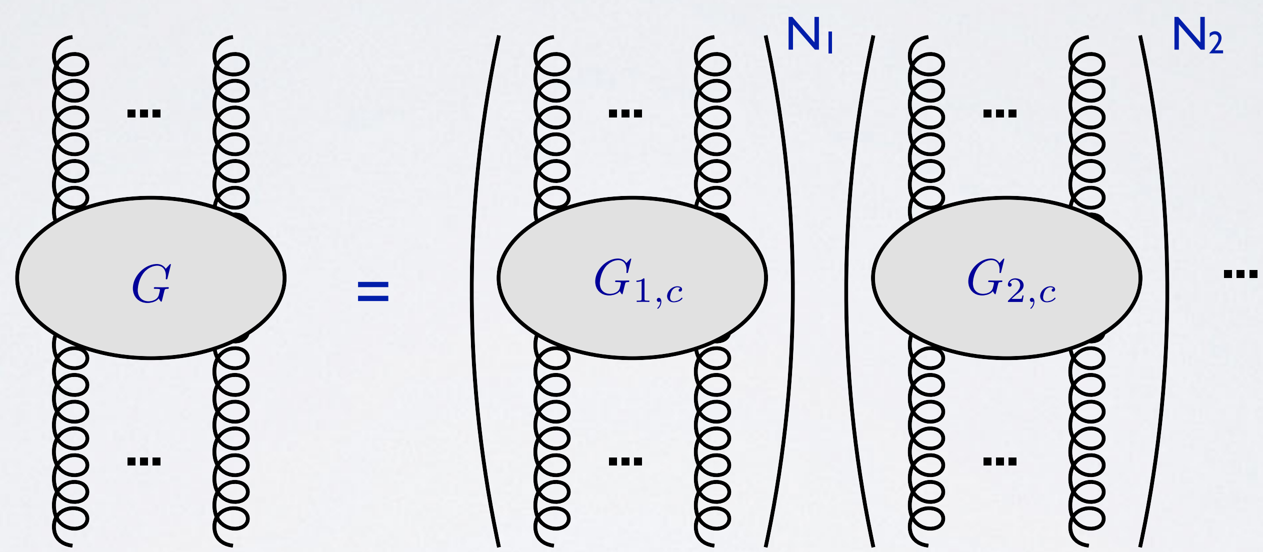
= uncorrelated emission

SOFT RADIATION AT THE EIKONAL LEVEL: ABELIAN



- A matrix element (squared) involves soft interactions between two external lines:

$$\mathcal{F}_{AB} = \sum_G \left[\prod_i \frac{p_A^{\mu_i}}{p_A \cdot k_i}\right] \left[\prod_j \frac{p_B^{\nu_j}}{p_B \cdot l_j}\right] G_{\mu_1 \ldots \mu_n; \nu_1 \ldots \nu_m}(k_i, l_j) \,,$$



- Sum over possible connected subdiagrams, each occurring N_i times:

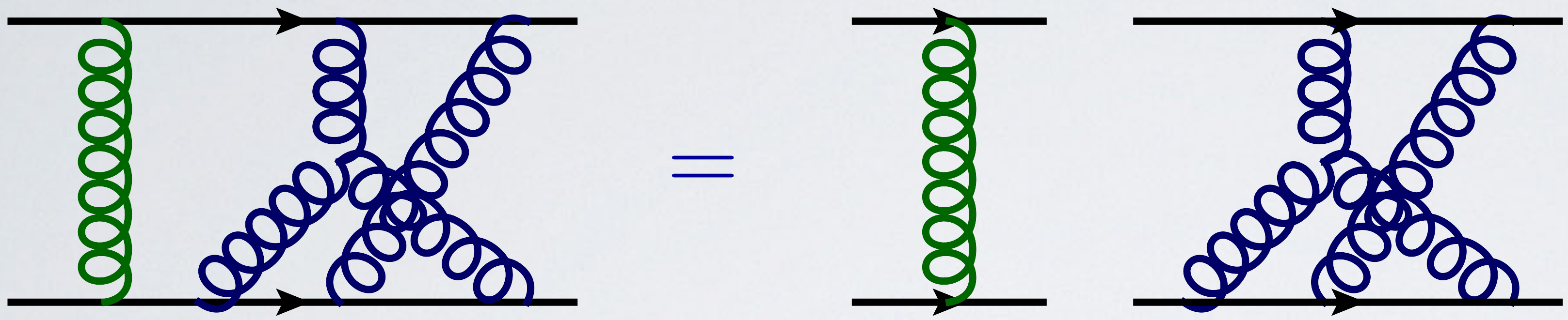
$$\mathcal{F}_{AB} = \sum_{\{N_i\}} \prod_i \frac{1}{N_i!} \left[\mathcal{F}_c^{(i)}\right]^{N_i} \,, \quad \mathcal{F}_c^{(i)} = \frac{1}{S_i} \left(\prod_q \frac{p_A^{\mu_q}}{p_A \cdot k_q}\right) \left(\prod_r \frac{p_B^{\nu_r}}{p_B \cdot l_r}\right) G^{(i)}_{\mu_1 \ldots \mu_{n_q}; \nu_1 \ldots \nu_{m_r}}$$

- This is actually an exponential:

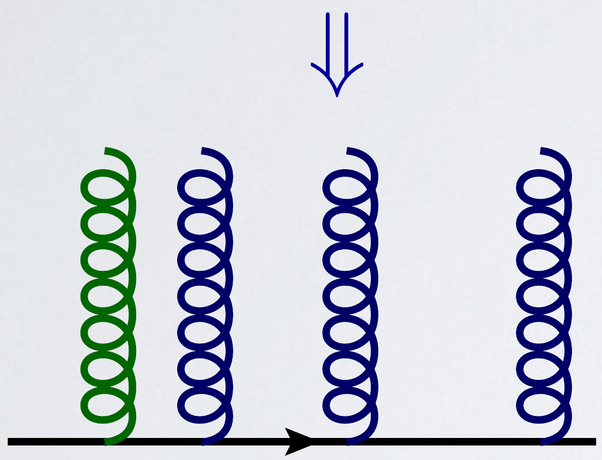
$$\mathcal{F}_{AB} = \exp\left[\sum_i \mathcal{F}_c^{(i)}\right] .$$

SOFT RADIATION AT THE EIKONAL LEVEL: NON-ABELIAN

- In case of non-abelian gluons, one has to face the non-commutative color matrices associated with each emission. We need to introduce first the concept of groups and webs:



- Web: two-eikonal irreducible diagram.
- Group: projection of a web onto a single eikonal line.



"shuffle product"

- One can only sums over permutations that do not affect the orderings of gluons within groups. The eikonal identity modifies according to

$$\sum_{\tilde{\pi}} \frac{1}{p\cdot k_{\tilde{\pi}_1}} \frac{1}{p\cdot (k_{\tilde{\pi}_1}+k_{\tilde{\pi}_2})} \cdots \frac{1}{p\cdot (k_{\tilde{\pi}_1}+\ldots+k_{\tilde{\pi}_n})}$$

$$= \prod_{\text{groups } g} \frac{1}{p\cdot k_{g_1}} \frac{1}{p\cdot (k_{g_1}+k_{g_2})} \cdots \frac{1}{p\cdot (k_{g_1}+\ldots+k_{g_m})}\,.$$

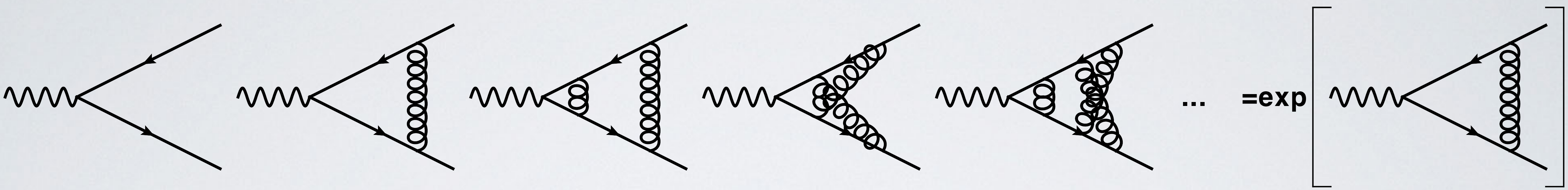
- Repeating the exercise (using induction, combinatorics and recursive definition of the color weights), one finds the replacement

$$\mathcal{F}_{AB} = \sum_G c_G\, \mathrm{E}(G) = \exp\left\{\sum_H \bar{c}_H\, \mathrm{E}(H)\right\}.$$

