Recent progress in HELAC for NLO and NNLO calculations

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Dyson-Schwinger Recursive Equations

From Feynman Diagrams to recursive equations: taming the n!

- 1999 HELAC: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles
- For QCD colour connection representation: revival of the 't Hooft ideas ('71) in the modern era.

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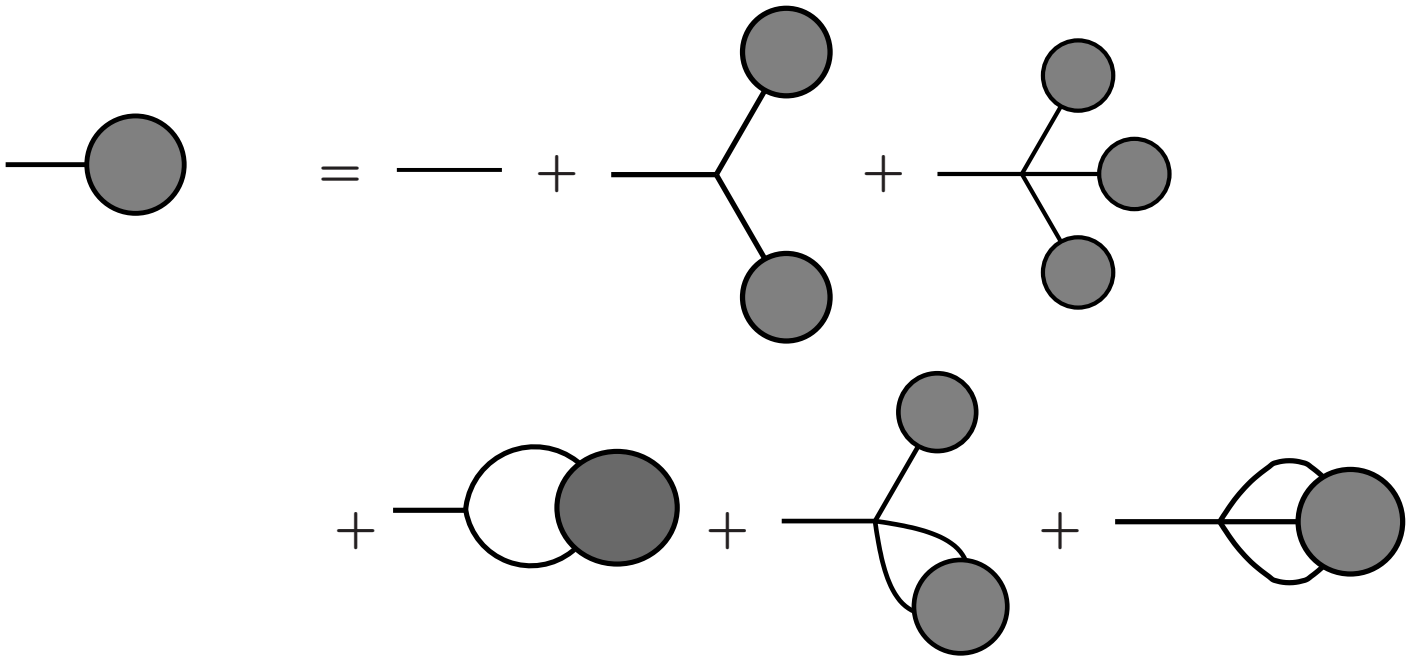
A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].

F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.

F. Caravaglios and M. Moretti, Phys. Lett. B **358** (1995) 332.



Unfortunately not so much on the second line !

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HELAC Color treatment

-• Colour flow or colour connection representation

$$\mathcal{M}^{a_1,i_2,\ldots,i_k}_{j_2,\ldots,j_k} t^{a_1}_{i_1 j_1} \to \mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}$$

$$\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k} = \sum_{\sigma} \delta_{i_{\sigma_1},j_1}\delta_{i_{\sigma_2},j_2}\ldots\delta_{i_{\sigma_k},j_k} A_\sigma \to n!$$

gluons $\to (i,j)$, quark $\to (i,0)$, anti-quark $\to (0,j)$, other $\to (0,0)$

$$\sum_{\sigma,\sigma'} A^*_\sigma \mathcal{C}_{\sigma,\sigma'} A_{\sigma'}$$

$$\mathcal{C}_{\sigma,\sigma\prime} \equiv \sum_{\{i\},\{j\}} \delta_{i_{\sigma_1},j_1}\delta_{i_{\sigma_2},j_2}\ldots\delta_{i_{\sigma_k},j_k}\delta_{i_{\sigma'_1},j_1}\delta_{i_{\sigma'_2},j_2}\ldots\delta_{i_{\sigma'_k},j_k} = N_c^{m(\sigma,\sigma\prime)}$$

HELAC Color treatment

- Colour configuration representation (Monte Carlo integration)

$$\sum_{\{i\},\{j\}} |\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}|^2 \to \beta^n$$

Partial solution $n < 6 - 7$

$$\mathcal{M}^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k} = \sum A_\sigma$$

Perturbative QCD at NLO

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m |M_m^{(0)}|^2 J_m(\Phi) \\
&+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}$$

$J_m(\Phi)$ jet function: Infrared safeness $J_{m+1} \to J_m$

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&+ \int_m d\Phi_m^{D=4} 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR}))J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1}^{D=4-2\epsilon_{IR}} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}$$

IR and UV divergencies, Four-Dimensional-Helicity scheme; scale dependence μ_R
QCD factorization$-\mu_F$ Collinear counter-terms when PDF are involved

NLO troubles

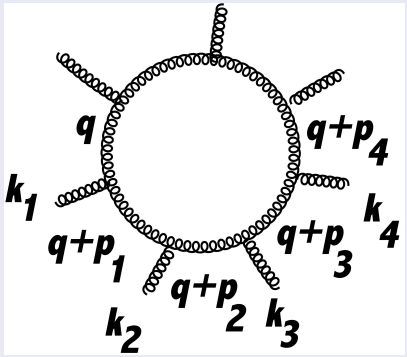
Problems arising in NLO calculations

- Large Number of Feynman diagrams
- Reduction to Scalar Integrals (or sets of known integrals)
- Numerical Instabilities (inverse Gram determinants, spurious phase-space singularities)
- Extraction of soft and collinear singularities (we need virtual and real corrections)

Get numerical predictions numerically !

One-loop amplitudes

Any m-point one-loop amplitude can be written as



$$\int d^D q\, A(\bar{q}) = \int d^D q\, \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

A bar denotes objects living in $n = 4 + \epsilon$ dimensions

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2$$

$$\bar{q}^2 = q^2 + \tilde{q}^2$$

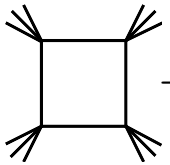
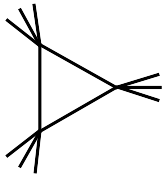
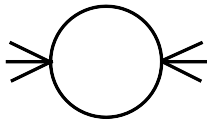
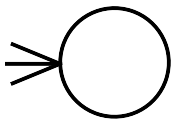
$$\bar{D}_i = D_i + \tilde{q}^2$$

The one loop paradigm

basis of scalar integrals:

G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217 [arXiv:hep-ph/9403226].

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \rightarrow$ cut-constructible part $\qquad R \rightarrow$ rational terms

$$\mathcal{A} = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

The old "master" formula

$$
\begin{aligned}
\int A &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3) \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) C_0(i_0 i_1 i_2) \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) B_0(i_0 i_1) \\
&+ \sum_{i_0}^{m-1} a(i_0) A_0(i_0) \\
&+ \text{rational terms}
\end{aligned}
$$

D_0, C_0, B_0, A_0, scalar one-loop integrals: 't Hooft and Veltman
`QCDLOOP` Ellis & Zanderighi ; `OneLOop` A. van Hameren

The old "master" formula

$$
\begin{aligned}
\int \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} a(i_0) \int \frac{1}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

Remove the integration !

The new "master" formula

$$
\begin{aligned}
\frac{N(q)}{\bar{D}_0\bar{D}_1\cdots\bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3)+\tilde{d}(q;i_0 i_1 i_2 i_3)}{\bar{D}_{i0}\bar{D}_{i1}\bar{D}_{i2}\bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2)+\tilde{c}(q;i_0 i_1 i_2)}{\bar{D}_{i0}\bar{D}_{i1}\bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1)+\tilde{b}(q;i_0 i_1)}{\bar{D}_{i0}\bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0)+\tilde{a}(q;i_0)}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

OPP "MASTER" FORMULA

Equation in a from "solvable" à la "unitarity"

General expression for the 4-dim $N(q)$ at the integrand level in terms of D_i

$$
\begin{aligned}
N(q) \quad = \quad & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
+ \quad & \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
+ \quad & \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
+ \quad & \sum_{i_0}^{m-1} \left[a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

A next to simple example

- Not only tensor integrals need reduction!

