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One-Loop Single-Real-Emission Contributions to $pp \rightarrow H + X$ at Next-to-Next-to-Next-to-Leading Order

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The Discovery of the Millennium

ATLAS and CMS have discovered a scalar boson with mass ~ 126 GeV!
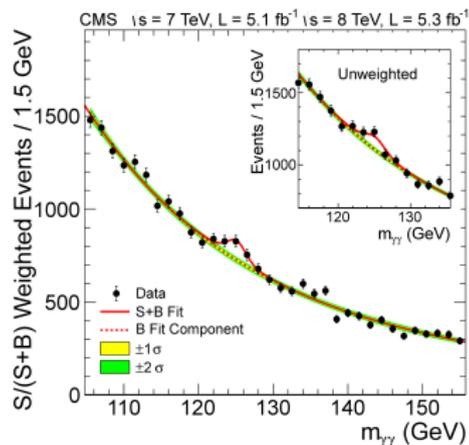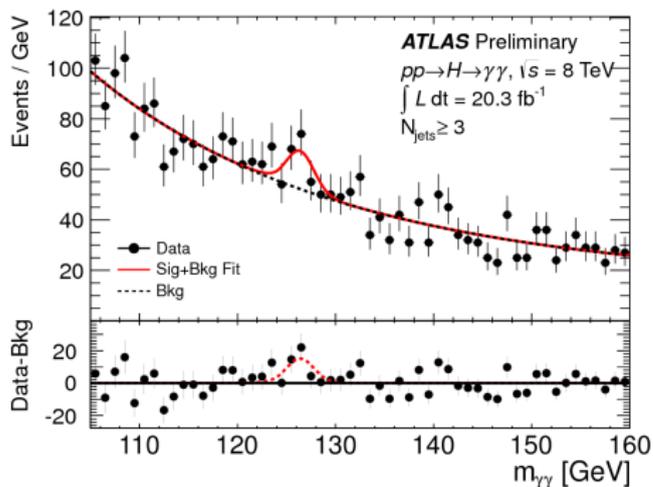
Is this *the* Higgs Boson[TM]?

It looks like a Higgs

It's still early in the LHC program and the measurements are not very precise,
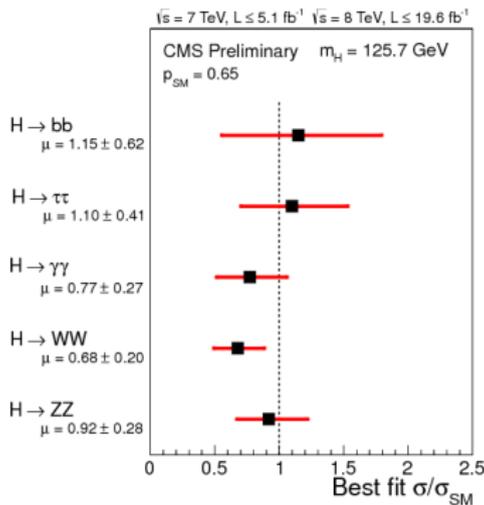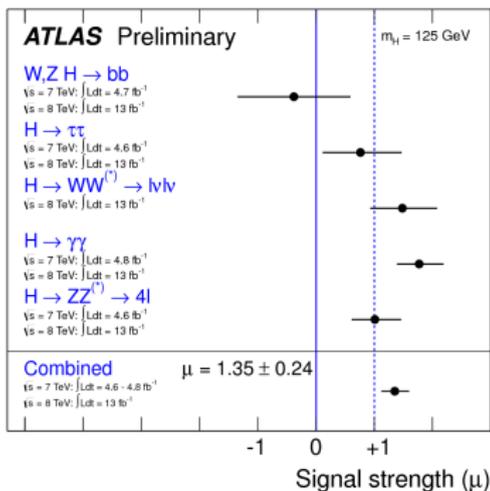
but the gross features look like the Higgs.

How can we tell?

How can we tell if this is the Higgs?

- Measure the cross section.
- Measure the couplings.
- Look for more Higgs bosons.
- Look for some other new physics.

The cross section measurement is limited by a large theoretical uncertainty. The couplings measurements will require a lot of data and will never by "precision" measurements. The high energy runs at LHC will have great reach in the search for new physics, but the most "natural" models have already been excluded.

