Multi-loop unitarity via computational algebraic geometry

Loopfest XIII, NYC, 2014

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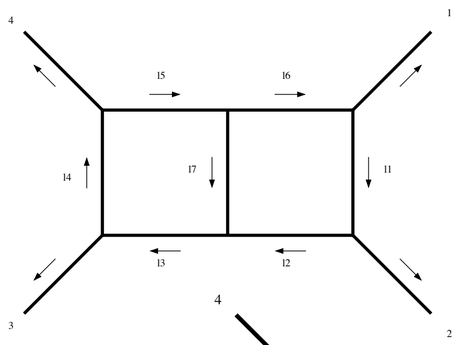
Based on

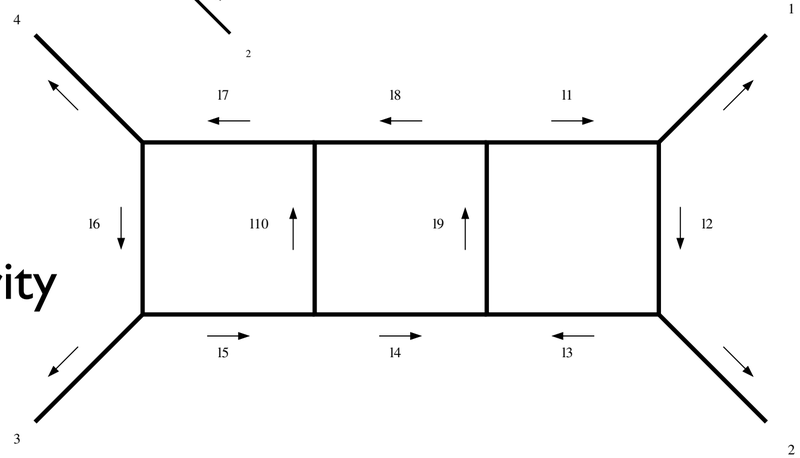
(2-loop 4-point)	arXiv:1202.2019, Simon Badger, Hjalte Frellesvig and YZ
(algebraic geometry methods)	arXiv:1205.5707, YZ
(3-loop 4-point)	arXiv:1207.2976, Simon Badger, Hjalte Frellesvig and YZ
(global structure)	arXiv:1302.1023, Rijun Huang and YZ
(2-loop 5-point QCD)	arXiv:1310.1051, Simon Badger, Hjalte Frellesvig and YZ
(maximal cut)	arXiv:1310.6006, Mads Sogaard and YZ

and works under progress...