$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}}$$

$$1 = \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \ldots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}} \sum \left[d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \ldots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left(\prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

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Rational Terms

Numerically treat $D = 4 - 2\epsilon$, means $4 \oplus 1$

Expand in D-dimensions ?

$$\bar{D}_i = D_i + \tilde{q}^2$$

$$\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2) \right] \prod^{m-1} \bar{D}_i + \tilde{P}(q) \prod^{m-1} \bar{D}_i
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\end{aligned}
$$

$$m_i^2 \to m_i^2 - \tilde{q}^2$$

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\end{aligned}
$$

$$
m_i^2 \to m_i^2 - \tilde{q}^2
$$

Rational Terms

In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$.

$$\begin{aligned} \mathrm{R}_1 &= -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\ &- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) . \end{aligned}$$

G. Ossola, C. G. Papadopoulos and R. Pittau,arXiv:0802.1876 [hep-ph]

Rational Terms - R_2

A different source of Rational Terms, called R_2, can also be generated from the ϵ-dimensional part of $N(q)$

$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, \epsilon; q)$$

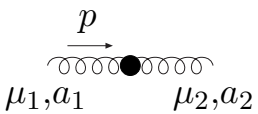
$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q} \frac{\tilde{N}(\tilde{q}^2, \epsilon; q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\,\mathcal{R}_2$$

$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

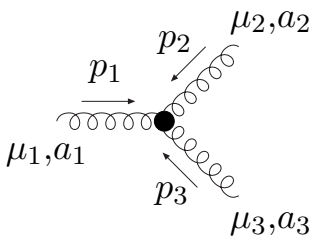
New vertices/particles or GKMZ-approach

HELAC R2 terms

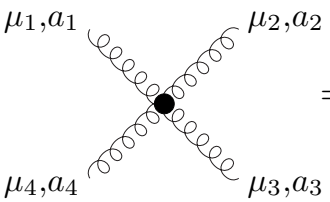
Contribution from $d-$dimensional parts in numerators:



$$= \frac{ig^2 N_{col}}{48\pi^2}\,\delta_{a_1 a_2}\left[\frac{p^2}{2} g_{\mu_1\mu_2} + \lambda_{HV}\left(g_{\mu_1\mu_2}p^2 - p_{\mu_1}p_{\mu_2}\right) + \frac{N_f}{N_{col}}\left(p^2 - 6\,m_q^2\right) g_{\mu_1\mu_2}\right]$$



$$= -\frac{g^3 N_{col}}{48\pi^2}\left(\frac{7}{4} + \lambda_{HV} + 2\frac{N_f}{N_{col}}\right) f^{a_1 a_2 a_3}\, V_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3)$$

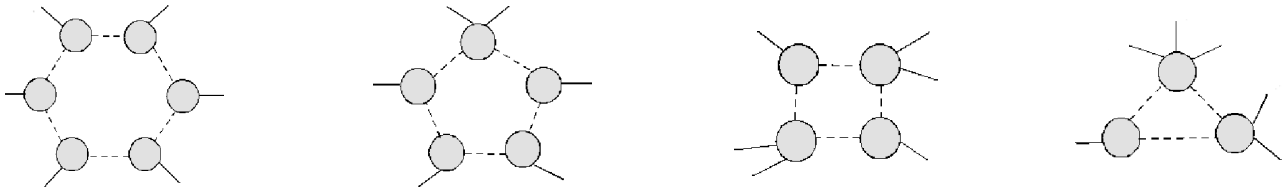


$$= -\frac{ig^4 N_{col}}{96\pi^2}\sum_{P(234)}\left\{\left[\frac{\delta_{a_1a_2}\delta_{a_3a_4} + \delta_{a_1a_3}\delta_{a_4a_2} + \delta_{a_1a_4}\delta_{a_2a_3}}{N_{col}} + 4\,Tr(t^{a_1}t^{a_3}t^{a_2}t^{a_4} + t^{a_1}t^{a_4}t^{a_2}t^{a_3})\,(3+\lambda_{HV}) - Tr(\{t^{a_1}t^{a_2}\}\{t^{a_3}t^{a_4}\})\,(5+2\lambda_{HV})\right] g_{\mu_1\mu_2}g_{\mu_3\mu_4} + 12\frac{N_f}{N_{col}} Tr(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\left(\frac{5}{3}g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_2\mu_3}g_{\mu_1\mu_4}\right)\right\}$$

The one-loop calculation in a nutshell

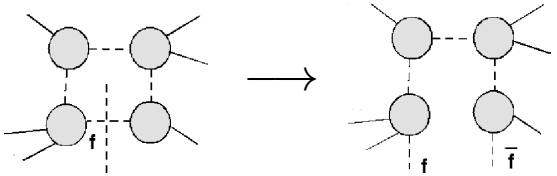
The computation of $pp(p\bar{p}) \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, `HELAC` evaluates numerically the numerators $N_i^6(q), N_i^5(q), \ldots$ with the values of the loop momentum q provided by `CutTools`

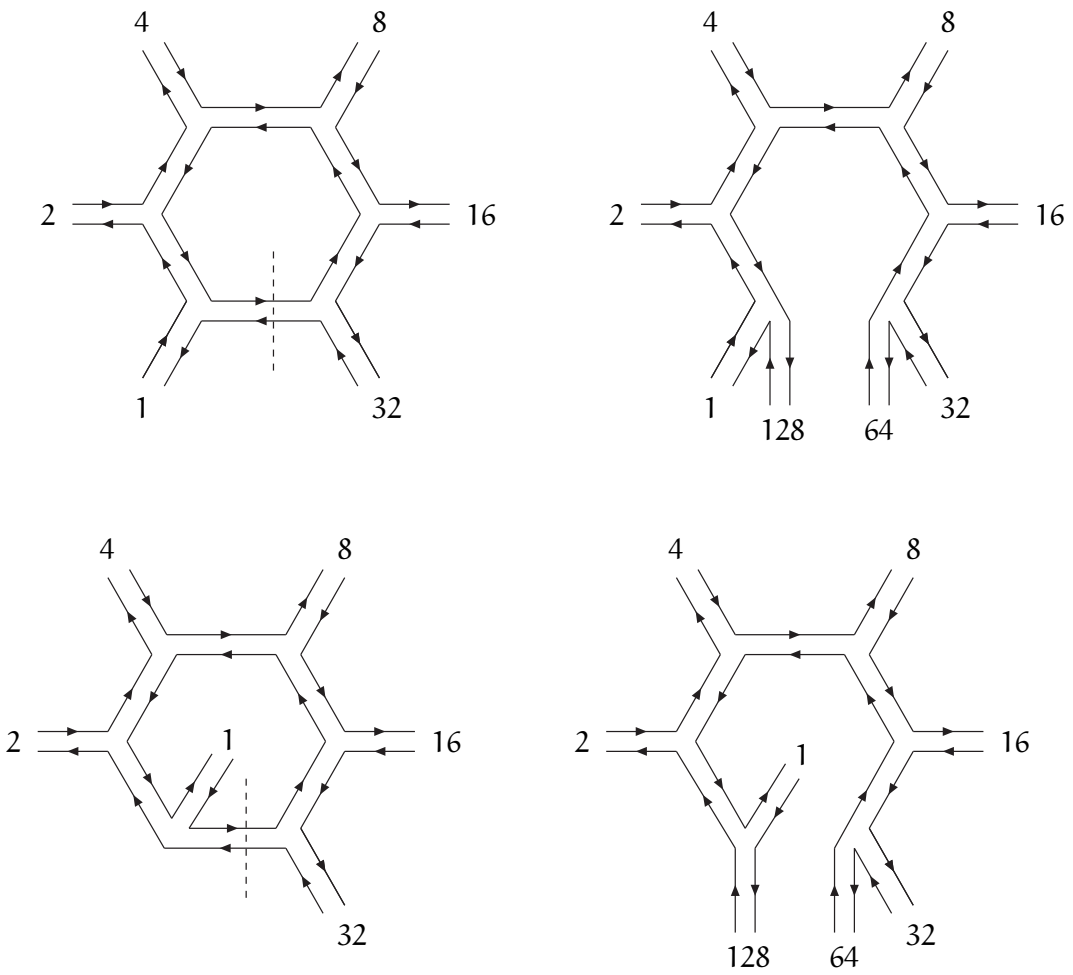
- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop (q is fixed) to get a $n+2$ *tree-like* process



The R_2 contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

Color treatment

HELAC is using color-connection representation of amplitudes + color-flow Feynman rules (Kanaki & Papadopoulos) - valid also at one loop

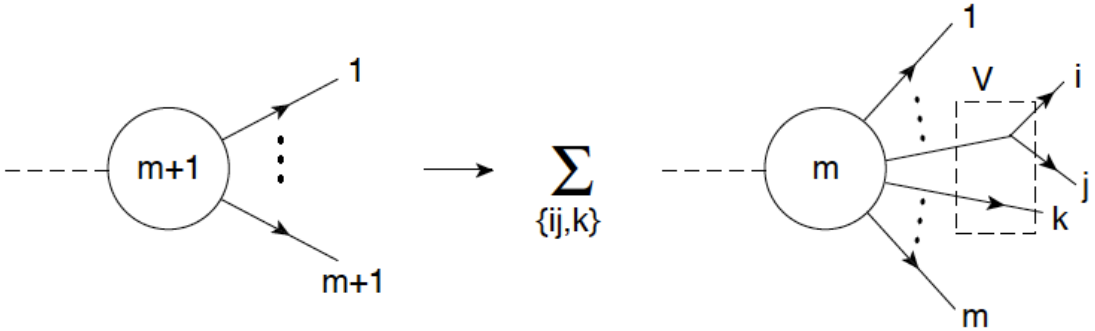


Real corrections

Real corrections: $D \to 4$ dimensions (Catani & Seymour)

$$\int_{m+1} d\sigma^R + \int_m d\sigma^V$$

$$\sigma^{NLO} = \int_{m+1} \left[\left(d\sigma^R\right)_{\epsilon=0} - \left(d\sigma^A\right)_{\epsilon=0} \right] + \int_m \left[d\sigma^V + \int_1 d\sigma^A \right]_{\epsilon=0}$$