SOFT RADIATION AT THE EIKONAL LEVEL

- To give more feeling with abelian vs. non-abelian "webs": consider soft form factor:

Abelian

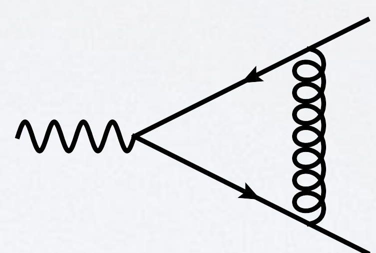


Gatheral, 1983;
Frenkel, Taylor, 1984

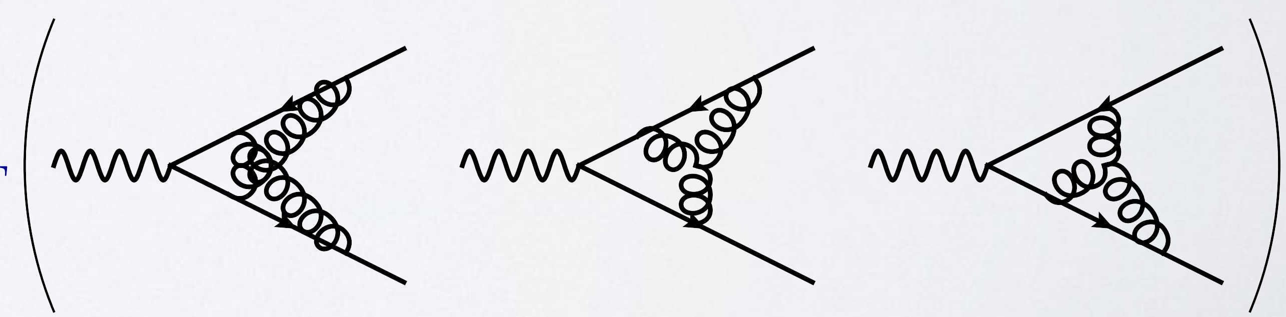
non-Abelian

Color factors associated with non-abelian webs:

C_F



$-\frac{C_A}{2}C_F$



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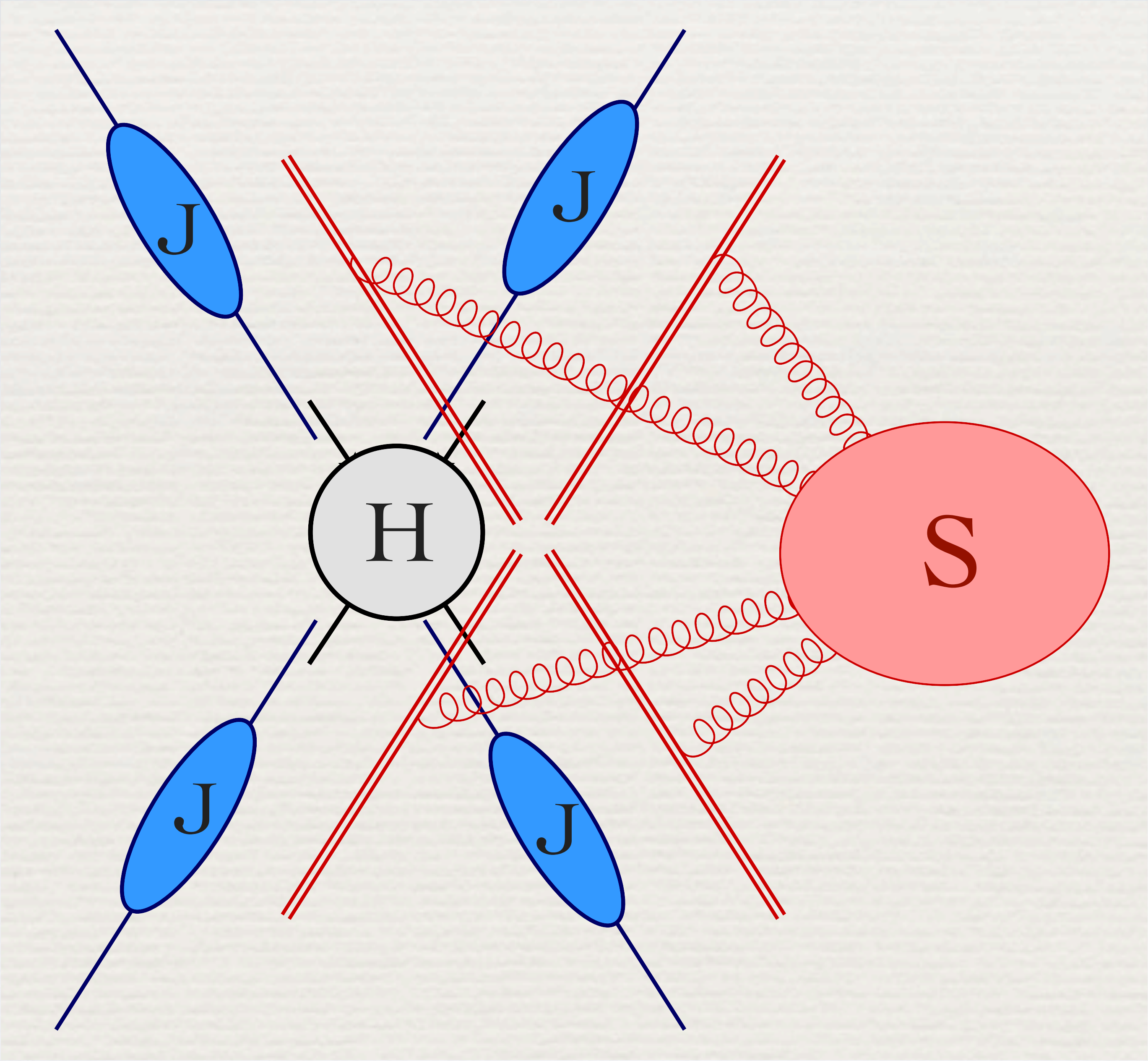
FACTORIZATION AT THE NEXT-TO-EIKONAL LEVEL



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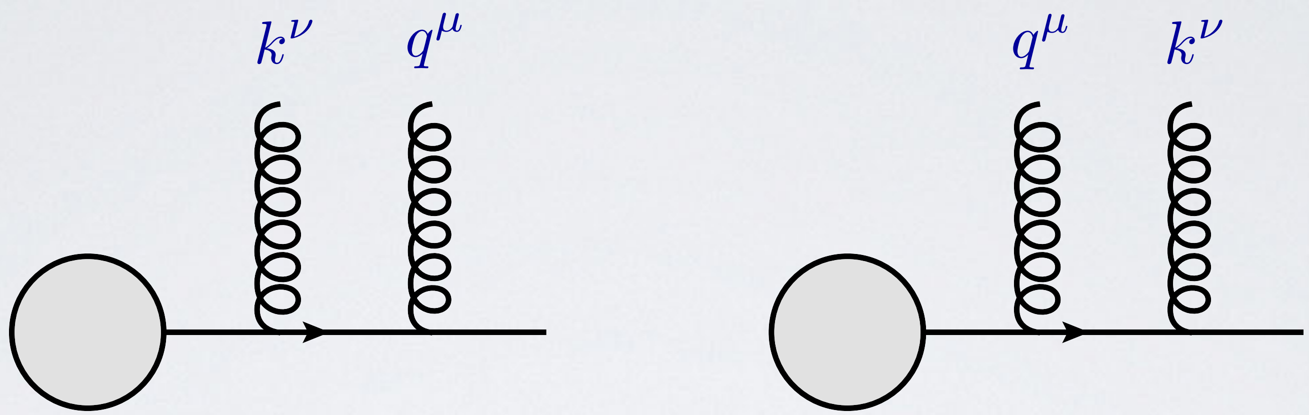
SOFT RADIATION AT THE NEXT-TO-EIKONAL LEVEL