Higgs production at the LHC

The inclusive Higgs production cross section is dominated by the gluon fusion channel.
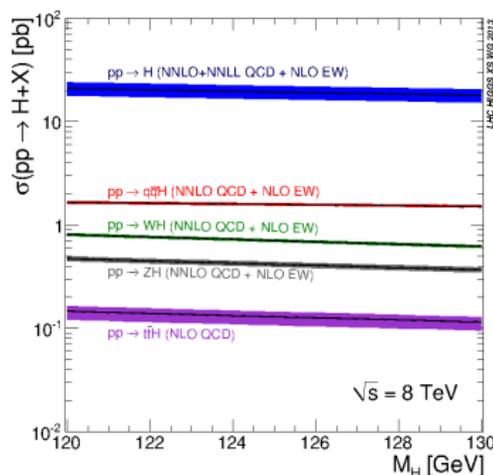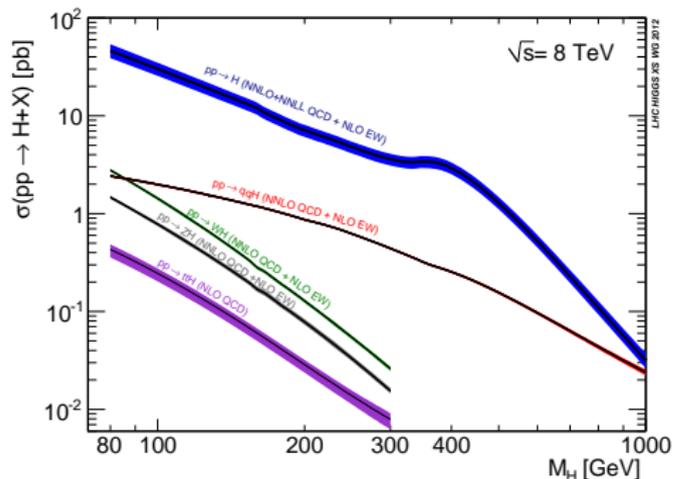
Cuts can bias the event selection toward the vector boson fusion channel, but there are also forward jets in gluon fusion so one can never get a pure sample.

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Theoretical Uncertainty

Even at NNLO, the theoretical uncertainty is very large.
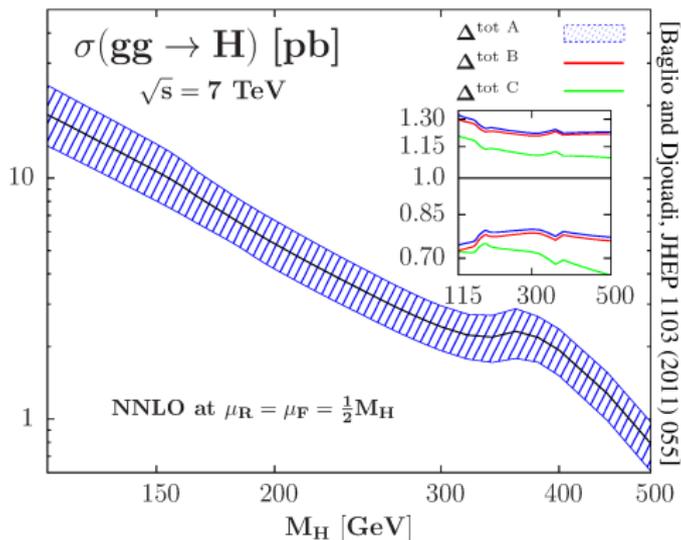
The dominant contributions are scale uncertainty of the partonic cross sections and parton distribution uncertainty.

Improving the theoretical uncertainty

How can we improve the theoretical uncertainty?

- Reduce the uncertainty in the parton distributions.
  - Collect more data to improve the fits.
  - Compute the N^3LO splitting kernels to improve parton evolution.
- Reduce the scale uncertainty in the partonic cross sections.
  - Compute the partonic cross sections at N^3LO.

In this talk, I will be concerned with improving the theoretical prediction for the total cross section.

Improving the partonic cross sections

Improving the calculation of the partonic cross sections, however, is a real challenge.

The partonic cross sections are already known at next-to-next-to-leading order. No hadronic scattering process is known to higher order and only a handful of processes are known at NNLO.

The difference is that Higgs production through gluon fusion is anomalously slow to converge and has unusually large scale dependence.