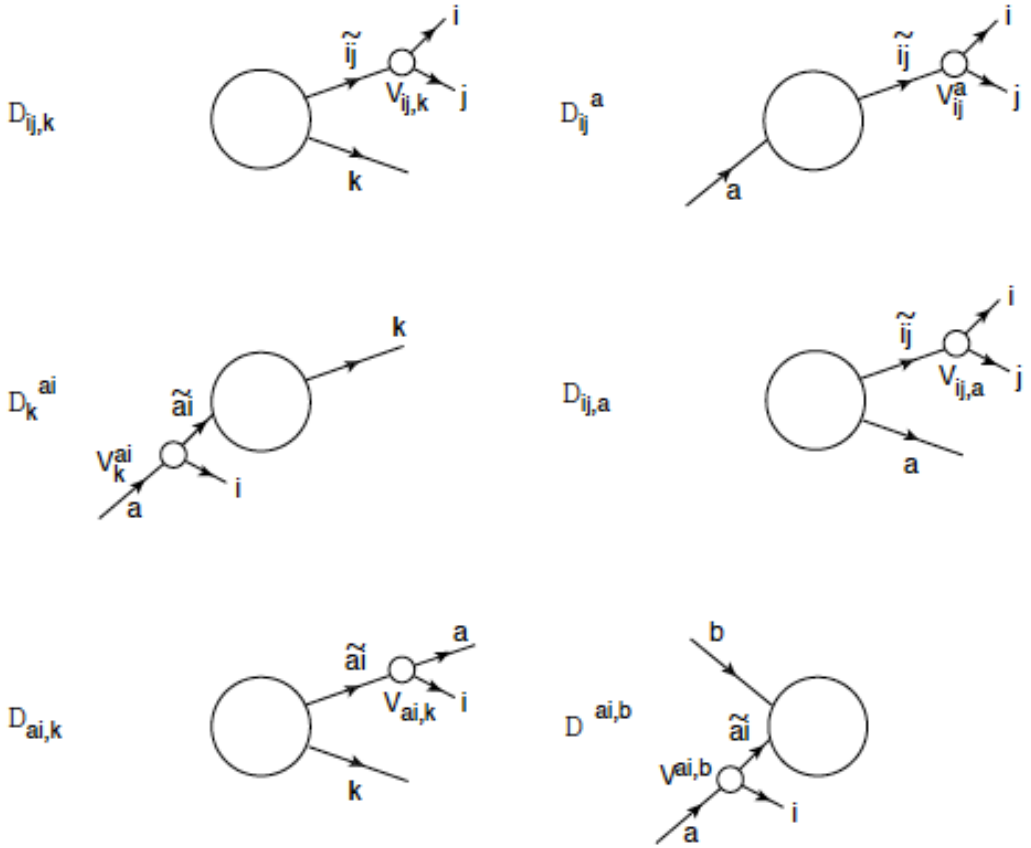
$$\tilde{p}_k^\mu = \frac{1}{1 - y_{ij,k}} p_k^\mu \ , \quad \tilde{p}_{ij}^\mu = p_i^\mu + p_j^\mu - \frac{y_{ij,k}}{1 - y_{ij,k}} p_k^\mu$$

$$d\phi(p_i, p_j, p_k; Q) = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \frac{d^d p_j}{(2\pi)^{d-1}} \delta_+(p_j^2) \frac{d^d p_k}{(2\pi)^{d-1}} \delta_+(p_k^2) \ (2\pi)^d \delta^{(d)}(Q - p_i - p_j - p_k)$$

$$d\phi(p_i, p_j, p_k; Q) = d\phi(\tilde{p}_{ij}, \tilde{p}_k; Q) \ [dp_i(\tilde{p}_{ij}, \tilde{p}_k)]$$

Real corrections

Dipoles in real life



Real corrections

Dipoles in real life: the formulae

$$
\begin{aligned}
d\sigma^A &= \mathcal{N}_{in} \sum_{\{m+1\}} d\phi_{m+1}(p_1, ..., p_{m+1}; Q) \, \frac{1}{S_{\{m+1\}}} \\
&\cdot \sum_{\substack{\text{pairs} \\ \text{i,j}}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_1, ..., p_{m+1}) \, F_J^{(m)}(p_1, ..\widetilde{p}_{ij}, \widetilde{p}_k, .., p_{m+1})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_{ij,k}\,(p_1, ..., p_{m+1}) &= -\frac{1}{2p_i \cdot p_j} \\
\cdot \quad {}_m\!< 1, .., \widetilde{ij}, .., \widetilde{k}, .., m+1| \, \frac{\boldsymbol{T}_k \cdot \boldsymbol{T}_{ij}}{\boldsymbol{T}_{ij}^2} \, \boldsymbol{V}_{ij,k} \, |1, .., \widetilde{ij}, .., \widetilde{k}, .., m+1 >_m
\end{aligned}
$$

$$
\begin{aligned}
d\sigma^R - d\sigma^A &= \mathcal{N}_{in} \sum_{\{m+1\}} d\phi_{m+1}(p_1, ..., p_{m+1}; Q) \, \frac{1}{S_{\{m+1\}}} \\
&\cdot \Big\{ |\mathcal{M}_{m+1}(p_1, ..., p_{m+1})|^2 \, F_J^{(m+1)}(p_1, ..., p_{m+1}) \\
&- \sum_{\substack{\text{pairs} \\ \text{i,j}}} \sum_{k \neq i,j} \mathcal{D}_{ij,k}(p_1, ..., p_{m+1}) \, F_J^{(m)}(p_1, ..\widetilde{p}_{ij}, \widetilde{p}_k, .., p_{m+1}) \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\int_{m+1} d\sigma^A &= -\int_m \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1, .., p_m; Q) \, \frac{1}{S_{\{m\}}} \, F_J^{(m)}(p_1, ..., p_m) \\
&\cdot \sum_i \sum_{k \neq i} |\mathcal{M}_m^{i,k}(p_1, ..., p_m)|^2 \, \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{2p_i \cdot p_k} \right)^{\epsilon} \, \frac{1}{\boldsymbol{T}_i^2} \, \mathcal{V}_i(\epsilon) \quad ,
\end{aligned}
$$

HELAC-NLO: Recent developments

Complete implementation of Nagy-Soper subtraction for NLO calculations in QCD

- ❑ **HELAC-DIPOLES** package extended
- ❑ Massive and massless cases included
- ❑ Integrated subtraction terms for massless and massive partons worked out
- ❑ Overall performance tested
- ❑ Comparison to results based on Catani-Seymour subtraction performed
- ❑ Random polarization and color sampling implemented and tested

G. Bevilacqua, M. Czakon, M. Kubocz and M. Worek (2013)

❑ Used for NLO QCD calculation for 4b-jets with massive bottom quarks at the LHC

G. Bevilacqua, M. Czakon, M. Krämer, M. Kubocz and M. Worek (2013)

❑ Based on a novel parton shower with an improved treatment of color and spin

Z. Nagy, D. Soper (2007-2014)

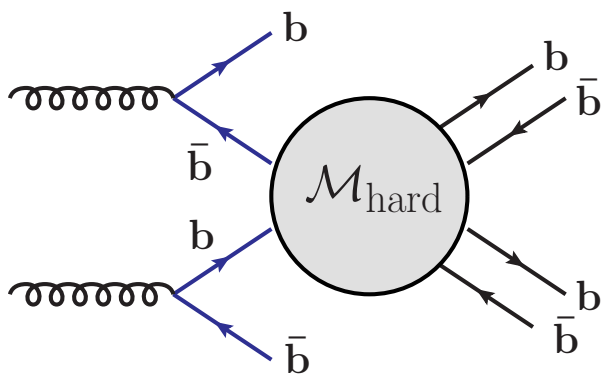
❑ Matching between a fixed order NLO calculation and this new parton shower under construction

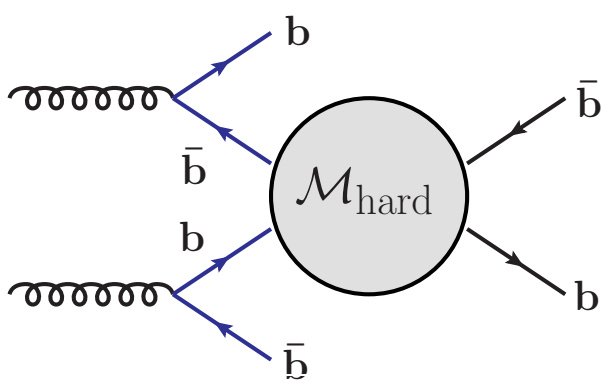
M. Czakon, B. Hartanto, M. Kraus and M. Worek (in preparation)

Massive bottom [4FS]

- Bottom quarks appear only in the final state and are massive
- PDF does not contain bottom quark, **n_l= 4 (u, d, c, s)**
  - Do not enter in the computation of the running of α_s
  - Do not enter in the evolution of the PDFs