Outline

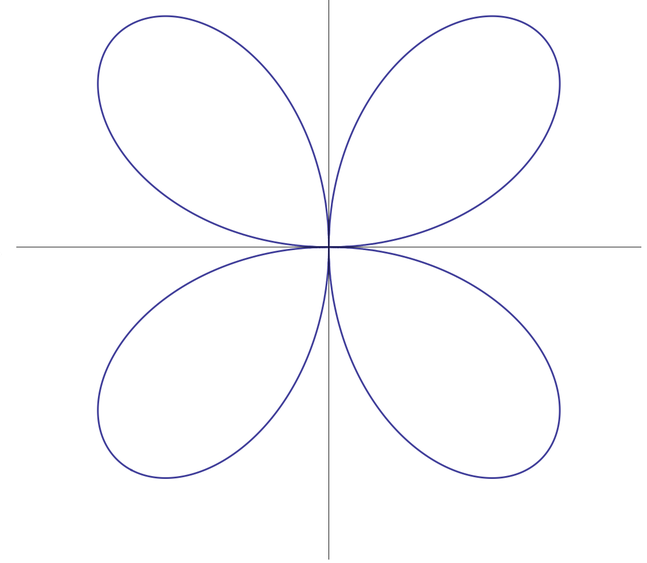


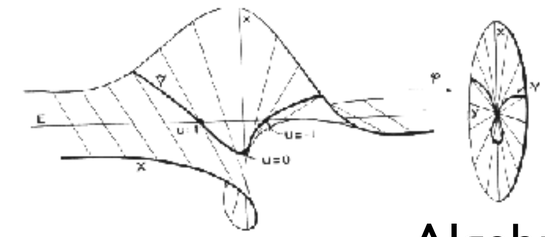
High-loop Amplitudes



Unitarity

$\Longrightarrow$





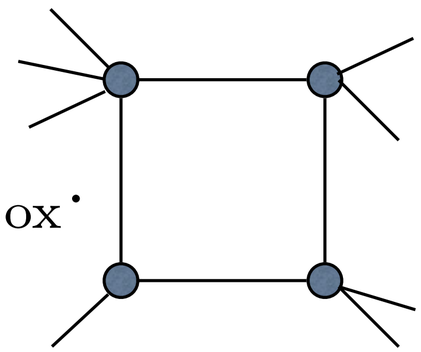
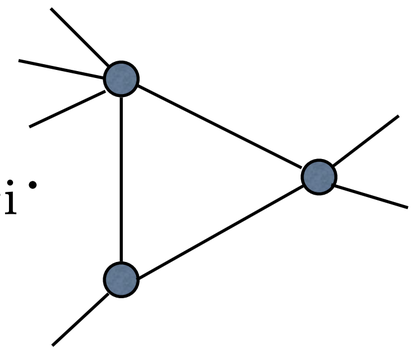
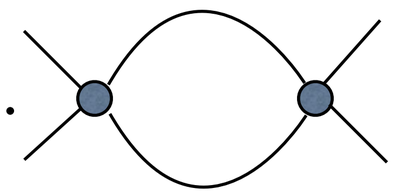
Algebraic geometry

Gröbner Basis
Primary Decomposition
Affine Variety Structure
Multivariate residue

- Integrand reduction at one loop, review
- Integrand reduction at n loop by algebraic geometry
- Examples: 2-loop 5-gluon planar QCD, 3-loop 4-point triple-box ...

Unitarity at one-loop

$D = 4$

$A^{(1)} = c_{\text{box}} \cdot$  $+ c_{\text{tri}} \cdot$  $+ c_{\text{bub}} \cdot$  $+ \ldots$

- no pentagon, hexagon ...
- scalar integral (numerator is one.)

Unitarity:

Determine 'c' coefficients from on-shell cut solutions and tree amplitudes

quadruple cut $\rightarrow c_{\text{box}}$

triple cut $\rightarrow c_{\text{tri}}$

double cut $\rightarrow c_{\text{bub}}$

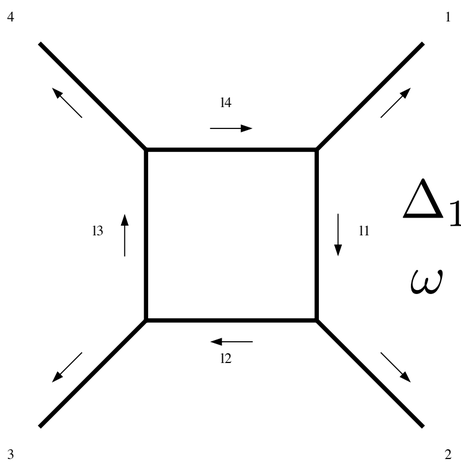
Integrand reduction: box

Integrand-level reduction, Ossola, Papadopoulos and Pittau (OPP), 2006
Giele, Kunszt, Melnikov, 2008

$$A^{(1)} = \int \frac{d^4k}{(2\pi)^4} \frac{N(k)}{D_1 D_2 D_3 D_4}$$

$$N(k) = \Delta_{1234}(k) + \sum_{i_1<i_2<i_3} \Delta_{i_1 i_2 i_3}(k) \prod_{i \neq i_1,i_2,i_3} D_i + \sum_{i_1<i_2} \Delta_{i_1 i_2}(k) \prod_{i \neq i_1,i_2} D_i$$

$$= \Delta_{1234}(k) + O(D_1, D_2, D_3, D_4)$$



$\Delta_{1234}(k)$ is a polynomial in scalar products (SP). $\mathbb{SP} = \{k \cdot P_1, k \cdot P_2, k \cdot P_3, k \cdot \omega\}$

ω is auxiliary, $(\omega \cdot P_i) = 0$, $i = 1, 2, 3, 4$

reducible scalar product (RSP) — irreducible scalar product (ISP)

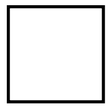
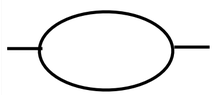
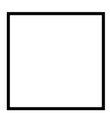
$$\begin{aligned} 2(k \cdot P_1) &= D_4 - D_1 - P_1^2 \\ 2(k \cdot P_2) &= D_1 - D_2 + P_2^2 \\ 2(k \cdot P_3) &= D_2 - D_3 + 2P_2 \cdot P_3 + P_3^2 \end{aligned}$$

$$\Delta_{1234}(k) = \sum_i c_i (k \cdot \omega)^i$$

$\Delta_{1234}(k)$ is a polynomial in ISP only.