- Ready to go: at the next-to-eikonal (NE) level one needs to take into account one NE insertion for each diagram. Consider for simplicity a fermion line with two-gluon emissions:

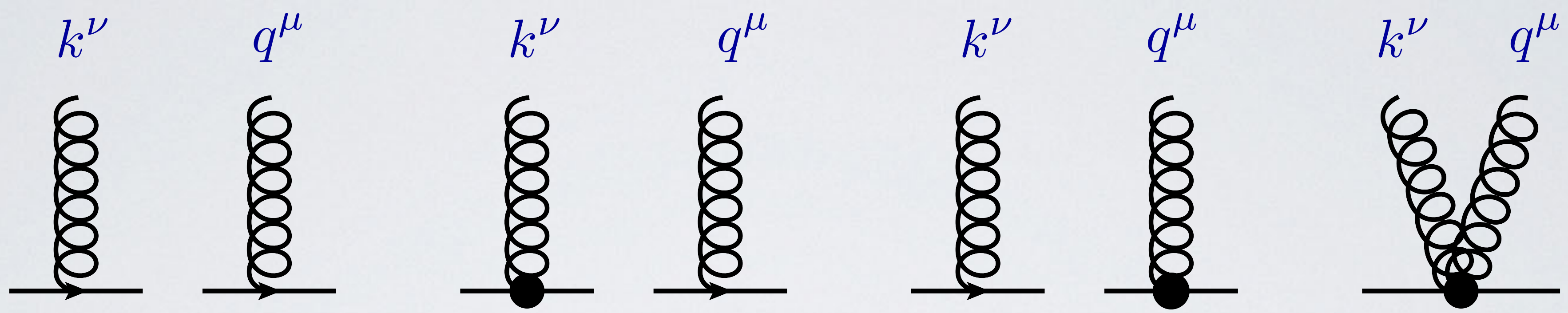


$$\left(\frac{\not{p}+\not{q}+\not{k}}{(p+q+k)^2}\,\gamma^\nu\,\frac{\not{p}+\not{q}}{(p+q)^2}\,\gamma^\mu + \frac{\not{p}+\not{q}+\not{k}}{(p+q+k)^2}\,\gamma^\mu\,\frac{\not{p}+\not{k}}{(p+k)^2}\,\gamma^\nu \right) u(p),$$

- Expanding in the soft gluon momenta one get

$$\frac{p^\mu}{p\cdot k}\frac{p^\nu}{p\cdot k} + \frac{p^\nu}{p\cdot k}\left(\frac{\not{q}\gamma^\mu}{2p\cdot q} - \frac{q^2 p^\mu}{2(p\cdot q)^2}\right) + \frac{p^\mu}{p\cdot q}\left(\frac{\not{k}\gamma^\nu}{2p\cdot k} - \frac{k^2 p^\nu}{2(p\cdot k)^2}\right)$$
$$+\frac{p^\nu k^\mu (p\cdot q) + p^\mu q^\nu (p\cdot k) - (p\cdot k)(p\cdot q) g^{\mu\nu} - p^\mu p^\nu (q\cdot k)}{p\cdot(q+k)p\cdot kp\cdot q},$$

SOFT RADIATION AT THE NEXT-TO-EIKONAL LEVEL



- Emission splits into different contributions (abelian case):

$$V_{\mathrm{E}}^{\mu}(p,k) = \frac{p^{\mu}}{p\cdot k},$$

$$V_{\mathrm{NE}}^{\mu\nu}(p,k) = \frac{p^{\nu}}{p\cdot k}\left(\frac{\not{q}\gamma^{\mu}}{2p\cdot q} - \frac{q^2 p^{\mu}}{2(p\cdot q)^2}\right),$$

$$R^{\mu\nu}(p,q,k) = \frac{p^{\nu}k^{\mu}(p\cdot q) + p^{\mu}q^{\nu}(p\cdot k) - (p\cdot k)(p\cdot q)g^{\mu\nu} - p^{\mu}p^{\nu}(q\cdot k)}{p\cdot(q+k)p\cdot kp\cdot q}.$$

- V_{NE} is a factorized product of an eikonal and a NE emission: the Dirac structure denote that at NE soft gluon emission are sensititive to the spin (magnetic moment) of the emitter. V_{NE} may involve sums over more gluon momenta.

- R gives an effective two-gluon vertex not present in the original theory. The two-gluon emission cannot be disentangled.

SOFT RADIATION AT THE NEXT-TO-EIKONAL LEVEL

- What does it imply for exponentiation? Remember that the key element for exponentiation at the eikonal level is the eikonal identity,

$$\sum_{\pi} \mathrm{E}(\pi) = \prod_{g} \mathrm{E}(g)$$

- In the same notation, it is possible to prove that, at the NE level,

$$\sum_{\pi} \mathrm{NE}(\pi) = \sum_{h} \left[\mathrm{NE}(h) \prod_{g \neq h} \mathrm{E}(g) \right] + \sum_{g \neq h} \left[R(g,h) \prod_{f \neq g,h} \mathrm{E}(f) \right] .$$

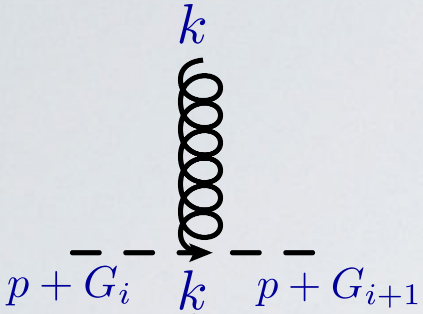
- Proof by induction, considering separately individual terms in the Feynman rules written in the last slide. Based on this results, it is possible to prove that soft real emission exponentiate:

$$\sum_{G} c_G \Big[\mathrm{E}(G) + \mathrm{NE}(G) \Big] = \exp\left[\sum_{H} \bar{c}_H \left(\mathrm{E}(H) + \mathrm{NE}(H) \right) \right]$$
$$= \exp\left[\sum_{H} \bar{c}_H \, \mathrm{E}(H) \right] \left[1 + \sum_{K} \bar{c}_K \, \mathrm{NE}(K) + \sum_{K,L} \bar{c}_K \bar{c}_L \, R(K,L) \right] .$$

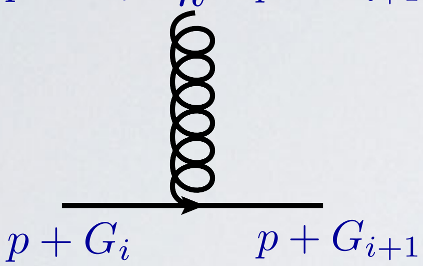
Laenen, Magnea, Stavenga, White, 2010

SOFT RADIATION AT THE NEXT-TO-EIKONAL LEVEL

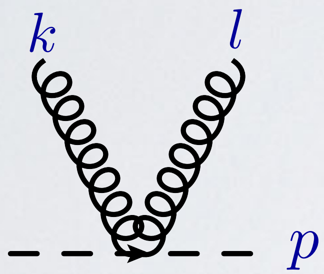
- Webs can be computed by means of effective Feynman rules: (here also for scalar particles)