"Normal" Perturbative Convergence

The theoretical predictions for both Drell-Yan and $t\bar{t}$ production show dramatic improvement when computed to NNLO.

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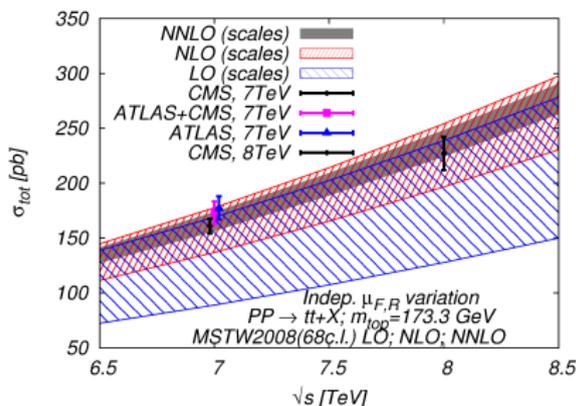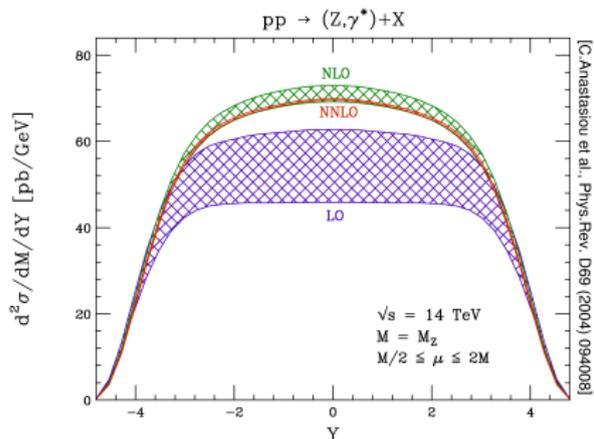
Higgs Perturbative Convergence

Higgs production through gluon fusion converges much more slowly and has large scale dependence.
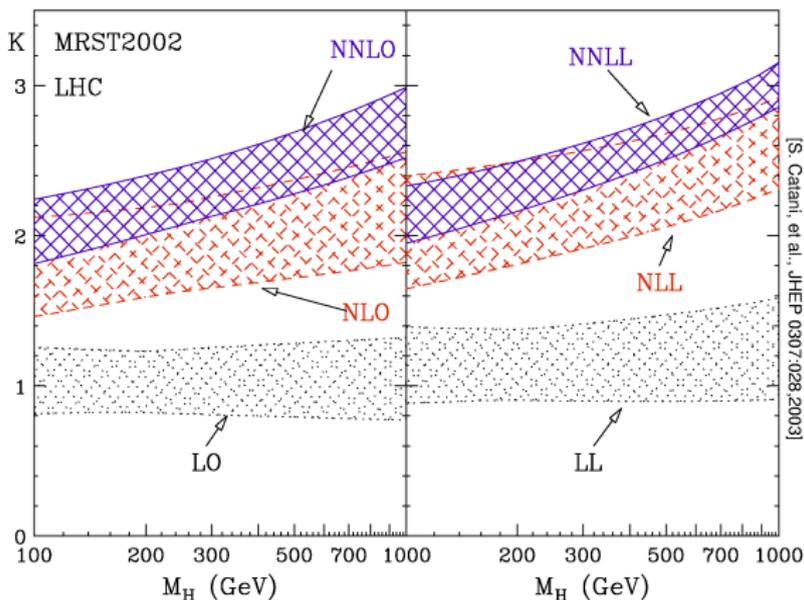
The only way to improve this situation is to push to higher order.