integrand basis $\boxed{\Delta_{1234}(k) = c_0 + c_1 (k \cdot \omega)}$

One loop, other diagrams

Dimension	Diagram	# SP (ISP+RSP)	#terms in integrand basis (non-spurious + spurious)	# Solutions (dimension)
4		4 (1+3)	2 (1+1)	2 (0)
4		4 (2+2)	7 (1+6)	1 (1)
4		4 (3+1)	9 (1+8)	1(2)
4-2ε		5 (2+3)	5 (3+2)	1(1)

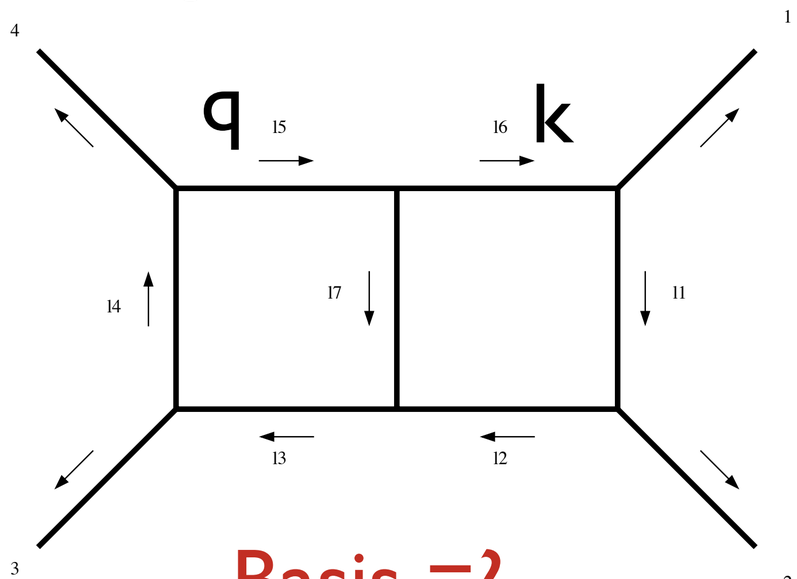
- straightforward to obtain integrand basis, unitarity cut solutions
- all one-loop master integrals are known
- c coefficients can be automatically computed by public codes

  - 'NGluon', Badger, Biedermann, and Uwer
  - 'CutTools', Ossola, Papadopoulos, and Pittau
  - 'GoSam', Cullen, Greiner, Heinrich, Luisoni, and Mastrolia
  - ...

Generalization to higher loops?

Example: 4D massless two-loop hepta cut

P. Mastrolia, G. Ossola, 2011
S. Badger, H. Frellesvig, YZ, 2012



Basis =?

7 cut-equations in 8 SP's

$$\mathbb{SP} = \{k \cdot P_1, k \cdot P_2, k \cdot P_4, k \cdot \omega, q \cdot P_1, q \cdot P_2, q \cdot P_4, q \cdot \omega\}$$

4 cut-equations to identify 4 RSP's

4 ISP's $\quad \mathbb{ISP} = \{k \cdot P_4, k \cdot \omega, q \cdot P_1, q \cdot \omega\}$

3 cut-equations for ISP's

$$(k \cdot \omega)^2 = (k \cdot P_4 - t/2)^2 \quad (1)$$

$$(q \cdot \omega)^2 = (q \cdot P_1 - t/2)^2 \quad (2)$$

$$(k \cdot \omega)(q \cdot \omega) = -\frac{t^2}{4} + \frac{t(k \cdot P_4)}{2} + \frac{t(q \cdot P_1)}{2} + \left(1 + \frac{2t}{s}\right)(k \cdot P_4)(q \cdot P_1) \quad (3)$$

Naive guessing: all renormalizable monomials which do NOT contain $(k \cdot \omega)^2$, $(q \cdot \omega)^2$ or $(k \cdot \omega)(q \cdot \omega)$.

$$\Delta_{\text{dbox}} = (k \cdot P_4)^m (q \cdot P_1)^n (k \cdot \omega)^\alpha (q \cdot \omega)^\beta$$
$$m + \alpha \leq 4, n + \beta \leq 4, m + n + \alpha + \beta \leq 6$$
$$(\alpha, \beta) = (0,0), (1,0), (0,1)$$

56 terms? wrong...