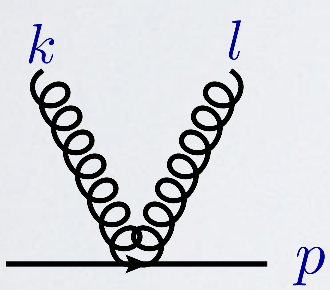
$$= t^A\left(\frac{2G_i^\mu - k^\mu}{2p\cdot G_i} - \frac{G_i^2\, p^\mu}{2(p\cdot G_i)^2}\right) = \frac{t^A}{2p\cdot G_i}\left(G_i^\mu + G_{i+1}^\mu - \frac{G_i^2\, p^\mu}{p\cdot G_i}\right),$$



$$= t^A\left(\frac{2G_i^\mu + \gamma^\mu \not{k}}{2p\cdot G_i} - \frac{G_i^2\, p^\mu}{2(p\cdot G_i)^2}\right) = \frac{t^A}{2p\cdot G_i}\left(G_i^\mu + G_{i+1}^\mu + \not{k}\,\gamma^\mu - k^\mu - \frac{G_i^2\, p^\mu}{p\cdot G_i}\right),$$

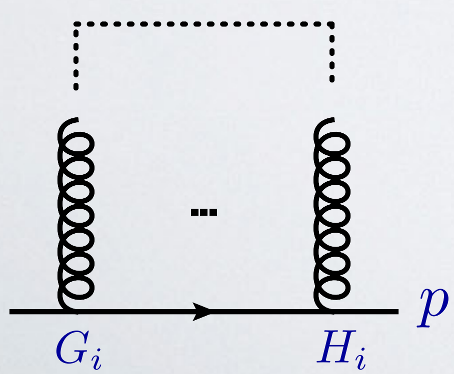


$$= \frac{g^{\mu\nu}\,\{t^A, t^B\}}{2p\cdot(k+l)}$$



$$= \frac{g^{\mu\nu}\,\{t^A, t^B\}}{p\cdot(k+l)} + \frac{[\gamma^\mu, \gamma^\nu]\,\left[t^A, t^B\right]}{2p\cdot(k+l)},$$

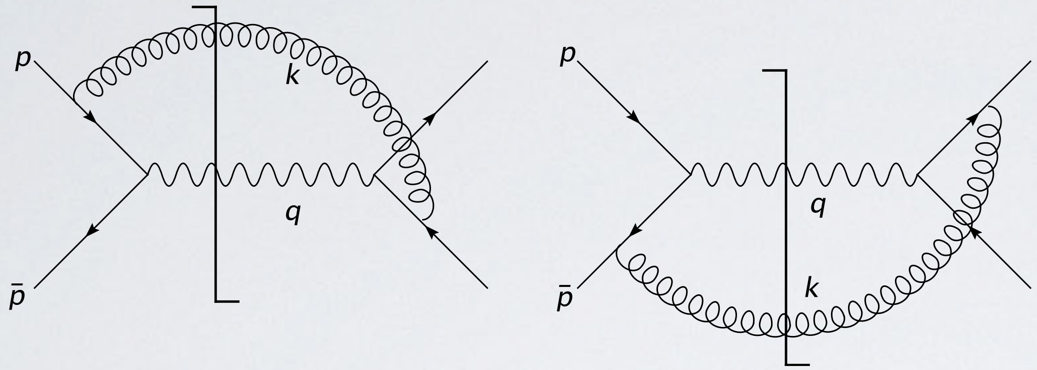
Laenen, Magnea, Stavenga, White, 2010



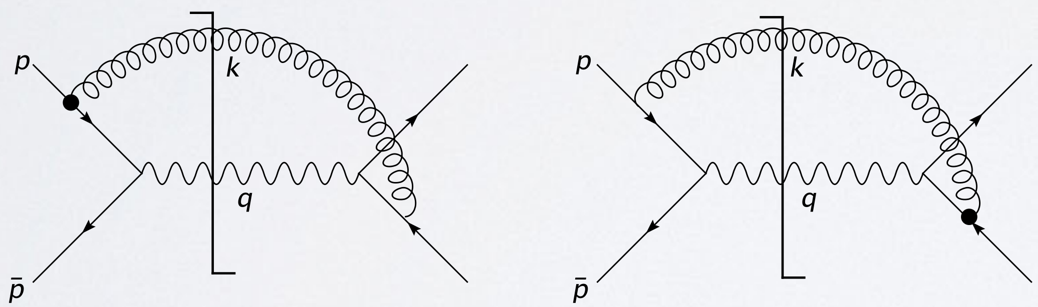
$$= t^A \otimes t^B \left(\frac{-\,G_i^\mu\, p^\nu\,(p\cdot H_j) \;-\; H_j^\nu\, p^\mu\,(p\cdot G_i) \;+\; p^\mu p^\nu\, G_i\cdot H_j}{2p\cdot(G_i + H_j)\, p\cdot G_i\, p\cdot H_j}\right)$$

SOFT RADIATION AT THE NEXT-TO-EIKONAL LEVEL

- check result for NNLO real emission in Drell-Yan, calculated using the effective Feynman rules above:

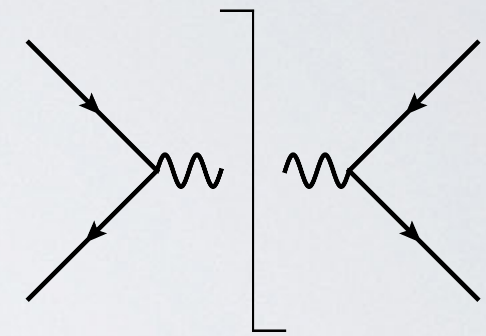


$$K^{(1)}_{\rm eik}(z) = \frac{\alpha_s}{4\pi}\, C_F \left\{ -\frac{8}{\epsilon}\mathcal{D}_0(z) + 16\mathcal{D}_1(z) - \frac{8\log(z)}{1-z} \right.$$
$$\left. - 4\epsilon \left[4\mathcal{D}_2(z) - 3\zeta_2\mathcal{D}_0(z) - \frac{4\log z \log(1-z)}{(1-z)} + \frac{\log^2 z}{1-z} \right] \right\},$$



$$K^{(1)}_{\rm NE}(z) = \frac{\alpha_s}{4\pi}\, C_F \left\{ \frac{8}{\epsilon} - 16\log(1-z) + 8\log z \right.$$
$$\left. - 4\epsilon \left[-4\log^2(1-z) + 4\log z\log(1-z) - \log^2 z + 3\zeta_2 \right] \right\},$$