Contributions to Higgs Production at N^3LO

Virtual production through 3 loops,
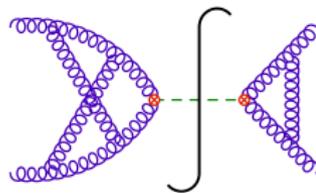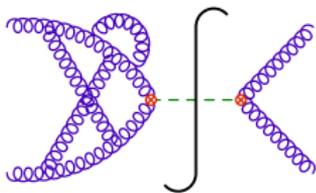
Single-real-emission through 2 loops,
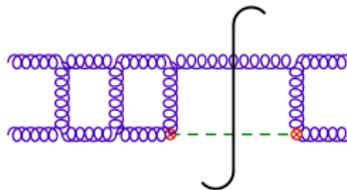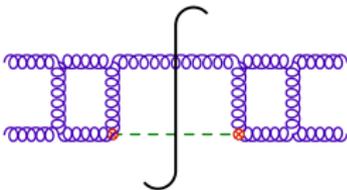
Double-real-emission through 1 loop and Triple-real-emission at tree-level.
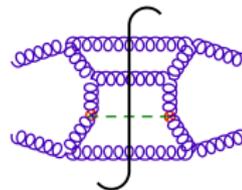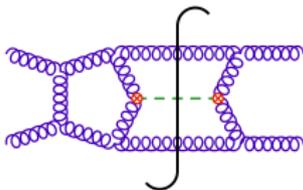
One-Loop Single-Real-Emissions Contributions

I will discuss only one set of contributions to the N^3LO cross-section: those arising from one-loop single-real-emission
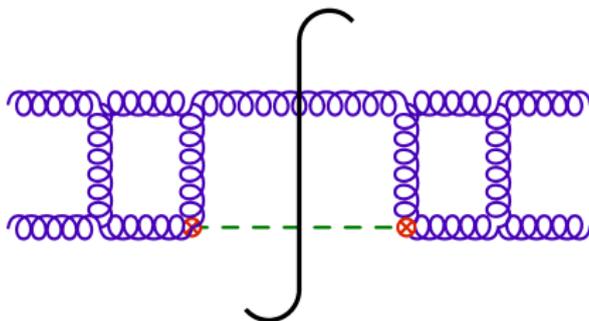

Although drawn as a cut loop integral, I will compute this contribution as a squared amplitude integrated over phase space.

- One-loop amplitudes can be computed in closed form.
- The phase-space element for single-real-emission is very simple.

This gives strong analytic control of the functions involved and allows one to attack the problem from multiple directions.

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Single-real-emission Amplitudes

The single-real-emission amplitudes ($H\,g\,g\,g$, $H\,g\,q\,\overline{q}$) can be written, at any loop order, in terms of linearly independent gauge invariant tensors.

$$\mathscr{M}(H;g_1,g_2,g_3) = \frac{g}{v}\,C_1(\alpha_s)f^{ijk}\varepsilon_{1\mu}^i\varepsilon_{2\nu}^j\varepsilon_{3\rho}^k\sum_{n=0}^{3}A_n\,\mathscr{Y}_n^{\mu\nu\rho}\,,$$

$$\mathscr{M}(H;g,q,\overline{q}) = i\frac{g}{v}\,C_1(\alpha_s)\,(T^g)_j^{\bar{i}}\varepsilon_\mu(p_g)\left(B_1\,\mathscr{X}_1^\mu + B_2\,\mathscr{X}_2^\mu\right)\,,$$

I calculate the amplitudes as follows: I generate the Feynman diagrams with QGRAF and use FORM to implement the Feynman rules and contract the diagrams with projectors onto the gauge invariant tensors. I use REDUZE2 to reduce Feynman integrals to master integrals by means of Integration-by-Parts [Chetyrkin, Tkachov] identities coupled with the LaPorta algorithm.

One-loop Master Integrals

There are only two master Feynman integrals at one loop, the one-loop bubble and the one-loop box with a single massive external leg.
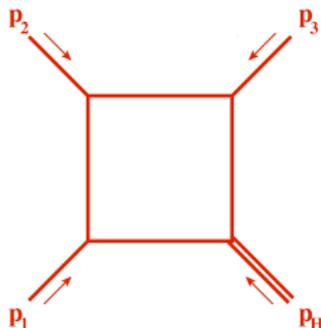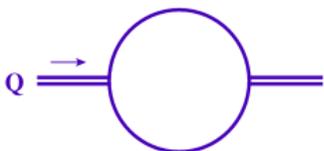
$$\mathscr{I}_2^{(1)}(Q^2) = \frac{i c_\Gamma}{\varepsilon(1-2\varepsilon)} \left(\frac{\mu^2}{-Q^2} \right)^\varepsilon$$