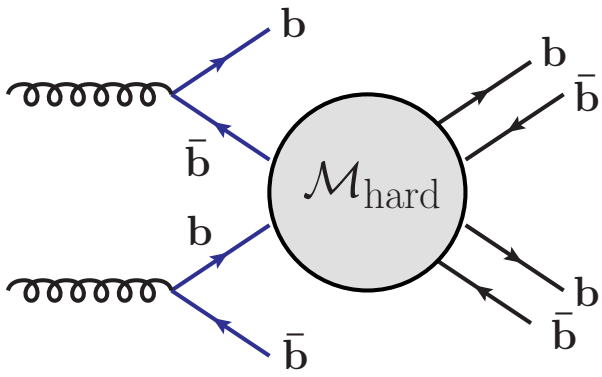
- Finite-m_b effects enter via:
  - Power corrections of the type $\mathcal{O}[(m_b/\mathcal{Q})^n]$
  - Logarithms of the type $\mathcal{O}[\log^n(m_b/\mathcal{Q})]$

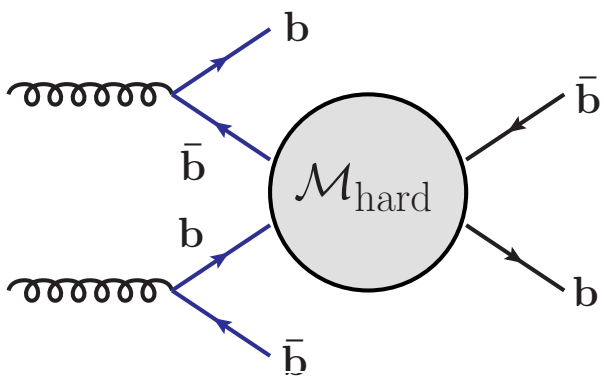




Massive bottom [4FS]

- At the LHC, typically $(\mathbf{m_b}/\mathcal{Q}) \ll \mathbf{1}$ and power corrections are suppressed
- While logarithms could be large (can be of initial or final state nature)
- For inclusive observables such as b-jets, logarithms can only originate from nearly collinear initial-state $\mathbf{g \to b\bar{b}}$ splitting
- Large logarithms could spoil the convergence of the fixed order calculations
- Resummation could be needed





Up to NLO Accuracy Potentially Large Logarithms

$$\log(\mathbf{m_b}/\mathcal{Q}) \to \log(\mathbf{p_{T,b}^{min}}/\mathcal{Q})\,, \qquad \mathbf{m_b} \ll \mathbf{p_{T,b}^{min}} \lesssim \mathcal{Q}$$

And are Less Significant Numerically

Massless bottom [5FS]

- Under the approximation that bottom quarks from splittings have small p_T towers of $\log^{\mathbf{n}}(\mathbf{m_b}/\mathcal{Q})$ explicitly resummed into bottom PDF
- For consistency with the factorization theorem, one should set $\mathbf{m_b = 0}$ in the calculation of the matrix element

- PDF contains bottom quark, $\mathbf{n_l = 5}$ **(u, d, c, s, b)**
  - bottom quarks enter in the computation of the running of α_s
  - bottom quarks enter in the evolution of the PDFs

- To all orders in perturbation theory two schemes are identical for log effects
- The way of ordering the perturbative expansion is different and at any finite order the results might not match

Maltoni, Ridolfi, Ubiali (2012)
Harlander, Krämer, Schumacher (2011)
Frederix, Re, Torrielli (2012)
...

4b-jets @ LHC

- Four flavor scheme with $m_b \neq 0$ (4FS) versus five flavor scheme with $m_b = 0$ (5FS)

$pp \to b\bar{b}b\bar{b} + X$	σ_{LO} [pb]	σ_{NLO} [pb]	$K = \sigma_{NLO}/\sigma_{LO}$
MSTW2008LO/NLO (5FS)	$99.9^{+58.7(59\%)}_{-34.9(35\%)}$	$136.7^{+38.8(28\%)}_{-30.9(23\%)}$	1.37
MSTW2008LO/NLO (4FS)	$84.5^{+49.7(59\%)}_{-29.6(35\%)}$	$118.3^{+33.3(28\%)}_{-29.0(24\%)}$	1.40

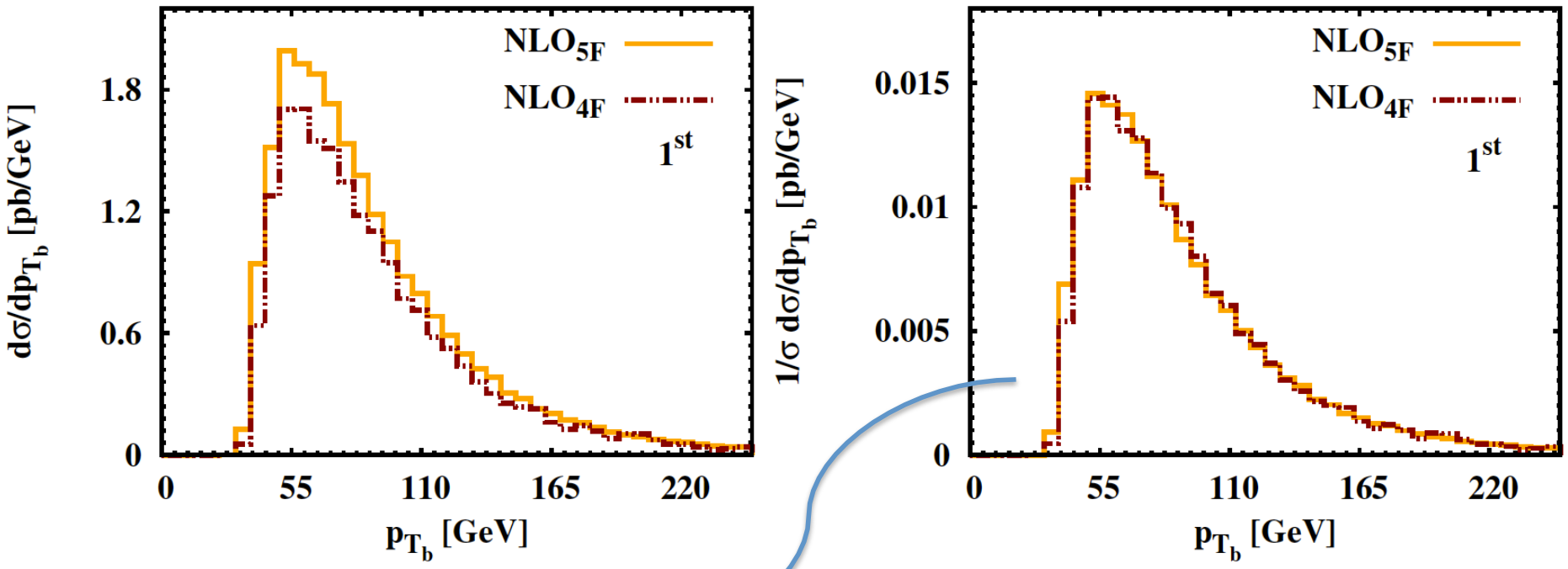
- Cross section predictions in LO and NLO for $\mu = H_T$ and **m_b=4.75 GeV**
- K-factor and residual scale dependence at NLO similar

- Comparing 4FS with 5FS bottom mass effects decrease the cross section by:
  - **18% at LO** & **16% at NLO**
  - Genuine bottom mass effects, for **$p_{T,b}$ > 30 GeV** of the order **~10%**
  - Strong dependence on $p_{T,b}$ cut, for **$p_{T,b}$ > 100 GeV** only **~1%**
  - Scheme dependence **~5%**, different PDFs and α_s

G. Bevilacqua, M. Czakon, M. Krämer, M. Kubocz and M. Worek (2013)

4b-jets @ LHC

- Transverse momentum of the hardest bottom jet in 5FS & 4FS
- Absolute prediction at NLO QCD
- Predictions normalized to inclusive cross sections



Shape Differences Very Small

G. Bevilacqua, M. Czakon, M. Krämer, M. Kubocz and M. Worek (2013)

Next-to-Leading Order QCD + Parton Shower effects on heavy quark production and evolution at the Tevatron and LHC

M.V. Garzelli, A. Kardos, C.G. Papadopoulos, Z. Trocsanyi

MTA-DE Particle Physics Research Group, Debrecen - NCSR Demokritos Athens

project status - June 2014

PowHel = HELAC-NLO + POWHEG-BOX

Interface between different event generators:

- All LO and NLO matrix-elements: HELAC-NLO
  `http://helac-phegas.web.cern.ch/helac-phegas/`
- Subtraction of IR divergencies and matching NLO + PS: POWHEG-BOX
  `http://powhegbox.mib.infn.it/`
- Parton and photon shower emissions: SMC codes (PYTHIA and HERWIG)
- Hadronization and hadron decay: SMC codes (PYTHIA and HERWIG)

OUTPUT:

Les Houches event files
and predictions at both parton and hadron level
with NLO QCD + Parton Shower accuracy
for p-p and p-$\bar{p}$ processes

PowHel + SMC: processes studied so far at LHC/Tevatron

- pp and $p\bar{p} \to t\bar{t}$
- pp and $p\bar{p} \to t\bar{t}j$ [arXiv:1101.2672]
- $pp \to t\bar{t}H/t\bar{t}A$ [arXiv:1108.0387], [arXiv:1201.3084]
- $pp \to t\bar{t}Z$ [arXiv:1111.1444], [arXiv:1208.2665]
- $pp \to t\bar{t}W^+$, $t\bar{t}W^-$ [arXiv:1208.2665]
- $pp \to t\bar{t}b\bar{b}$ [arXiv:1303.6291], [arXiv:1307.1347] + in preparation
- pp and $p\bar{p} \to (t\bar{t} \to W^+W^-b\bar{b}) \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ [arXiv:1405.5859]

All these processes involve the production of a $t\bar{t}$ pair.

pp and $p\bar{p} \to (t\bar{t} \to W^+W^-b\bar{b}) \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$

∗ Important background to $Hb\bar{b}$ and to New Physics searches.