Example: 4D massless two-loop hepta cut

S. Badger, H. Frellesvig, YZ, 2012

3 cut-equations for ISP's, and their combinations

$$
\begin{aligned}
(k\cdot\omega)^2 &= (k\cdot P_4 - t/2)^2 && (1)\\
(q\cdot\omega)^2 &= (q\cdot P_1 - t/2)^2 && (2)\\
(k\cdot\omega)(q\cdot\omega) &= -\frac{t^2}{4} + \frac{t(k\cdot P_4)}{2} + \frac{t(q\cdot P_1)}{2} + \left(1+\frac{2t}{s}\right)(k\cdot P_4)(q\cdot P_1) && (3)
\end{aligned}
$$

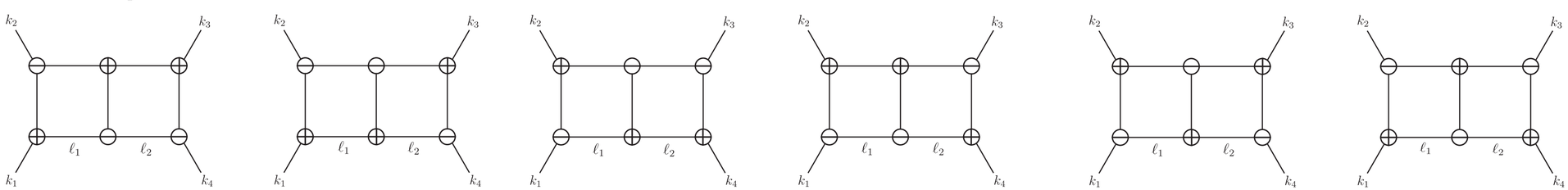
reduced

$$\boxed{(1)\times(2)-(3)^2}$$

$$4(k\cdot P_4)^2(q\cdot P_1)^2 = -2s(k\cdot P_4)^2(q\cdot P_1) - 2s(k\cdot P_4)(q\cdot P_1)^2 - st(k\cdot P_4)(q\cdot P_1)$$

We have to "exhaust" all combinations...

Finally, we determine that the basis contains 32 terms



6 families of hepta-cut solutions, Laurant series contains 38 terms

Solving 38 linear equations for 32 coefficients, done!

Messy, not automatic!

Gröbner basis and integrand basis

arXiv:1205.5707, YZ
arXiv:1205.7087, Mastrolia, Mirabella, Ossola and Peraro

$$I = \langle D_1, \dots D_k \rangle = \{ \sum_{i=1}^{k} g_i D_i | \quad \forall g_i \in R \}$$

$$\int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \frac{N}{D_1 D_2 \dots D_7}, \quad N = Q + \Delta_{\text{dbox}}, \quad Q \in I$$

Synthetic polynomial division

N divided by $\{D_1, \dots D_k\}$:

Define a monomial order, and recursively preform $N/D_1, \dots, N/D_k$. Finally, the division process will stop and we have

Euclidean division

$$N = f_1 D_1 + \dots f_k D_k + r'$$

where r' is the remainder. $\Delta_{\text{dbox}} = r'$???

In most cases, it does not work since it stops too early,
unless we are using Gröbner basis.

$$I = \langle D_1, \dots, D_k \rangle = \langle g_1, \dots, g_m \rangle$$

Gröbner basis
'good' generators

$$N = q_1 g_1 + \dots q_k g_k + r$$

- r is uniquely determined.
- If $N \in I$, $r = 0$.

$$\boxed{\Delta_{\text{dbox}} = r}$$

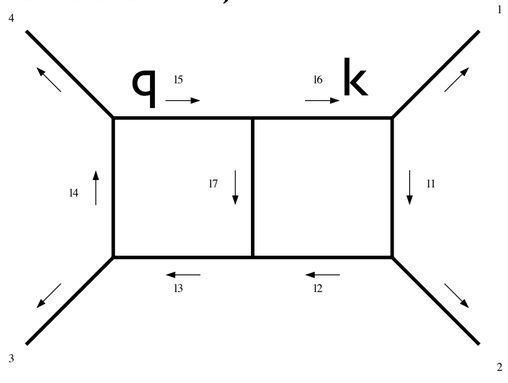
$$\begin{pmatrix} y^3 & x - 2y^2 \end{pmatrix} = \begin{pmatrix} x^3 - 2xy & x^2y - 2y^2 + x \end{pmatrix} \begin{pmatrix} -\frac{1}{4} - \frac{1}{4}xy - \frac{1}{2}y^3 & y^2 \\ \frac{1}{4}x^2 - \frac{1}{2}y + \frac{1}{2}xy^2 & 1 - xy \end{pmatrix}$$

Toy Model: $N = xy^3$, $I = \langle x^3 - 2xy, x^2y - 2y^2 + x \rangle$. Direct synthetic division of N towards $\{x^3 - 2xy, x^2y - 2y^2 + x\}$ gives $r' = xy^3$.

But the Gröbner basis is $I = \langle y^3, x - 2y^2 \rangle$, and the synthetic division of N on Gröbner basis gives $r = 0$. So $N \in I$.