$$\mathscr{I}_4^{(1)}(s_{12}, s_{23}; M_H^2) = \frac{2 i c_\Gamma}{s_{12} s_{23}} \frac{1}{\varepsilon^2} \left[\left(\frac{\mu^2}{-s_{12}} \right)^\varepsilon {}_2F_1\left(1, -\varepsilon; 1-\varepsilon; -\frac{s_{31}}{s_{23}} \right) \right.$$

$$\left. + \left(\frac{\mu^2}{-s_{23}} \right)^\varepsilon {}_2F_1\left(1, -\varepsilon; 1-\varepsilon; -\frac{s_{31}}{s_{12}} \right) - \left(\frac{\mu^2}{-M_H^2} \right)^\varepsilon {}_2F_1\left(1, -\varepsilon; 1-\varepsilon; -\frac{M_H^2 s_{31}}{s_{12} s_{23}} \right) \right]$$

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Computing the Partonic Cross Sections

With the preliminaries out of the way, I will now describe the computation of the partonic cross section. This proceeds as follows:

- Squaring the amplitude and integrating over phase space
- Use Integration-by-Parts to map the full set of phase space integrals onto a small set of master phase space integrals
- Evaluate the master integrals.

The evaluation of the master integrals is greatly simplified by taking advantage of the nice mathematical properties of the Hypergeometric functions.

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Phase Space Integration

The partonic cross section is computed by squaring the production amplitudes, averaging (summing) over initial- (final-) state colors and spins, and integrating over phase space.

$$\sigma = \frac{1}{16\pi s_{12}} \frac{1}{\mathbb{S}} \sum_{\text{spin/color}} \int \left(\frac{4\pi\mu^2}{s_{12}}\right)^\varepsilon \frac{(s_{23}\,s_{31})^\varepsilon}{\Gamma(1-\varepsilon)} |\mathcal{M}|^2 \, ds_{23}$$

Defining $\hat{s} = s_{12}$, I introduce dimensionless parameters $x = M_H^2/\hat{s}$, $\bar{x} = 1-x$, $y = \frac{1}{2}(1-\cos\theta^*)$, $\bar{y} = 1-y$, to get

$$s_{12} = \hat{s}, \qquad\qquad\qquad M_H^2 = x\hat{s},$$
$$s_{23} = \bar{x}\,y\hat{s}, \qquad\qquad\qquad s_{31} = \bar{x}\,\bar{y}\,\hat{s},$$

$$\sigma = \frac{1}{16\pi\hat{s}} \frac{1}{\mathbb{S}} \sum_{\text{spin/color}} \left(\frac{4\pi\mu^2}{\hat{s}}\right)^\varepsilon \frac{\bar{x}^{1-2\varepsilon}}{\Gamma(1-\varepsilon)} \int_0^1 dy \, y^{-\varepsilon} \bar{y}^{-\varepsilon} |\mathcal{M}|^2.$$

Integration-by-Parts for Phase Space

Computing the partonic cross section involves a large number of phase space integrals. To simplify the problem, I borrow the idea of using IBP identities. For example,

$$\int_0^1 dy \, \frac{d}{dy} \left[y^{-\varepsilon} \bar{y}^{-2\varepsilon} {}_2F_1 \left(1, -\varepsilon; 1 - \varepsilon; \bar{x} \, y \right) \right] = 0$$

$$= (1 - 2\varepsilon) \int_0^1 dy \, y^{-\varepsilon} \bar{y}^{-2\varepsilon} {}_2F_1 \left(1, -\varepsilon; 1 - \varepsilon; \bar{x} \, y \right)$$

$$+ 2\varepsilon \int_0^1 dy \, y^{-\varepsilon} \bar{y}^{-1-2\varepsilon} {}_2F_1 \left(1, -\varepsilon; 1 - \varepsilon; \bar{x} \, y \right)$$

$$- \varepsilon \int_0^1 dy \, y^{-\varepsilon} \bar{y}^{-2\varepsilon} \left(1 - \bar{x} \, y \right)^{-1}$$

I get non-trivial relations among the phase space integrals allowing me to map the full calculation onto a small number of master phase space integrals.