– NLO level: G. Bevilacqua et al. [arXiv:1012.4230], A. Denner et al. [arXiv:1012.3975], [arXiv:1207.5018], R. Frederix [arXiv:1311.4893], F. Cascioli et al. [arXiv:1312.0546], G. Heinrich et al. [arXiv:1312.6659]

– NLO matched to Parton Shower: Garzelli, Kardos, Trocsanyi [arXiv:1405.5859] study at NLO level and after SMC in 3 different configurations:

1) Full NLO QCD corrections to $pp \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ (i.e. six particle final state): generation of LHEF events for this configuration, further evolution with SMC.

2) NLO QCD corrections just to $t\bar{t}$ on-shell production: generation of LHEF events for this configuration, further evolution with SMC which takes care of all decays in DCA (Decay Chain Approximation). Spin correlations are not included if the SMC neglects them.

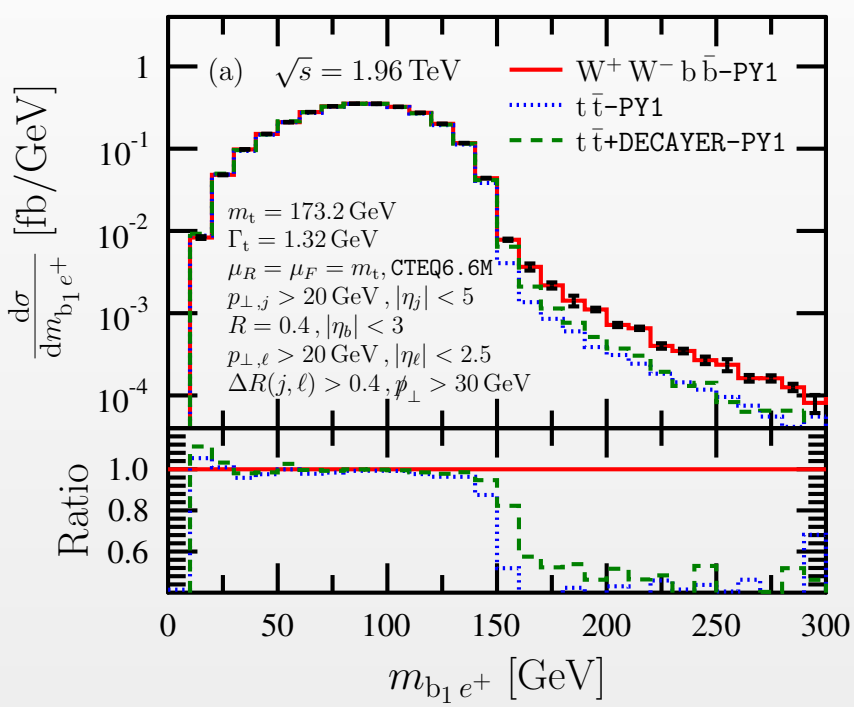
3) NLO QCD corrections to $t\bar{t}$ + `Decayer` + SMC: the $t\bar{t}$ events in the LHEF are decayed by `Decayer` code, to account for spin correlations in the NWA (Narrow Width Approximation), and then passed to the SMC code.

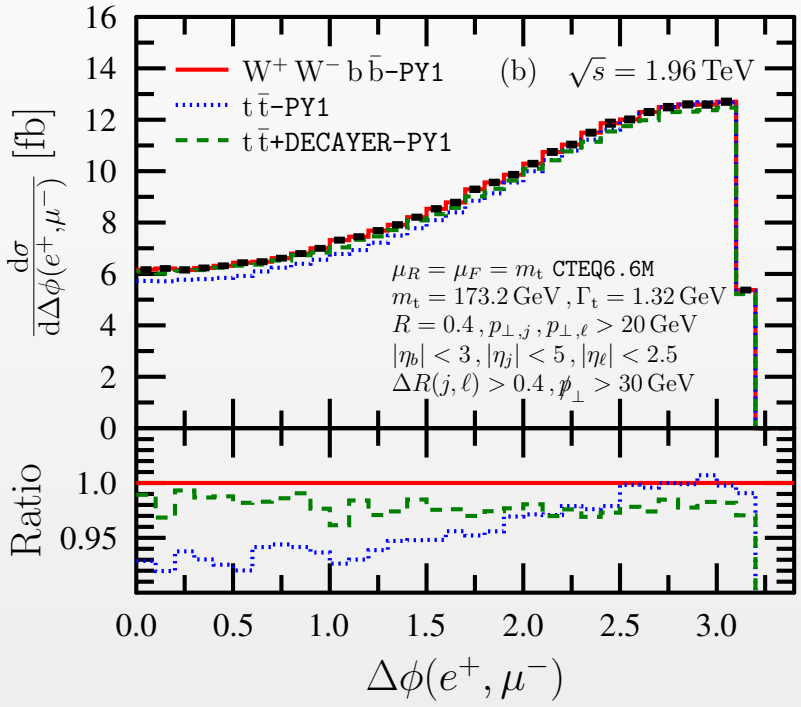
case 1) includes all resonant and non-resonant top contributions (top treated in the Complex Mass Scheme),
case 2) and 3) just include the resonant top contributions

$p\bar{p} \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ evoluted to the hadron level

Tevatron

Garzelli, Kardos, Trocsanyi [arXiv:1405.5859]





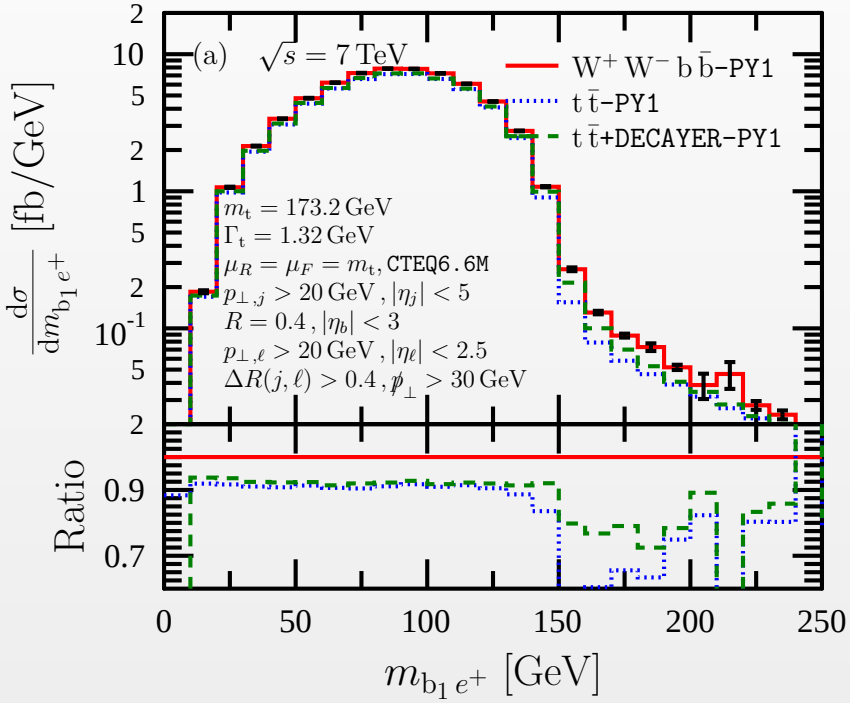
Distributions of a) invariant mass of hardest b-jet and the hardest isolated positron and of b) azimuthal separation between the hardest isolated positron and muon after full SMC. The lower inset shows the ratio of the predictions with decays of the t-quarks in DCA and Decayer compared to the complete *WWbb* computation.

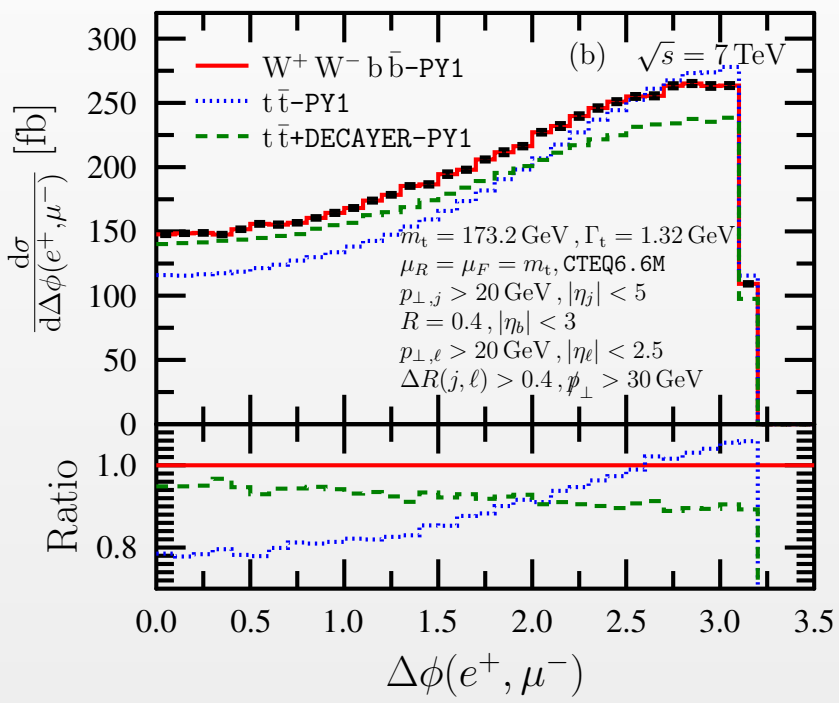
* In NWA at LO, limit $m^2_{inv}(b_1, e) < m_t^2 - m_W^2$, here instead high-energy tail.
* Spin correlations affect azimuthal angle distributions.