Grobner basis: dbox example

arXiv:1205.5707, YZ



4 ISP's $\mathbb{ISP} = \{k \cdot P_4, k \cdot \omega, q \cdot P_1, q \cdot \omega\}$

$$N = q_1 g_1 + \ldots q_k g_k + \Delta_{\text{dbox}}$$

N contains 160 terms where Δ_{dbox} contains 32 terms.

In principle, it works for arbitrary number of loops, any dimension.
Automated by the package: 'BasisDet'

http://www.nbi.dk/~zhang/BasisDet.html, YZ 2012

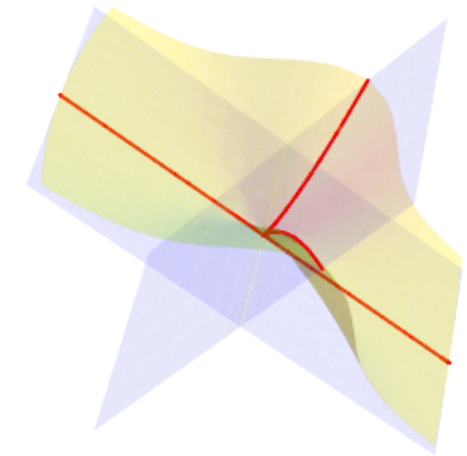
Dimension propagators, kinematics → Integrand basis

Can also find ISP automatically!

Primary decomposition

arXiv:1205.5707, YZ

Find the number of branches of unitarity solutions



$I = \langle x^2 - y^2, x^3 + y^3 - z^2 \rangle$. How many (irreducible) curves are there in $\mathcal{Z}(I)$.

Primary decomposition:

$$I = I_1 \bigcap I_2 \quad I_1 = \langle x + y, z^2 \rangle, \quad I_2 = \langle x - y, 2y^3 - z^2 \rangle$$

- AG software 'Macaulay 2'
- Numeric algebraic geometry methods

$$I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5 \cap I_6$$

4D massless dbox hepta-cut: 6 branches of solutions

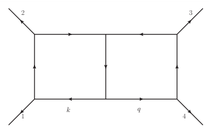
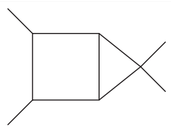
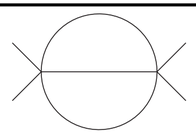
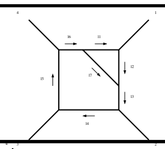
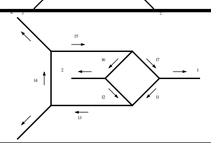
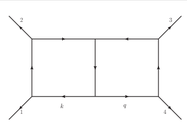
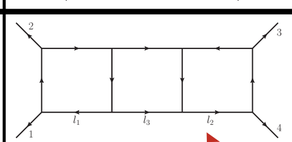
dictionary

Algebra	Geometry	
height I	$\dim \mathcal{Z}(I) = n - \text{height } I$	(# free parameters)
arithmetic genus	(geometric) genus	(topology)

High genus examples: arXiv:1302.1203, Rijun and YZ

works for arbitrary number of loops, any dimension

More examples

Dimension	Diagram	# SP (ISP+RSP)	#terms in integrand basis (non-spurious + spurious)	# Solutions (dimension)
4		8 (4+4)	32 (16+16)	6 (1)
4		8 (5+3)	69 (18+51)	4 (2)
4		4 (3+1)	42 (12+30)	1(5)
4		8 (3+5)	20 (10+10)	2 (2)
4		8 (4+4)	38 (19+19)	8 (1)
4-2ε		11 (7+4)	160 (84+76)	1(4)
4		12 (7+5)	398 (199+199)	14 (2)

Nontrivial dimension

Non-planar

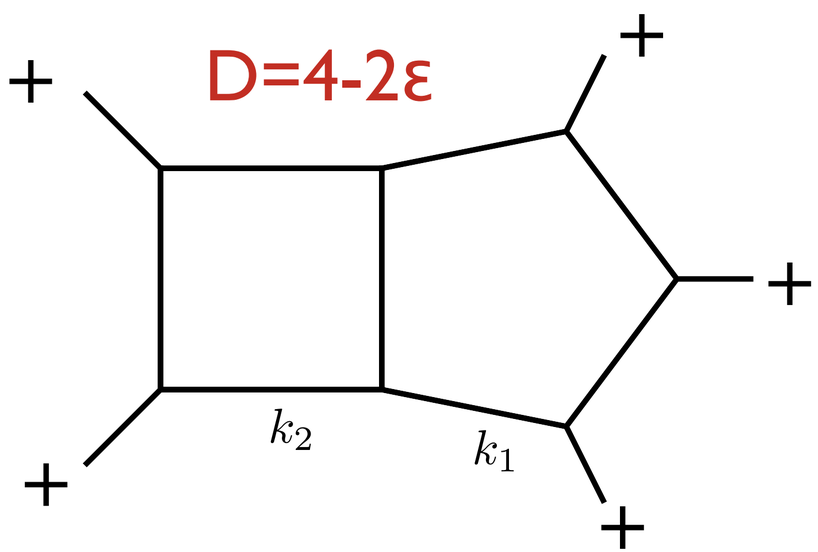
Three-loop!