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Harmonic Polylogarithms

The class of functions called harmonic polylogarithms (HPLs) [Goncharov; Vermaseren,Remiddi] are ubiquitous in the solution of Feynman integrals. The standard HPLs are defined in terms of weight functions, f_{+1}, f_0, and f_{-1}:

$$f_{+1}(x) = \frac{1}{1-x}, \qquad f_0(x) = \frac{1}{x}, \qquad f_{-1}(x) = \frac{1}{1+x}$$

The rank 1 HPLs are defined by

$$H(0;x) = \ln x, \qquad H(\pm 1;x) = \int_0^x dz\, f_{\pm 1}(z).$$

Higher ranks are defined by iterated integrations against weight functions:

$$H(w_n, w_{n-1}, \ldots, w_0; x) = \int_0^x dz\, f_{w_n}(z)\, H(w_{n-1}, \ldots, w_0; z),$$

The HPLs include the classic polylogarithms, $\text{Li}_n(x)$ as special cases. For example, $\text{Li}_1(x) = H(1;x)$, $\text{Li}_2(x) = H(0,1;x)$, $\text{Li}_3(x) = H(0,0,1;x)$, etc.

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More HPLs

An important property is that the HPLs form a shuffle algebra, so that

$$H(\vec{w}_1; x) H(\vec{w}_2; x) = \sum_{\vec{w}' \in \vec{w}_1 \, \amalg \, \vec{w}_2} H(\vec{w}'; x),$$

where $\vec{w}_1 \amalg \vec{w}_2$ is the set of shuffles, or mergers of the sequences $\vec{w}_1$ and $\vec{w}_2$ that preserve their relative orderings.

It is also possible to "unshuffle" in order to separate out $H(\vec{0}; x)$ or $H(\vec{1}; x)$, thereby exposing the logarithmic dependence on x or $1 - x$, respectively.

To see that HPLs are ubiquitous in the solution of Feynman Integrals, consider the 1-loop master integrals, which involved functions like

$$_2F_1\left(1, -\varepsilon; 1 - \varepsilon; z\right) = 1 - \sum_{n=1}^{\infty} \varepsilon^n \, \mathrm{Li}_n(z) = 1 - \sum_{n=1}^{\infty} \varepsilon^n \, H(n; z),$$

$$z^{-\varepsilon} = \sum_{n=0}^{\infty} \frac{(-\varepsilon)^n}{n!} \ln^n(z) = \sum_{n=0}^{\infty} (-\varepsilon)^n \, H(\vec{0}_n; z).$$

Master Phase Space Integrals

After the application of IBP identities, I find that there are 24 master integrals at N³LO. Of these, two involve only y and $\bar{y}$. Since they do not involve the external scale, x, I call these scale-free integrals.

$$M_{10} = -\varepsilon \int_0^1 dy \, y^{-1-\varepsilon} \, \bar{y}^{-\varepsilon} \, {}_2F_1\left(1, -\varepsilon; 1-\varepsilon; -\frac{y}{\bar{y}}\right)^2,$$

$$M_{11} = -2\,\varepsilon \int_0^1 dy \, y^{-1-\varepsilon} \, \bar{y}^{-\varepsilon} \, {}_2F_1\left(1, -\varepsilon; 1-\varepsilon; -\frac{y}{\bar{y}}\right) {}_2F_1\left(1, -\varepsilon; 1-\varepsilon; -\frac{\bar{y}}{y}\right).$$

While they cannot be evaluated in closed form, they can be readily evaluated to arbitrary order in ε by expanding the hypergeometric functions in terms of HPLs and making use of the shuffle identity.

Master integrals at N^3LO

The rest of the master integrals depend on the physical scale parameter x. These integrals, which often involve the products of hypergeometric functions, generally cannot be evaluated in closed form. Instead I will evaluate these integrals by performing an extended threshold expansion.

Most terms in the integrands can be expanded using the series representation of the hypergeometric function

$$_2F_1\left(1, -\varepsilon; 1-\varepsilon; z\right) = 1 - \varepsilon \sum_{n=1}^{\infty} \frac{z^n}{n-\varepsilon} \, ,$$

Other terms can make use of the Taylor Series expansion,

$$_2F_1\left(1, -\varepsilon; 1-\varepsilon; -x\frac{y}{\bar{y}}\right) = \sum_{n=0}^{\infty} \frac{\bar{x}^n}{n!} \left[\frac{d^n}{d\bar{x}^n} {}_2F_1\left(1, -\varepsilon; 1-\varepsilon; (\bar{x}-1)\frac{y}{\bar{y}}\right)\right]_{\bar{x}=0} \, .$$

The result of these expansions is a sum of powers of $\bar{x}$ multiplying scale-free master integrals.