$$K^{(n)}(z) = \frac{1}{\sigma^{(0)}} \frac{d\sigma^{(n)}(z)}{dz}$$

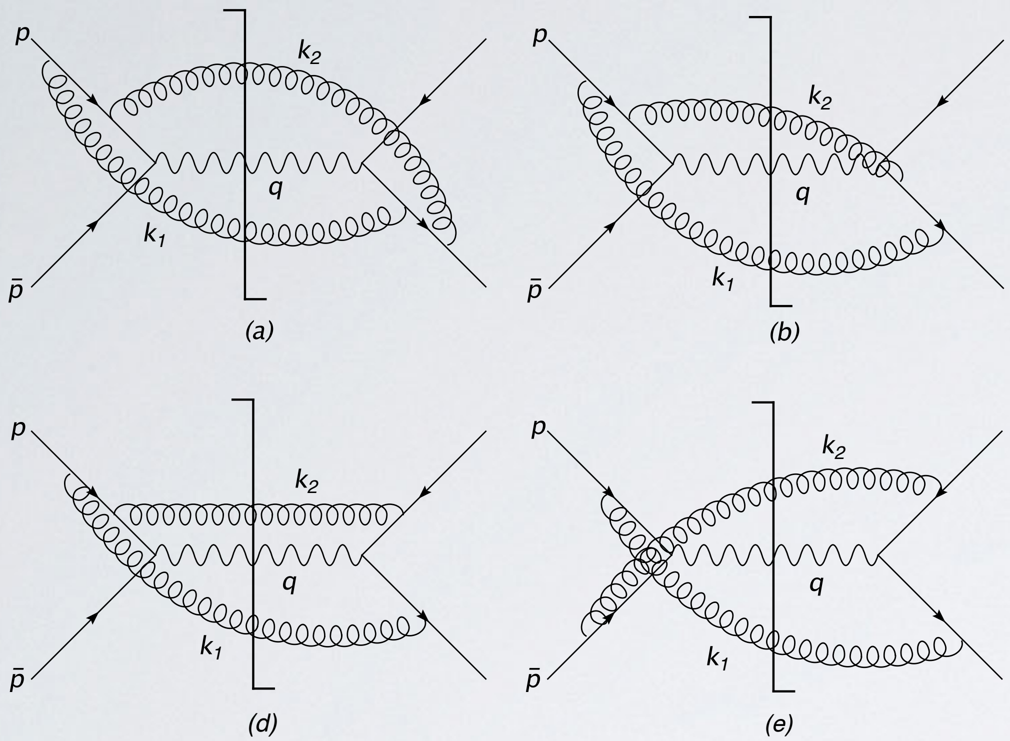


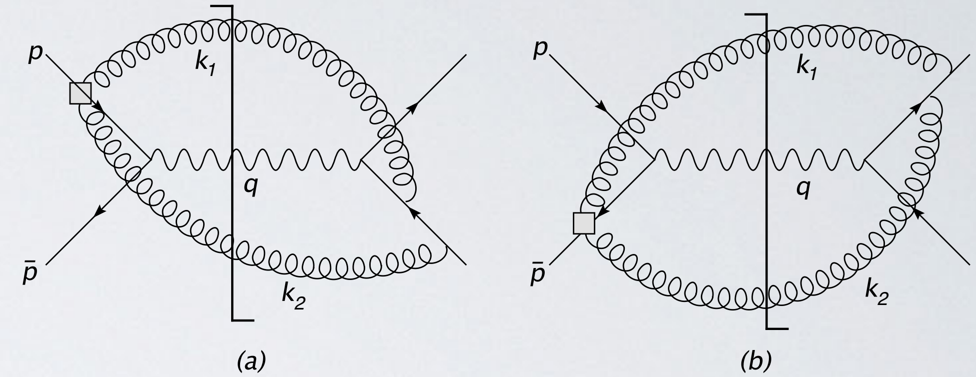
$$K^{(0)}(z) = \delta(1-z)$$

$$\mathcal{D}_p(z) = \left. \frac{\log^p(1-z)}{1-z} \right|_+$$

Laenen, Magnea, Stavenga, White, 2010

SOFT RADIATION AT THE NEXT-TO-EIKONAL LEVEL



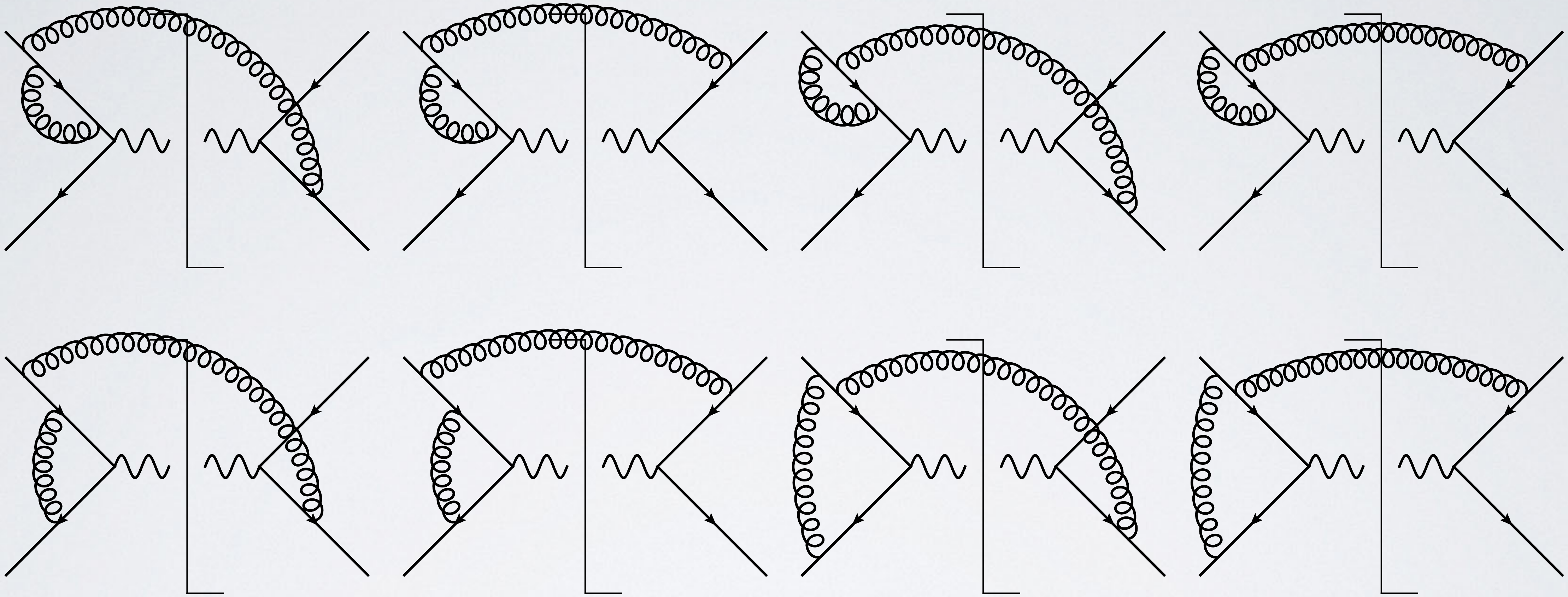


$$K^{(2)}_{\rm eik}(z) = \left(\frac{\alpha_s}{4\pi}\,C_F\right)^2\,\Bigg[-\frac{32}{\epsilon^3}\,\mathcal{D}_0(z) + \frac{128}{\epsilon^2}\,\mathcal{D}_1(z) - \frac{256}{\epsilon}\,\mathcal{D}_2(z) - \frac{320}{\epsilon}\log(1-z) + \frac{1024}{3}\,\mathcal{D}_3(z) + 640\log^2(1-z) + \ldots\Bigg]\,,$$

$$K^{(2)}_{\rm NE}(z) = \left(\frac{\alpha_s}{4\pi}C_F\right)^2\Bigg[-\frac{32}{\epsilon^3}\,\mathcal{D}_0(z) + \frac{128}{\epsilon^2}\,\mathcal{D}_1(z) - \frac{128}{\epsilon^2}\log(1-z) - \frac{256}{\epsilon}\,\mathcal{D}_2(z) + \frac{256}{\epsilon}\log^2(1-z) - \frac{320}{\epsilon}\log(1-z) + \frac{1024}{3}\,\mathcal{D}_3(z) - \frac{1024}{3}\log^3(1-z) + 640\log^2(1-z)\Bigg]\,.$$

NE SOFT RADIATION: VIRTUAL + REAL EMISSION

- Is this the end of the story? No: at NNLO, one has to face virtual + real radiation.



- Virtual radiation cannot be described in terms of soft gluon only.
- Building an effective field theory describing the process requires one to individuate the relevant momentum modes. This can be done easily by means of a momentum region analysis.