Even more examples:
arXiv:1209.3747 Bo Feng and Rijun Huang

D-dim integrand reduction

2-loop 5-point QCD

arXiv: 1310.1051: Simon Badger, Hjalte Frellesvig and YZ



$$\mu_{11} = k^2_{[-2\epsilon],1},\ \mu_{22} = k^2_{[-2\epsilon],2} \text{ and } \mu_{12} = 2(k_{[-2\epsilon],1} \cdot k_{[-2\epsilon],2})$$

$$\mu_{33} = \mu_{11} + \mu_{22} + \mu_{12}$$

$$\Delta_{431}(1^+, 2^+, 3^+, 4^+, 5^+) = \frac{i\, s_{12} s_{23} s_{45} F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12})}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}(tr_+(1345)(k_1 + p_5)^2 + s_{15}s_{34}s_{45})$$

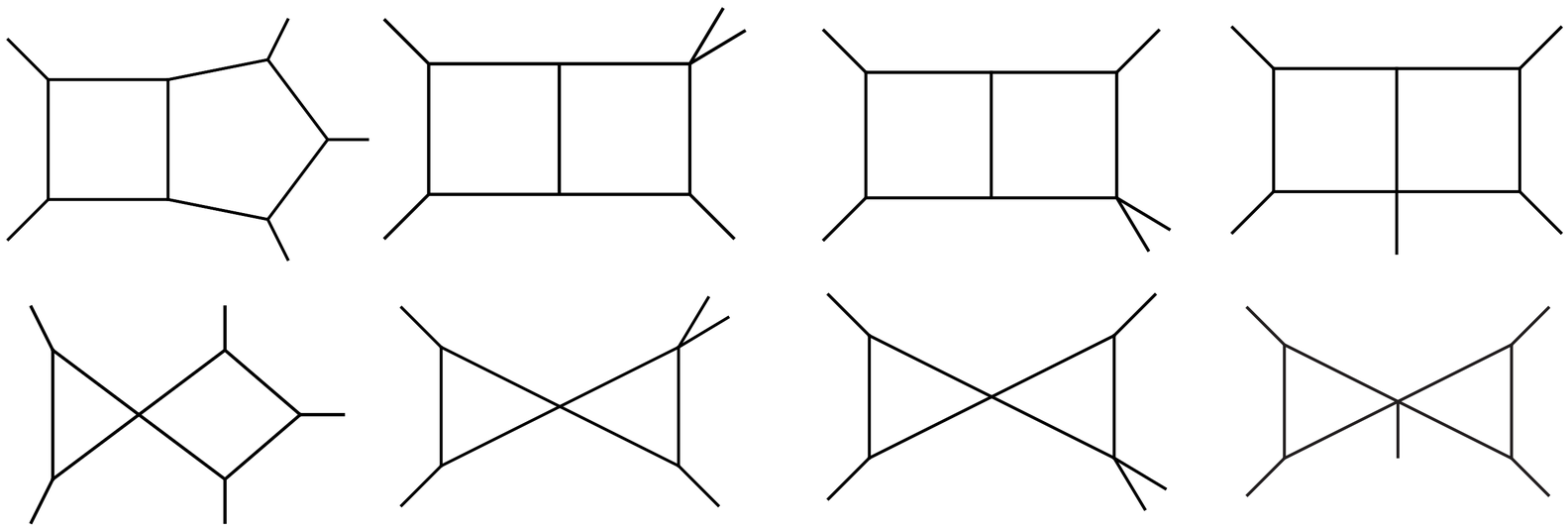
$$F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12}) = (D_s - 2)(\mu_{11}\mu_{22} + \mu_{11}\mu_{33} + \mu_{22}\mu_{33}) + 4(\mu_{12}^2 - 4\mu_{11}\mu_{22})$$

- Feynman rules + cut solution
- 6D spinor helicity formalism

Momentum-twistor parametrization

$(\lambda, \tilde{\lambda}) \longrightarrow (\lambda, \mu)$ (Andrew Hodges)