Threshold Expansion

Once the scale-free integrals have been evaluated, one is left with a pure power series in $\bar{x}$. For phenomenological purposes, only a few terms are needed.
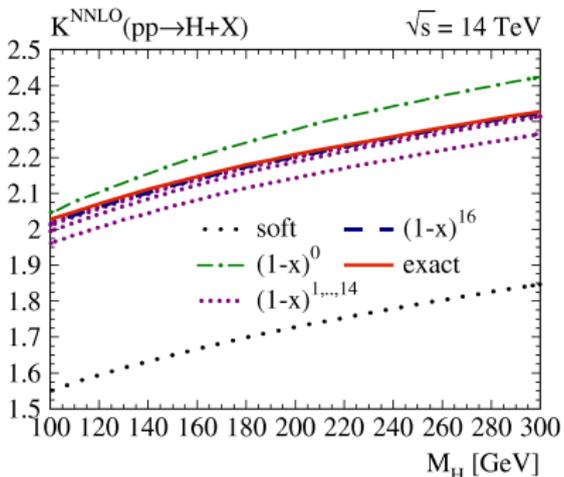
But with enough terms, one can do much better.

Finding a Basis

One expects that the result of the calculation can be expressed in closed form in terms of Harmonic Polylogarithms, and it is simple to expand the HPLs in powers of $\bar{x}$.

$$H(\vec{w}; \bar{x}) = \sum_{i=0}^{\infty} \bar{x}^i \, Z_i(\vec{w}) \,.$$

Therefore, one should be able to create a mapping between the threshold expansions for the master integrals and a linear combination of HPLs.

Once a basis of functions is chosen, I form a matrix $\mathbb{M}$ of coefficients, with each column corresponding to a different function, and each row to a different order in $\bar{x}$, and I invert $\mathbb{M}$. The solution to the integral $I(\bar{x})$ is then found to be

$$I(\bar{x}) = \vec{f} \cdot \mathbb{M}^{-1} \cdot \vec{\iota} \,,$$

where $\vec{f}$ is a row-vector of the basis functions, and $\vec{\iota}$ is a column-vector consisting of the threshold expansion coefficients of the integral $I(\bar{x})$.

Series Inversion

All of the expansion coefficients are computed analytically as rational numbers. When the proper basis is used for $\mathbb{M}$, the solution $I(\bar{x})$ contains only simple, rational coefficients. The size of the basis determines the number of terms needed in the threshold expansion. For the low-rank inversions (higher poles), the number of terms needed is quite small.

Since the inversion is being performed using a fixed basis, one always gets a result. A "good" result consists of simple, rational coefficients and usually does not saturate the basis. A "bad" result usually does saturate the basis and consists of unwieldy rational numbers (e.g. ratios of > 100 digit integers). A bad result calls for expanding the basis.

One can verify the inversion by checking expansion coefficients that were not used in the inversion. If I use 32 coefficients in the inversion and then verify the 33rd-37th terms, I have a very strong check on the result.

Results

The result is not physical and contains infrared divergences. Here is the "soft" part of the calculation

$$
\sigma^{3,B,\text{soft}}_{gg \to Hg} = \frac{C_1^2\, \pi}{64\, v^2}\, C_A^3 \left(\frac{g^2 (4\pi)^\varepsilon}{4\pi^2 \exp(\varepsilon\, \gamma_E)} \right)^3 \left(\frac{\mu^2}{M_H^2} \right)^{3\varepsilon} \left\{ \frac{1}{\varepsilon^6}\, \frac{23}{72}\, \delta(\bar{x}) + \frac{1}{\varepsilon^5} \left[\frac{23}{72}\, \delta(\bar{x}) - \frac{19}{24}\, \mathscr{D}_0(\bar{x}) \right] \frac{1}{\varepsilon^4} \left[\delta(\bar{x}) \left(\frac{23}{72} - \frac{247}{144}\, \zeta(2) \right) \right. \right.
$$