NE SOFT RADIATION: VIRTUAL + REAL EMISSION

- Decompose momenta along the light-cone directions of the external momenta:

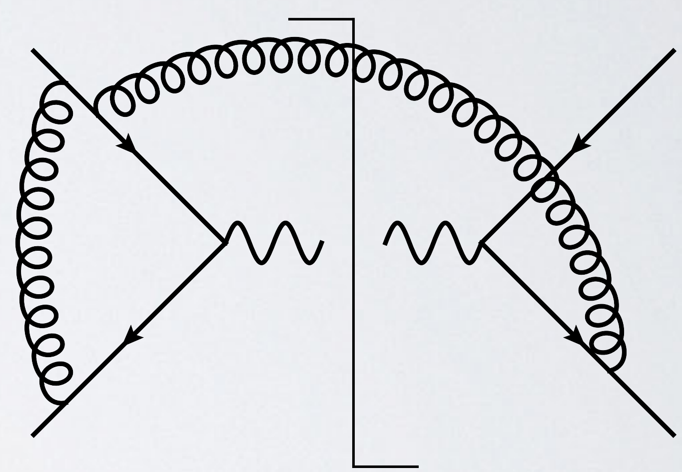
$$l^\mu = (n_- l)\frac{n_+^\mu}{2} + (n_+ l)\frac{n_-^\mu}{2} + l_\perp, \quad \Rightarrow \quad l \sim (n_- l, l_\perp, n_+ l), \quad n_+^2 = n_-^2 = 0, \quad n_- \cdot n_+ = 2.$$

- External momenta have definite scaling in the small parameter $\lambda = \sqrt{\frac{E_{\rm soft}}{\sqrt{\hat{s}}}}$:

$$p^\mu = n_- p \frac{n_+^\mu}{2} = \sqrt{\hat{s}}\frac{n_+^\mu}{2}, \quad \Rightarrow \quad p \sim (1,0,0);$$

$$\bar{p}^\mu = n_+ \bar{p} \frac{n_-^\mu}{2} = \sqrt{\hat{s}}\frac{n_-^\mu}{2}, \quad \Rightarrow \quad \bar{p} \sim (0,0,1);$$

$$k_2 \sim \frac{\sqrt{\hat{s}}}{2}(\lambda^2, \lambda^2, \lambda^2), \quad p \cdot \bar{p} = \frac{\hat{s}}{2}$$



- Virtual gluon can scale according to:

$$\text{Hard:} \quad k_1 \sim \frac{\sqrt{\hat{s}}}{2}(1,1,1);$$

$$\text{Collinear:} \quad k_1 \sim \frac{\sqrt{\hat{s}}}{2}(1,\lambda,\lambda^2);$$

$$\text{Anti-collinear:} \quad k_1 \sim \frac{\sqrt{\hat{s}}}{2}(\lambda^2,\lambda,1);$$

$$\text{Soft:} \quad k_1 \sim \frac{\sqrt{\hat{s}}}{2}(\lambda^2,\lambda^2,\lambda^2).$$

- Expand amplitude in the small parameters appearing in each region.

NE SOFT RADIATION: VIRTUAL + REAL EMISSION

- Calculation can be automatised with FORM; loop integrals in each region are quite easy.
- One finds that contributions arise from the region where the internal gluon is hard or collinear:

$$
\begin{aligned}
K^{\mathrm{NNLO_{tot}}}_{\mathrm{E,\,h}}(z) = \frac{\alpha_s}{4\pi}\,C_F\Bigg\{ & -\frac{256\mathcal{D}_0(z)}{\epsilon^3} + \frac{-256+192\mathcal{D}_0(z)-256\mathcal{D}_1(z)}{\epsilon^2} \\
& + \frac{192-256\mathcal{D}_0(z)+192\mathcal{D}_1(z)-128\mathcal{D}_2(z)-256\log(1-z)}{\epsilon} - 256 + 256\mathcal{D}_0(z) \\
& - 256\mathcal{D}_1(z) + 96\mathcal{D}_2(z) - \frac{128\mathcal{D}_3(z)}{3} + 192\log(1-z) - 128\log^2(1-z)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
K^{\mathrm{NNLO_{tot}}}_{\mathrm{NE,\,h}}(z) = \frac{\alpha_s}{4\pi}\,C_F\Bigg\{ & \frac{256}{\epsilon^3} + \frac{-64+256\log(1-z)}{\epsilon^2} + \frac{160-64\log(1-z)+128\log^2(1-z)}{\epsilon} - 128 \\
& + 160\log(1-z) - 32\log^2(1-z) + \frac{128}{3}\log^3(1-z)\Bigg\}.
\end{aligned}
$$

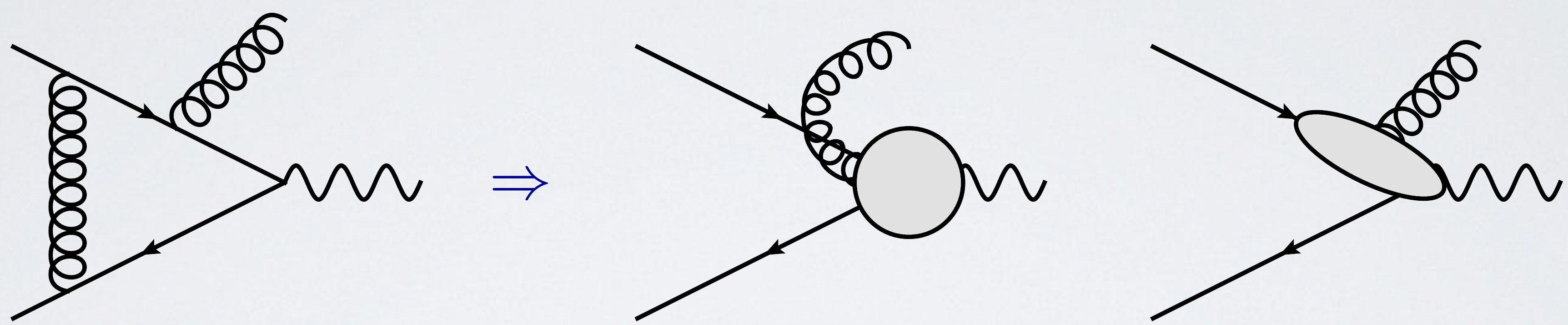
$$
K^{\mathrm{NNLO_{vertex}}}_{\mathrm{NE,\,c+\bar{c}}}(z) = \frac{\alpha_s}{4\pi}\,C_F\left\{16 - \frac{32}{\epsilon^2} - \frac{48\log(1-z)}{\epsilon} - 36\log^2(1-z)\right\},
$$

$$
K^{\mathrm{NNLO_{external\,legs}}}_{\mathrm{NE,\,c+\bar{c}}}(z) = \frac{\alpha_s}{4\pi}\,C_F\left\{-40 - \frac{32}{\epsilon^2} + \frac{40-48\log(1-z)}{\epsilon} + 60\log(1-z) - 36\log^2(1-z)\right\}.
$$

- The sum reproduces the full QCD result.
- Similar conclusion obtained in the context of Higgs production by Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 2013.

NE SOFT RADIATION: VIRTUAL + REAL EMISSION

- The momentum region analysis (and old results, see below) shows that there are contributions from the region where the internal gluon is hard or collinear: schematically



- This contribution (emission of a soft gluon from the hard vertex) cannot occur at the eikonal level, because the Compton Wavelenght of the soft photon cannot resolve the hard interaction.

- They occur at the NE level, however, and have been studied by Low for massive scalars, then generalised to spinors by Burnett, Kroll, and then generalised to the case of small mass by Del Duca (1990). Here we need the limit $m \to 0$.

Low, 1958;
Brunett, Kroll, 1968;
Del Duca, 1990

NE SOFT RADIATION: VIRTUAL + REAL EMISSION

- Consider factorisation of the quark form factor: (Collins, Korchemsky)

$$\Gamma\left(\frac{Q^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)=\mathcal{H}\left(\frac{Q^2}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s(\mu^2),\epsilon\right)\times\mathcal{S}\left(\beta_1\cdot\beta_2,\alpha_s(\mu^2),\epsilon\right)\times\prod_{i=1}^{2}\left[\frac{J\left(\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s(\mu^2),\epsilon\right)}{\mathcal{J}\left(\frac{(\beta_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s(\mu^2),\epsilon\right)}\right].$$

- where the soft function collects infrared singularities associated with the eikonal term in the momentum expansion of emitted gluons,

$$\mathcal{S}(\beta_1\cdot\beta_2,\alpha_s(\mu^2),\epsilon)=\langle 0\,|\Phi_{\beta_2}(\infty,0)\Phi_{\beta_1}(0,-\infty)|\,0\rangle\,,\quad \text{with}\quad \Phi_n(\lambda_2,\lambda_1)=\mathcal{P}\exp\left[ig_s\int_{\lambda_1}^{\lambda_2}n\cdot A(\lambda n)\right],$$

- and the partonic jet functions are defined as

$$J\left(\frac{(p\cdot n)^2}{n^2\mu^2},\alpha_s(\mu^2),\epsilon\right)u(p)=\langle 0\,|\Phi_n(\infty,0)\psi(0)|\,p\rangle\,,$$

$$\mathcal{J}\left(\frac{(\beta\cdot n)^2}{n^2\mu^2},\alpha_s(\mu^2),\epsilon\right)u(p)=\langle 0\,|\Phi_n(\infty,0)\Phi_{\beta_1}(0,-\infty)|\,0\rangle\,.$$

- where $\psi(x)$ is a wavefunction for the external parton, and the auxiliary vector n ensures that the definition is gauge-covariant. One must divide by eikonal jet functions in order to avoid the double-counting of soft and collinear contributions.