$pp \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ evoluted to the hadron level

LHC

Garzelli, Kardos, Trocsanyi [arXiv:1405.5859]



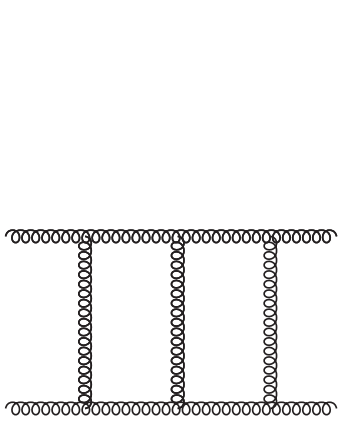


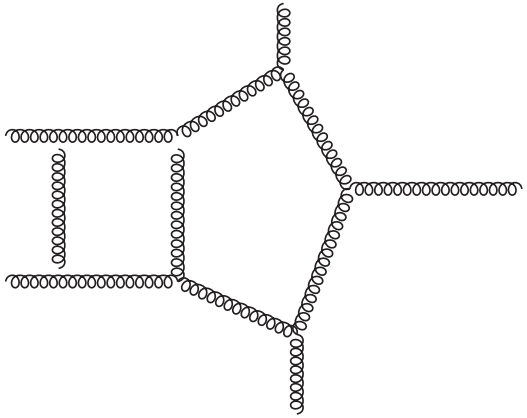
Same as in previous slide, as for the LHC.

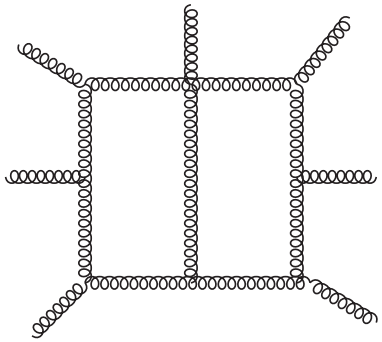
∗ At LHC, the high energy tail in the $m_{inv}(b_1, e)$ distribution is more pronounced than at the Tevatron.

∗ Spin correlations effects are also more important at the LHC than at the Tevatron.

NNLO







Repeat the one-loop "success story" ?

Reduction at the integrand level

Over the last few years very important activity to extend unitarity and integrand level reduction ideas beyond one loop

J. Gluza, K. Kajda and D. A. Kosower, "Towards a Basis for Planar Two-Loop Integrals," Phys. Rev. D **83** (2011) 045012 [arXiv:1009.0472 [hep-th]].

D. A. Kosower and K. J. Larsen, "Maximal Unitarity at Two Loops," Phys. Rev. D **85** (2012) 045017 [arXiv:1108.1180 [hep-th]].

P. Mastrolia and G. Ossola, "On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes," JHEP **1111** (2011) 014 [arXiv:1107.6041 [hep-ph]].

S. Badger, H. Frellesvig and Y. Zhang, "Hepta-Cuts of Two-Loop Scattering Amplitudes," JHEP **1204** (2012) 055 [arXiv:1202.2019 [hep-ph]].

Y. Zhang, "Integrand-Level Reduction of Loop Amplitudes by Computational Algebraic Geometry Methods," JHEP **1209** (2012) 042 [arXiv:1205.5707 [hep-ph]].

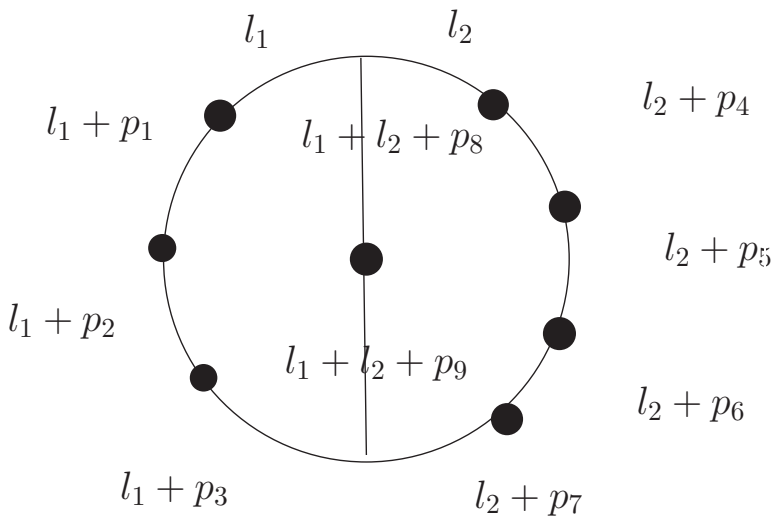
P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, "Integrand-Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division," arXiv:1209.4319 [hep-ph].

P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, "Multiloop Integrand Reduction for Dimensionally Regulated Amplitudes," arXiv:1307.5832 [hep-ph].

Two-loop amplitudes

- Reduction at the integrand level → helicity amplitudes
- Generic two-loop graph: iGraph

R. H. P. Kleiss, I. Malamos, C. G. Papadopoulos and R. Verheyen, arXiv:1206.4180 [hep-ph].

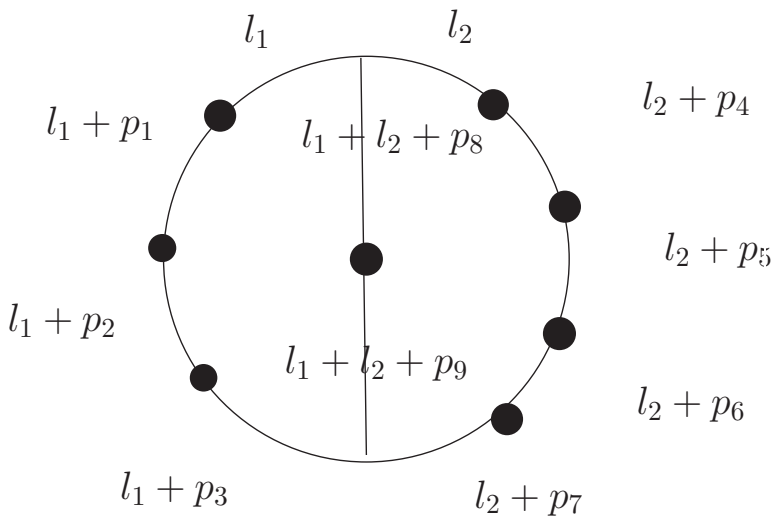


$$D(l_1 + p_i)\ ,\ D(l_2 + p_j)\ ,\ D(l_1 + l_2 + p_k)$$

Two-loop amplitudes

- Reduction at the integrand level → helicity amplitudes
- Generic two-loop graph: iGraph

R. H. P. Kleiss, I. Malamos, C. G. Papadopoulos and R. Verheyen, arXiv:1206.4180 [hep-ph].



$$D(l_1 + p_i)\ ,\ D(l_2 + p_j)\ ,\ D(l_1 + l_2 + p_k)$$

The simplest case: $n \to n-1$ reduction

The general strategy consists in finding polynomials $\Pi_j \equiv \Pi_j(l_1, l_2)$

$$\sum_{j=1}^{n_1} \Pi_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} \Pi_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} \Pi_j D(l_2 + p_j) = 1 \ .$$

Is this plausible at all ?

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Hilbert's Nullstellensatz theorem

Hilbert's Nullstellensatz (German for "theorem of zeros," or more literally, "zero-locus-theorem" see Satz) is a theorem which establishes a fundamental relationship between geometry and algebra. This relationship is the basis of algebraic geometry, an important branch of mathematics. It relates algebraic sets to ideals in polynomial rings over algebraically closed fields. This relationship was discovered by David Hilbert who proved Nullstellensatz and several other important related theorems named after him (like Hilbert's basis theorem).

$$1 = g_1 f_1 + \cdots + g_s f_s \quad g_i, f_i \in k[x_1, \ldots, x_n]$$

Janos Kollar, J. Amer. Math. Soc., Vol. 1, No. 4. (Oct., 1988), pp 963-975

$$\deg g_i f_i \leq \max\{3, d\}^n \quad d = \max \deg f_i \quad 3^8 = 6561$$

M. Sombra, Adv. in Appl. Math. 22 (1999), 271-295

$$\deg g_i f_i \leq 2^{n+1} \quad 2^9 = 512$$

OPP at two loops

- Write the "OPP-type" equation at two loops

$$\frac{N\left(l_1, l_2; \{p_i\}\right)}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$S_{m;n}$ stands for all subsets of m indices out of the n ones

Multivariate Division and Groebner Basis

D. Cox, J. Little, D. O'Shea *Ideals, Varieties and Algorithms*

- Given any ideal I we can define a unique Groebner basis up to ordering $< g_1, \ldots, g_s >$

$$f = h_1 g_1 + \ldots + h_n g_n + r$$

multivariate polynomial division

Strategy:

- Start with a set of polynomials $I =< d_1, \ldots, d_n >$
- Find the GB, $G =< g_1, \ldots, g_s >$
- Perform the division of an arbitrary polynomial N

$$N = h_1 g_1 + \ldots + h_n g_s + v$$

- Express back g_i in terms of d_i

$$N = \tilde{h}_1 d_1 + \ldots + \tilde{h}_n d_n + v$$

OPP at two loops

- Planar topology (4,1,4)

$$
\begin{aligned}
&D_1 = l_1^2 - M_1^2, D_2 = (l_1 + p_1)^2 - M_2^2, \\
&D_3 = (l_1 + p_2)^2 - M_3^2, D_4 = (l_1 + p_3)^2 - M_4^2, \\
&D_5 = l_2^2 - M_5^2, D_6 = (l_2 + p_4)^2 - M_6^2, \\
&D_7 = (l_2 + p_5)^2 - M_7^2, D_8 = (l_2 + p_6)^2 - M_8^2, \\
&D_9 = (l_1 + l_2)^2 - M_9^2
\end{aligned}
$$

•: l_1

$$
v_1^\mu = \frac{\delta^{\mu p_2 p_3}_{p_1 p_2 p_3}}{\Delta} \quad v_2^\mu = \frac{\delta^{p_1 \mu p_3}_{p_1 p_2 p_3}}{\Delta} \quad v_3^\mu = \frac{\delta^{p_1 p_2 \mu}_{p_1 p_2 p_3}}{\Delta} \quad \eta^\mu = \frac{\varepsilon^{\mu p_1 p_2 p_3}}{\sqrt{\Delta}}
$$

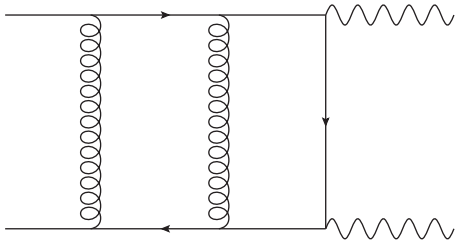
with

$$
\Delta = \delta^{p_1 p_2 p_3}_{p_1 p_2 p_3} = \varepsilon^{p_1 p_2 p_3} \varepsilon_{p_1 p_2 p_3} = \begin{vmatrix} p_1 \cdot p_1 & p_1 \cdot p_2 & p_1 \cdot p_3 \\ p_2 \cdot p_1 & p_2 \cdot p_2 & p_2 \cdot p_3 \\ p_3 \cdot p_1 & p_3 \cdot p_2 & p_3 \cdot p_3 \end{vmatrix}
$$

OPP at two loops

•: l_2, the same as above with p_4, p_5, p_6 replacing p_1, p_2, p_3 accordingly. The momenta $p_i, i = 1, \ldots, 6$ are arbitrary. The basis coefficients may be read as $l_1^\mu = \sum_{i=1}^{3} z_i v_i^\mu + z_4 \eta^\mu$, with $z_i = l_1 \cdot p_i, i = 1 \ldots, 3$ (l_2, with w_i replacing z_i).

As an example I reduced a two-loop 7-propagator graph contributing to $q\bar{q} \to \gamma^* \gamma^*$



$$n1 := 155$$

$$1, 1, \text{" ---- "}, \frac{4347392}{81} - \frac{2891776}{243}\sqrt{2}\sqrt{3} + \frac{425984}{27} z3 - \frac{134144}{27}\sqrt{3}\sqrt{2}\, z3 + \frac{1921024}{27} w3 - \frac{1358848}{81}\sqrt{3}\sqrt{2}\, w3 + \frac{66560}{3} w4\, z4 - \frac{20480}{3} w4\, z4\sqrt{2}\sqrt{3} + \frac{16384}{3} z3\, w4\, z4$$
$$+ \frac{524288}{81} w4^2\, w3 + \frac{131072}{243}\sqrt{3}\, w4^2\sqrt{2}\, w3 - \frac{32768}{27}\sqrt{3}\sqrt{2}\, z3\, w4^2 - \frac{16384}{27} w4^2\, z3 + \frac{702464}{81} w4^2 - \frac{53248}{243} w4^2\sqrt{2}\sqrt{3} + 4608\, z4^2, \text{"------"}, 16$$

$$2, dd_3\, dd_5\, dd_1, \text{" ---- "}, \frac{3136}{15}\sqrt{2}\sqrt{3} + \frac{2048}{27} w4^2\sqrt{2}\sqrt{3} + \frac{8192}{9} w4^2, \text{"------"}, 3$$

$$3, dd_3\, dd_8\, dd_5\, dd_4, \text{" ---- "}, \frac{1970176}{5625} w3 - \frac{2363392}{1875} w2 + \frac{575488}{16875}\sqrt{3}\sqrt{2}\, w3 - \frac{956416}{1875}\sqrt{2}\sqrt{3} - \frac{618496}{5625}\sqrt{2}\sqrt{3}\, w2 - \frac{1407232}{625}, \text{"------"}, 6$$

$$4, dd_3\, dd_8\, dd_5\, dd_9, \text{" ---- "}, -\frac{2048}{9} w3 + \frac{2048}{3} w2 - \frac{512}{27}\sqrt{3}\sqrt{2}\, w3 + \frac{512}{9}\sqrt{2}\sqrt{3}\, w2 + \frac{512}{3}\sqrt{2}\sqrt{3} + 512, \text{"------"}, 6$$

$$5, dd_6\, dd_8\, dd_4\, dd_3, \text{" ---- "}, -\frac{309248}{16875}\sqrt{3}\sqrt{2}\, w3 + \frac{323584}{5625}\sqrt{2}\sqrt{3}\, w2 - \frac{1181696}{5625} w3 + \frac{1312768}{1875} w2 + \frac{1167232}{1875} + \frac{956416}{5625}\sqrt{2}\sqrt{3}, \text{"------"}, 6$$

$$6, dd_5\, dd_9\, dd_6\, dd_1, \text{" ---- "}, -\frac{7168}{27}\sqrt{2}\sqrt{3} + \frac{24320}{9}, \text{"------"}, 2$$

$$7, dd_5\, dd_9\, dd_6\, dd_2, \text{" ---- "}, \frac{2176}{9}\sqrt{2}\sqrt{3} - \frac{8192}{3} + \frac{2560}{81}\sqrt{3}\sqrt{2}\, z3 + \frac{10240}{27} z3, \text{"------"}, 4$$

$$8, dd_5\, dd_9\, dd_6\, dd_3, \text{" ---- "}, 256, \text{"------"}, 1$$

$$9, dd_5\, dd_9\, dd_8\, dd_2, \text{" ---- "}, 1536 + 128\sqrt{2}\sqrt{3} - \frac{1024}{3} w2 - \frac{256}{9}\sqrt{2}\sqrt{3}\, w2 + \frac{1024}{9} w3 + \frac{256}{27}\sqrt{3}\sqrt{2}\, w3, \text{"------"}, 6$$

$$10, dd_5\, dd_9\, dd_8\, dd_4, \text{" ---- "}, -\frac{135808}{5625}\sqrt{2}\sqrt{3} - \frac{418816}{1875}, \text{"------"}, 2$$

$$11, dd_6\, dd_9\, dd_4\, dd_3, \text{" ---- "}, -\frac{157312}{16875}\sqrt{2}\sqrt{3} - \frac{615424}{5625}, \text{"------"}, 2$$

$$12, dd_6\, dd_9\, dd_4\, dd_8, \text{" ---- "}, \frac{418816}{1875} + \frac{135808}{5625}\sqrt{2}\sqrt{3}, \text{"------"}, 2$$

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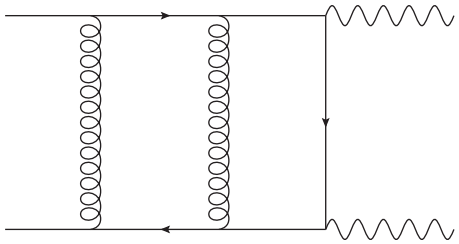

$$\frac{\Pi\left(\{z_i\}, \{w_j\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}} \to \text{spurious} \oplus \text{nonscalar integrals}$$

- IBPI to Master Integrals

OPP at two loops - Rational terms

S. Badger, talk in Amplitudes 2013

- Rational terms

$$l_1 \to l_1 + l_1^{(2\varepsilon)},\ l_2 \to l_2 + l_2^{(2\varepsilon)},\ l_{1,2} \cdot l_{1,2}^{(2\varepsilon)} = 0$$

$$\left(l_1^{(2\varepsilon)}\right)^2 = \mu_{11},\quad \left(l_2^{(2\varepsilon)}\right)^2 = \mu_{22},\quad l_1^{(2\varepsilon)} \cdot l_2^{(2\varepsilon)} = \mu_{12}$$

$$\left\{l_1^{(4)}, l_2^{(4)}\right\} \to \left\{l_1^{(4)}, l_2^{(4)}, \mu_{11}, \mu_{22}, \mu_{12}\right\}$$

Welcome: $I = \sqrt{I}$ prime ideals

- R_2 terms

Master Integrals: The current approach

- m independent momenta l loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products $D_i = (\{k, l\} + p_i)^2 - M_i^2$
- $F[a_1, \dots, a_N]$

$$\int d^d k d^d l \;\; \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left(\frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0$$

- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations

Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

J. M. Henn, Phys. Rev. Lett. **110** (2013) 25, 251601 [arXiv:1304.1806 [hep-th]].

Differential Equations Approach

- Library of MI à la one-loop
- Iterated Integrals K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
- Polylogarithms, Symbol algebra

A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

C. Duhr, H. Gangl and J. R. Rhodes, JHEP **1210** (2012) 075 [arXiv:1110.0458 [math-ph]].

$$G\left(a_n, \ldots, a_1, x\right) = \int_0^x dt \frac{1}{t - a_n} G\left(a_{n-1}, \ldots, a_1, t\right)$$

with the special cases, $G(x) = 1$ and

$$G\left(\underbrace{0, \ldots 0}_{n}, x\right) = \frac{1}{n!} \log^n(x)$$

Differential Equations Approach

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_0 D_1 \ldots D_{n-1}}$$

with $D_i = (k + p_0 + \ldots + p_i)^2$ and take for convenience $p_0 = 0$. It can be considered as a function of the external momenta p_i. It belongs to the topology defined by

$$G_{a_1 \ldots a_n} = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_0^{a_1} D_1^{a_2} \ldots D_{n-1}^{a_n}}$$

namely $G_{1 \ldots 1}$.