$$
+ -\frac{19}{24}\, \mathscr{D}_0(\bar{x}) + \frac{9}{4}\, \mathscr{D}_1(\bar{x}) \Big] + \frac{1}{\varepsilon^3} \left[\delta(\bar{x}) \left(\frac{127}{144} - \frac{247}{144}\, \zeta(2) - \frac{125}{36}\, \zeta(3) \right) + \mathscr{D}_0(\bar{x}) \left(-\frac{19}{24} + \frac{275}{48}\, \zeta(2) \right) + \frac{9}{4}\, \mathscr{D}_1(\bar{x}) - \frac{15}{4}\, \mathscr{D}_2(\bar{x}) \right]
$$

$$
+ \frac{1}{\varepsilon^2} \left[\delta(\bar{x}) \left(\frac{185}{72} - \frac{247}{144}\, \zeta(2) - \frac{125}{36}\, \zeta(3) + \frac{3029}{384}\, \zeta(4) \right) + \mathscr{D}_0(\bar{x}) \left(-\frac{49}{24} + \frac{275}{48}\, \zeta(2) + \frac{269}{24}\, \zeta(3) \right) + \mathscr{D}_1(\bar{x}) \left(\frac{9}{4} - \frac{169}{8}\, \zeta(2) \right) \right.
$$

$$
\left. -\frac{15}{4}\, \mathscr{D}_2(\bar{x}) + \frac{29}{6}\, \mathscr{D}_3(\bar{x}) \right] + \frac{1}{\varepsilon} \left[\delta(\bar{x}) \left(\frac{937}{144} - \frac{1151}{288}\, \zeta(2) - \frac{125}{36}\, \zeta(3) + \frac{3029}{384}\, \zeta(4) - \frac{553}{20}\, \zeta(5) + \frac{2125}{72}\, \zeta(2)\, \zeta(3) \right) \right.
$$

$$
+ \mathscr{D}_0(\bar{x}) \left(-\frac{139}{24} + \frac{275}{48}\, \zeta(2) + \frac{269}{24}\, \zeta(3) - \frac{3841}{128}\, \zeta(4) \right) + \mathscr{D}_1(\bar{x}) \left(\frac{21}{4} - \frac{169}{8}\, \zeta(2) - \frac{171}{4}\, \zeta(3) \right) + \mathscr{D}_2(\bar{x}) \left(-\frac{15}{4} + \frac{335}{8}\, \zeta(2) \right)
$$

$$
\left. + \frac{29}{6}\, \mathscr{D}_3(\bar{x}) - \frac{21}{4}\, \mathscr{D}_4(\bar{x}) \right] \delta(\bar{x}) \left(\frac{547}{36} - \frac{1561}{144}\, \zeta(2) - \frac{1193}{144}\, \zeta(3) + \frac{3029}{384}\, \zeta(4) - \frac{553}{20}\, \zeta(5) + \frac{2125}{72}\, \zeta(2)\, \zeta(3) - \frac{84281}{3072}\, \zeta(6) \right.
$$

$$
\left. + \frac{4607}{144}\, \zeta(3)^2 \right) + \mathscr{D}_0(\bar{x}) \left(-\frac{349}{24} + \frac{593}{48}\, \zeta(2) + \frac{269}{24}\, \zeta(3) - \frac{3841}{128}\, \zeta(4) + \frac{4869}{40}\, \zeta(5) - \frac{5581}{48}\, \zeta(2)\, \zeta(3) \right)
$$

$$
+ \mathscr{D}_1(\bar{x}) \left(\frac{57}{4} - \frac{169}{8}\, \zeta(2) - \frac{171}{4}\, \zeta(3) + \frac{6777}{64}\, \zeta(4) \right) + \mathscr{D}_2(\bar{x}) \left(-\frac{31}{4} + \frac{335}{8}\, \zeta(2) + \frac{373}{4}\, \zeta(3) \right)
$$

$$
+ \mathscr{D}_3(\bar{x}) \left(\frac{29}{6} - \frac{701}{12}\, \zeta(2) \right) - \frac{21}{4}\, \mathscr{D}_4(\bar{x}) + \frac{149}{30}\, \mathscr{D}_5(\bar{x}) + \mathscr{O}(\varepsilon) \right\}
$$

Summary

- The discovery of the Higgs demands that we verify it in every possible way.
- The large theoretical uncertainty in the total cross section calls for an improved calculation at N^3LO.
- This is an enormous calculation and I have completed only one significant part, that due to one-loop single-real-emission amplitudes.
- The techniques developed for this calculation can be applied to the remaining parts of the N^3LO calculation.