NE SOFT RADIATION: VIRTUAL + REAL EMISSION

- The structure of the internal emission can be derived following Del Duca 1990: One split this contribution according to

$$\epsilon_\mu \Gamma^\mu_{\rm int.} = \epsilon_\mu \Gamma^\mu_H + \epsilon_\mu \Gamma^\mu_J$$

- (emission from the hard and jet functions respectively). Using Ward identities and introducing the "G" and "K" polarisation tensor:

$$k_\mu \Gamma^\mu_H = - k_\mu \Gamma^\mu_J, \qquad K_{\nu\mu}(p;k) = k_\nu \frac{(2p+k)_\mu}{2p\cdot k + k^2}, \quad G_{\nu\mu} = g_{\nu\mu} - K_{\nu\mu},$$

- the complete (K+G) emission from the hard function can be combined with the K emission for the jet, to give

$$\epsilon_\mu(k)\left(\Gamma^\nu_J K_\nu{}^\mu + \Gamma^\mu_H\right) = -\sum_{i=1}^{2} q_i\, G_{\nu\mu}(p_i,k) \left(-\frac{\partial}{\partial p_{i\nu}} \Gamma(\{p_i\}) + H(\{p_i\}) S(\{\beta_i\}) \frac{\partial}{\partial p_{i\nu}} \prod_{j=1}^{2} J_j(p_j,n_j) \right),$$

- i.e., internal emission contributions are generated by derivatives acting on the hard function with no emission, which in a sense shows an iterative structure as well. The remaining G-emission from the current reads

$$\Gamma^\nu_J\, G_{\nu\mu}\, \epsilon^\mu(k) = \sum_{i=1}^{n} H(\{p_i\}) S(\{\beta_i\}) J^\nu(p_i,k,n_i) G_{\mu\nu}(p_i;k) \prod_{j\neq i} J(p_j,n_j),$$

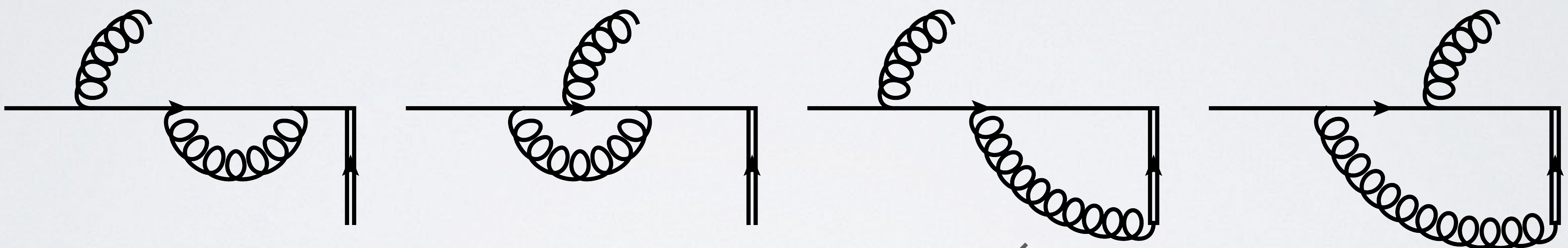
- which is not simply given by derivatives acting on the hard function, but depends on a universal
- function as well.

NE SOFT RADIATION: VIRTUAL + REAL EMISSION

- Calculation of

$$J_\mu\left(\frac{(p\cdot n)^2}{n^2\mu^2},\alpha_s(\mu^2),\epsilon;k\right)u(p)=\left\langle 0\left|\int d^dy e^{-i(p+k)\cdot y}\Phi_n(y,\infty)\psi(y)j_\mu(0)\right|p\right\rangle .$$

- is non trivial:



$$J^\mu_{[3]}=i\,2\,g_s^3\,\mathcal{C}\int[dk_1]\frac{(\not{p}-\not{k}_1-\not{k}_2)\not{n}(\not{p}-\not{k}_2)\gamma^\mu u(p)}{(p-k_1-k_2)^2(p-k_2)^2(2n\cdot k_1)k_1^2},$$

$$J^\mu_{[4]}=i\,2\,g_s^3\,\mathcal{C}\int[dk_1]\frac{(\not{p}-\not{k}_1-\not{k}_2)\gamma^\mu(\not{p}-\not{k}_1)\not{n}u(p)}{(p-k_1-k_2)^2(p-k_1)^2(2n\cdot k_1)k_1^2}.$$

NE SOFT RADIATION: VIRTUAL + REAL EMISSION

- Calculation is non trivial, but one expect a relatively simple result!
- For instance, n-dependence must cancel from the full result, and we expect it to occur already at the level of the amplitude:

$$\epsilon_\mu(k)\left(\Gamma^\nu_J ({K_\nu}^\mu + {G_\nu}^\mu) + \Gamma^\mu_H\right)$$

- this gives already quite some strong constraints on the structure of the result.
- However, this simplicity is not apparent in the calculation: for instance, one cannot expand in k_2 before integration over k_1: the integrals become ill-defined. Similarly, one encounters problems assuming some special choice of n, such as $n^2 = 0$.
- Result for J^μ is completed. For instance

$$\begin{aligned} J^\mu_{[4]} =& i2g_s^3\mathcal{C}\Big\{p^\mu\Big[4p\cdot n(I_0^{[4]} - 2I_{11}^{[4]} + I_{22}^{[4]}) - 2n^2(I_{13}^{[4]} - I_{26}^{[4]})\Big] + k_2^\mu\Big[4p\cdot n(I_{25}^{[4]} - I_{12}^{[4]}) + 2n^2 I_{27}^{[4]}\Big] \\ &+ \not{k}_2\gamma^\mu\Big[-2p\cdot n(I_0^{[4]} - I_{11}^{[4]}) + n^2 I_{13}^{[4]}\Big] + \gamma^\mu\not{n}\Big[2p\cdot k_2(I_{12}^{[4]} - I_{25}^{[4]}) - (d-2)I_{21}^{[4]} - 2n\cdot k_2 I_{27}^{[4]}\Big] \\ &+ \not{n}\gamma^\mu\Big[-2p\cdot n(I_{13}^{[4]} - I_{26}^{[4]}) + n^2 I_{24}^{[4]}\Big] - 2p^\mu\not{k}_2\not{n}\Big[I_{12}^{[4]} - I_{25}^{[4]}\Big] + 2k_2^\mu\not{k}_2\not{n}\Big[I_{23}^{[4]} + I_{12}^{[4]}\Big] + 2n^\mu\not{k}_2\not{n}\Big[I_{27}^{[4]}\Big]\Big\}u(p). \end{aligned}$$

- where, for instance,

NE SOFT RADIATION: VIRTUAL + REAL EMISSION

- Note: preliminar!