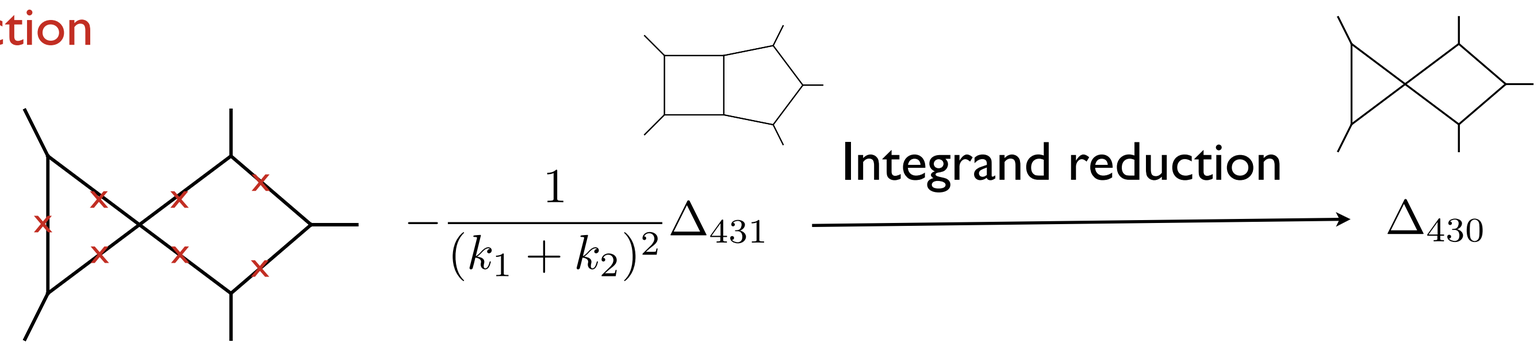
2-loop 5-gluon amplitude



arXiv: 1310.1051

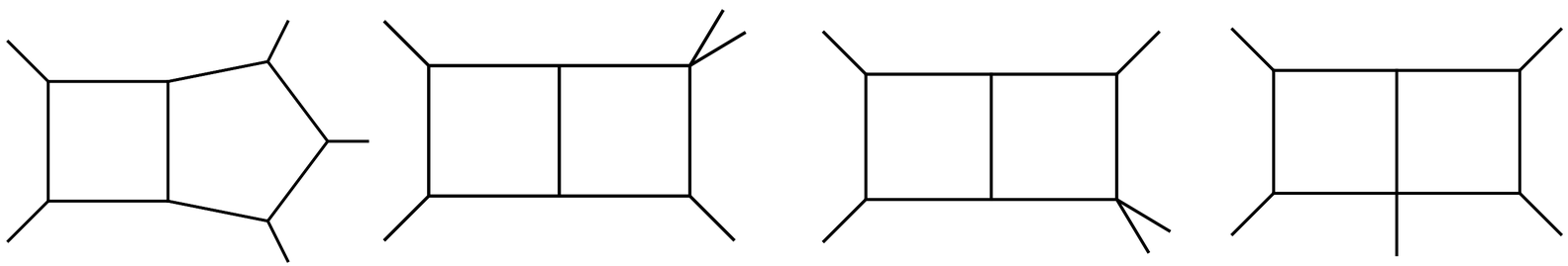
first result on 2-loop 5-gluon helicity amplitude in QCD

subtraction



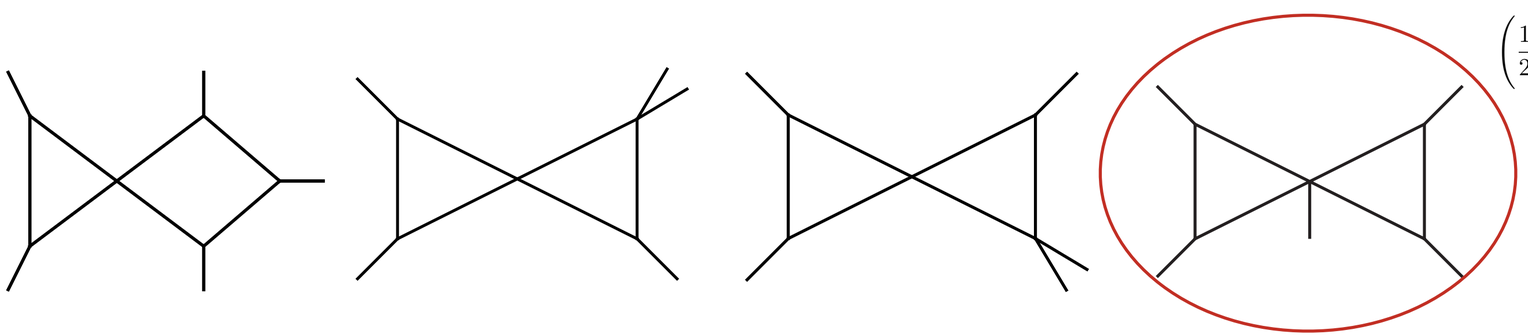
all coefficients are analytically found
IR structure: consistent with Catani's factorization

2-loop 5-gluon amplitude



arXiv: 1310.1051

$$= F_1(D_s, \mu_{11}, \mu_{22}, \mu_{12}) \times (\text{helicity factor}) \times (\mathcal{N} = 4 \text{ Integrand})$$



$$\begin{aligned}
\Delta_{330;5L}(1^+,2^+,3^+,4^+,5^+) \;=\; & -\frac{i}{\langle 12\rangle\langle 12\rangle\langle 12\rangle\langle 12\rangle\langle 12\rangle}\times \\
& \Bigg(\frac{1}{2}\left(\mathrm{tr}_+(1245) - \frac{\mathrm{tr}_+(1345)\mathrm{tr}_+(1235)}{s_{13}s_{35}}\right)\Bigg(2(D_s-2)(\mu_{11}+\mu_{22})\mu_{12} \\
& \quad + (D_s-2)^2\mu_{11}\mu_{22}\frac{4(k_1\cdot p_3)(k_2\cdot p_3) + (k_1+k_2)^2(s_{12}+s_{45}) + s_{12}s_{45}}{s_{12}s_{45}}\Bigg) \\
& + (D_s-2)^2\mu_{11}\mu_{22}\Bigg[(k_1+k_2)^2 s_{15} \\
& \quad + \mathrm{tr}_+(1235)\left(\frac{(k_1+k_2)^2}{2s_{35}} - \frac{k_1\cdot p_3}{s_{12}}\left(1 + \frac{2(k_2\cdot\omega_{453})}{s_{35}} + \frac{s_{12}-s_{45}}{s_{35}s_{45}}(k_2-p_5)^2\right)\right) \\
& \quad + \mathrm{tr}_+(1345)\left(\frac{(k_1+k_2)^2}{2s_{13}} - \frac{k_2\cdot p_3}{s_{45}}\left(1 + \frac{2(k_1\cdot\omega_{123})}{s_{13}} + \frac{s_{45}-s_{12}}{s_{12}s_{13}}(k_1-p_1)^2\right)\right)\Bigg]\Bigg).
\end{aligned} \tag{1}$$

No corresponding $\mathcal{N} = 4$ diagrams

similar for non-planar diagrams,
under progress

Simon Badger, Donal O'Connell, Hjalte Frellesvig and YZ

Momentum-twistor parametrization

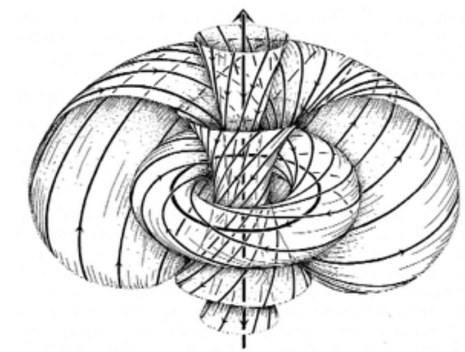
Analytic computation

Andrew Hodges

Spinor helicity formalism $(\lambda, \tilde{\lambda})$ ⟶ Momentum-twistor parametrization (λ, μ)

- momentum conservation
- Schouten identity
- Fierz identity
- ...

all constraints resolved



$$\boxed{\tilde{\lambda}_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle \langle i-1,\, i\rangle}}$$

5-point

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1}+\frac{1}{x_2} & \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_4 & 1 \\ 0 & 0 & 1 & 1 & \frac{x_5}{x_4} \end{pmatrix}$$

In the final result, it is easy to convert $\{x_1, x_2, x_3, x_4, x_5\}$ to s_{ij}, tr_5...

n-point, under progress

Conclusion

- Algebraic geometry approach to high-loop amplitudes
  - Gröbner Basis → Integrand basis
  - Primary decomposition → Global unitarity cut structure
  - Multivariate residues → maximal unitarity

Outlook

- Global residues via algebraic geometry

  works by K. Larsen, D. Kosower, M. Sogaard, YZ

- Integration-by-parts identities from the viewpoint of differential geometry

  YZ 1406.xxxx, to appear