$$
\begin{aligned}
(I_0^{[4]} - 2I_{11}^{[4]} + I_{22}^{[4]}) =& \frac{1}{\epsilon^2}\left\{-\frac{1}{8p\cdot k_2 p\cdot n}\right\} + \frac{1}{\epsilon}\left\{-2 - \frac{\log\left(\frac{n^2}{4(p\cdot n)^2}\right)}{8p\cdot k_2\, p\cdot n}\right\} \\
&+ \frac{-48+11\pi^2}{96p\cdot k_2\, p\cdot n} + \frac{3n^2\, p\cdot k_2 - 4\pi^2\, n^2\, p\cdot k_2 + 6n\cdot k_2\, p\cdot n}{12p\cdot k_2\,(p\cdot n)^3} - \frac{n^2\log(2p\cdot k_2)^2}{4(p\cdot n)^3} + \frac{\log(2p\cdot k_2)^2}{8p\cdot k_2\, p\cdot n} \\
&- \frac{(3n^2\, p\cdot k_2 - n\cdot k2\, p\cdot n)\log\left(\frac{n^2}{(2p\cdot n)^2}\right)}{4p\cdot k_2\,(p\cdot n)^3} - \frac{n^2\log\left(\frac{n^2}{(2p\cdot n)^2}\right)^2}{4(p\cdot n)^3} + \frac{\log\left(\frac{n^2}{(2p\cdot n)^2}\right)^2}{16p\cdot k_2\, p\cdot n} \\
&+ \frac{\log(2p\cdot k_2)\left(1+\log\left(\frac{n^2}{(2p\cdot n)^2}\right)\right)}{4p\cdot k_2\, p\cdot n} \\
&+ \frac{\log(2p\cdot k_2)\left(-3n^2\, p\cdot k_2 + n\cdot k_2\, p\cdot n - 2n^2\, p\cdot k_2\log\left(\frac{n^2}{(2p\cdot n)^2}\right)\right)}{4p\cdot k_2\,(p\cdot n)^3},
\end{aligned}
$$

$$
\begin{aligned}
(I_{13}^{[4]} - I_{26}^{[4]}) =& -\frac{\pi^2}{6(p\cdot n)^2} - \frac{\log(2p\cdot k_2)^2}{8(p\cdot n)^2)} - \frac{\log\left(\frac{n^2}{(2p\cdot n)^2}\right)}{2(p\cdot n)^2} - \frac{\log\left(\frac{n^2}{(2p\cdot n)^2}\right)^2}{8(p\cdot n)^2} \\
&- \frac{\log(2p\cdot k_2)\left(2+\log\left(\frac{n^2}{(2p\cdot n)^2}\right)\right)}{4(p\cdot n)^2},
\end{aligned}
$$

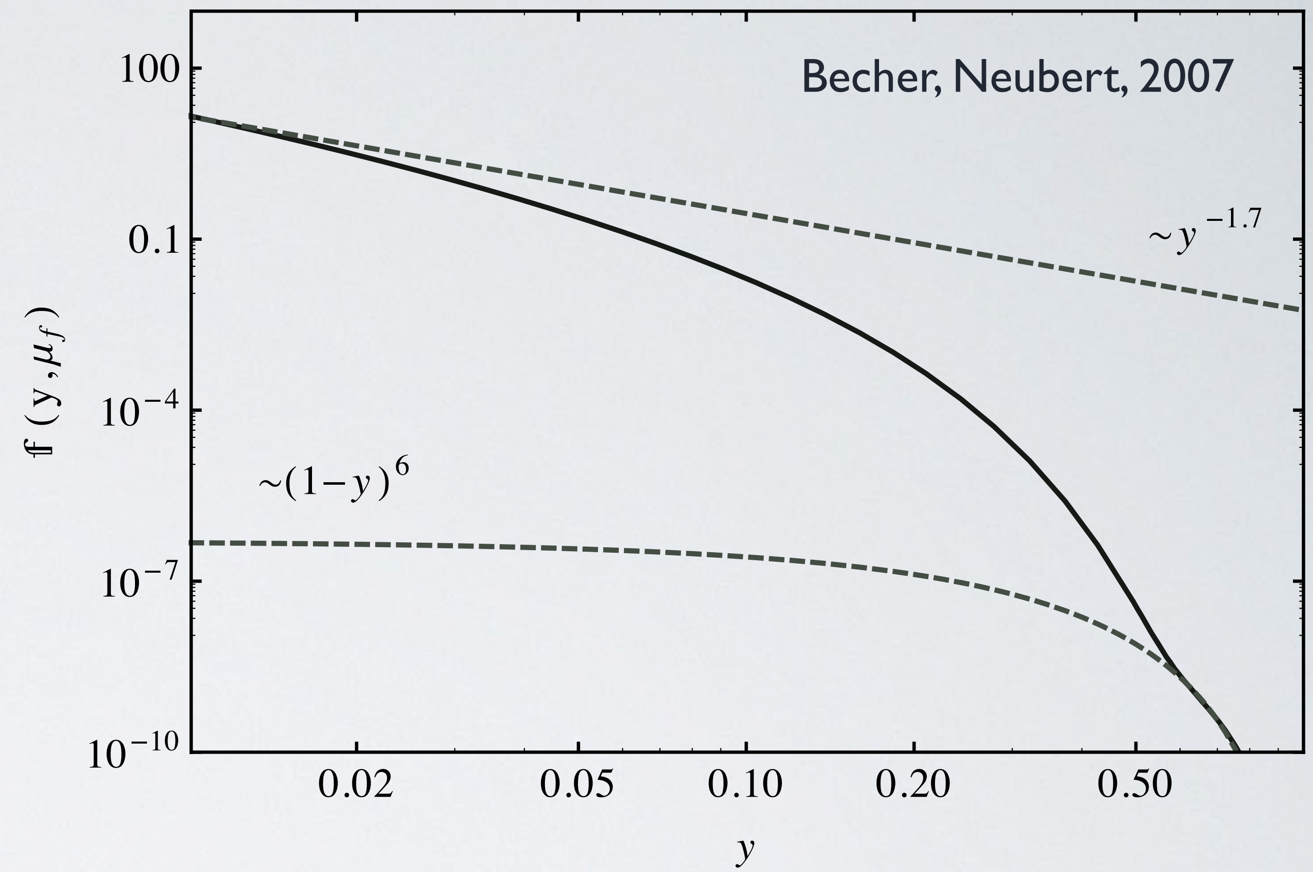
$\ldots$

OUTLOOK

- Soft gluons exponentiate at leading (eikonal) order. Their resummation is important to get precise prediction for scattering processes at hadron colliders.
- It has been observed that some of the logs originating from soft gluon at the sub-leading order (NE) exponentiate, too. Their inclusion can be phenomenologically relevant.
- We prove that soft gluon emission from external energetic partons exponentiate at the NE level, too, provided one extends the standard description in terms of soft gluon webs, introducing next-to-eikonal webs.
- We analysed Drell-Yan at NNLO, showing that reproducing all the logs at the NE level requires taking into account soft gluon emission from the hard and the hard-collinear interaction.
- This can be implemented in the contest of the Low-Burnett-Kroll-Del Duca formulation of factorisation at the sub-leading order. Verification that all the logs can be reproduced in this framework is in progress.

BACKUP

SOFT RADIATION IN DRELL-YAN AND ELECTROWEAK ANNIHILATION



$$\frac{d\sigma}{dQ^2} \sim \int_\tau^1 \frac{dz}{z}\, \hat{\sigma}_{q\bar{q}}(z, Q, \mu)\, \mathcal{L}_{q\bar{q}}\left(\frac{\tau}{z}, \mu\right),$$

- For $\tau \lesssim 0.05$ becomes

$$\frac{d\sigma}{dQ^2} \sim \mathcal{L}_{q\bar{q}}(\tau, \mu) \int_\tau^1 \frac{dz}{z}\, \hat{\sigma}_{q\bar{q}}(z, Q, \mu)\, z^{-a},$$

- While for $\tau \gtrsim 0.3$

$$\frac{d\sigma}{dQ^2} \sim \mathcal{L}_{q\bar{q}}(\tau, \mu) \int_\tau^1 \frac{dz}{z}\, \hat{\sigma}_{q\bar{q}}(z, Q, \mu) \left(\frac{1-\tau/z}{1-\tau}\right)^b,$$

- The **z** integral receives important contributions from the region $(1-z) < \frac{1-\tau}{b}$ with **b~10**.

- Even for **τ** values not near **1** there is a parametric enhancement of the partonic threshold region, which turns the threshold logarithms into logarithms of the exponent **b**.