Differential Equations Approach

The integral is a function of external momenta, so one can set-up differential equations by differentiating with respect to these

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum G[a_1', \ldots, a_n']$$

$$p_1^\mu \frac{\partial}{\partial p_1^\mu} (k + p_1)^2 = 2\,(k + p_1) \cdot p_1 = (k + p_1)^2 + p_1^2 - k^2$$

- Find the proper parametrization
- Boundary conditions
- Bring the system of equations in a form suitable to express the MI in terms of GPs

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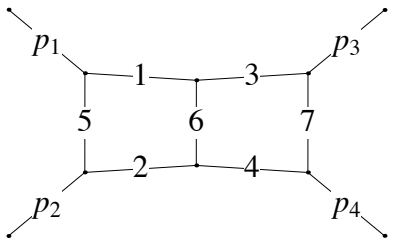
Differential Equations Approach

J. M. Henn, K. Melnikov and V. A. Smirnov, arXiv:1402.7078 [hep-ph].



$$S = (q_1 + q_2)^2 = (q_3 + q_4)^2, \quad T = (q_1 - q_3)^2 = (q_2 - q_4)^2, \quad U = (q_1 - q_4)^2 = (q_2 - q_3)^2;$$

$$\frac{S}{M_3^2} = (1+x)(1+xy), \quad \frac{T}{M_3^2} = -xz, \quad \frac{M_4^2}{M_3^2} = x^2 y.$$

$$d\,\vec{f}(x,y,z;\epsilon) = \epsilon\, d\,\tilde{A}(x,y,z)\,\vec{f}(x,y,z;\epsilon)$$

$$\tilde{A} = \sum_{i=1}^{15} \tilde{A}_{\alpha_i}\, \log(\alpha_i)$$

$$\alpha = \{x, y, z, 1+x, 1-y, 1-z, 1+xy, z-y, 1+y(1+x)-z, xy+z,$$
$$1+x(1+y-z), 1+xz, 1+y-z, z+x(z-y)+xyz, z-y+yz+xyz\}.$$

The Simplified Differential Equations Approach

C. G. Papadopoulos, arXiv:1401.6057 [hep-ph].

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

$$G_{11\ldots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\,p_1)^2\,(k + p_1 + p_2)^2 \ldots (k + p_1 + p_2 + \ldots + p_n)^2}$$

Now the integral becomes a function of x, which allows to define a differential equation with respect to x, schematically given by

$$\frac{\partial}{\partial x} G_{11\ldots1}(x) = -\frac{1}{x} G_{11\ldots1}(x) + x p_1^2 G_{12\ldots1} + \frac{1}{x} G_{02\ldots1}$$

and using IBPI we obtain

$$\begin{aligned} m_1 x G_{121} + \tfrac{1}{x} G_{021} \quad &= \left(\tfrac{1}{x-1} + \tfrac{1}{x - m_3/m_1}\right)\left(\tfrac{d-4}{2}\right) G_{111} \\ &+ \tfrac{d-3}{m_1 - m_3}\left(\tfrac{1}{x-1} - \tfrac{1}{x - m_3/m_1}\right)\left(\tfrac{G_{101} - G_{110}}{x}\right) \end{aligned}$$

The Simplified Differential Equations Approach

The integrating factor M is given by

$$M = x\,(1-x)^{\frac{4-d}{2}}\,(-m_3 + m_1 x)^{\frac{4-d}{2}}$$

and the DE takes the form, $d = 4 - 2\varepsilon$,

$$\frac{\partial}{\partial x} M G_{111} = c_\Gamma \frac{1}{\varepsilon}\,(1-x)^{-1+\varepsilon}\,(-m_3 + m_1 x)^{-1+\varepsilon}\left(\left(-m_1 x^2\right)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)$$

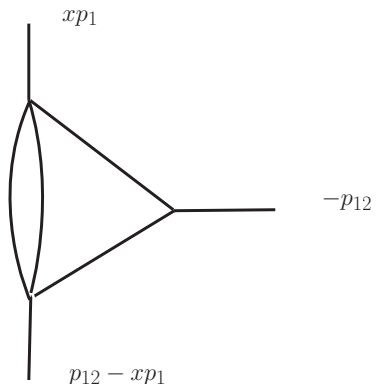
- Integrating factors $\epsilon = 0$ do not have branch points
- DE can be straightforwardly integrated order by order $\rightarrow$ GPs.

The Simplified Differential Equations Approach

$$G_{111} = \frac{c_\Gamma}{(m_1 - m_3)x}\mathcal{I}$$

$$\begin{aligned}
\mathcal{I} = {} & \frac{-(-m_1)^{-\varepsilon} + (-m_3)^{-\varepsilon} + \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)x_1}{\varepsilon^2} \\
& + \frac{\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)x_1 G\left(\frac{m_3}{m_1},1\right) - \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\left(G\left(\frac{m_3}{m_1},1\right) - G\left(\frac{m_3}{m_1},x\right)\right)}{\varepsilon} \\
& + \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\left(G\left(\frac{m_3}{m_1},1\right)G\left(\frac{m_3}{m_1},x\right) - G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) - G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\right) + x_1\left(-2G(0,1,x)(-m_1)^{-\varepsilon}\right. \\
& \quad + 2G\left(0,\frac{m_3}{m_1},x\right)(-m_1)^{-\varepsilon} + 2G\left(\frac{m_3}{m_1},1,x\right)(-m_1)^{-\varepsilon} + G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)(-m_1)^{-\varepsilon} - G\left(\frac{m_3}{m_1},x\right)\log(1-x)(-m_1)^{-\varepsilon} \\
& \quad - 2G\left(\frac{m_3}{m_1},x\right)\log(x)(-m_1)^{-\varepsilon} + 2\log(1-x)\log(x)(-m_1)^{-\varepsilon} - 2(-m_3)^{-\varepsilon}G\left(\frac{m_3}{m_1},1,x\right) - (-m_3)^{-\varepsilon}G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) \\
& \quad \left. - \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)G\left(\frac{m_3}{m_1},1\right)\left(G\left(\frac{m_3}{m_1},x\right) - \log(1-x)\right) + (-m_3)^{-\varepsilon}G\left(\frac{m_3}{m_1},x\right)\log(1-x)\right) \\
& + \varepsilon\left(\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\left(G\left(\frac{m_3}{m_1},x\right)G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) - G\left(\frac{m_3}{m_1},1\right)G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right) - G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)\right.\right. \\
& \quad \left. + G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\right) + \frac{1}{2}x_1\left(\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)G\left(\frac{m_3}{m_1},1\right)\left(\log^2(1-x) + 2G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\right)\right. \\
& \quad + G\left(\frac{m_3}{m_1},x\right)\left(4\log^2(x) - 2\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) + 2\left((-m_3)^{-\varepsilon} - (-m_1)^{-\varepsilon}\right)G\left(\frac{m_3}{m_1},1\right)\log(1-x)\right) \\
& \quad + 2\left(G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)(-m_1)^{-\varepsilon} + G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)\log(1-x)(-m_1)^{-\varepsilon} - 2\log(1-x)\log^2(x) - 4G(0,0,1,x)\right. \\
& + 4G\left(0,0,\frac{m_3}{m_1},x\right) - 2G(0,1,1,x) + 4G\left(0,\frac{m_3}{m_1},1,x\right) - 2G\left(0,\frac{m_3}{m_1},\frac{m_3}{m_1},x\right) + 2G\left(\frac{m_3}{m_1},0,1,x\right) - 2G\left(\frac{m_3}{m_1},0,\frac{m_3}{m_1},x\right) \\
& - (-m_3)^{-\varepsilon}G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},\frac{m_3}{m_1},1\right) - (-m_3)^{-\varepsilon}G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},1\right)\log(1-x) + \log^2(1-x)\log(x) \\
& \left.\left.\left. + 4G(0,1,x)\log(x) - 4G\left(0,\frac{m_3}{m_1},x\right)\log(x) - 4G\left(\frac{m_3}{m_1},1,x\right)\log(x) + 2G\left(\frac{m_3}{m_1},\frac{m_3}{m_1},x\right)\log(x)\right)\right)\right)
\end{aligned}$$

The Simplified Differential Equations Approach



We are interested in $G_{0101011}$. The DE involves also the MI $G_{0201011}$, so we have a system of two coupled DE, as follows:

$$\frac{\partial}{\partial x}\left(M_{0101011}G_{0101011}\right) = \frac{A_3(2-3\varepsilon)(1-x)^{-2\varepsilon}x^{\varepsilon-1}(m_1x-m_3)^{-2\varepsilon}}{2\varepsilon(2\varepsilon-1)} + \frac{m_1\varepsilon(1-x)^{-2\varepsilon}(m_1x-m_3)^{-2\varepsilon}}{2\varepsilon-1}g(x)$$

$$\frac{\partial}{\partial x}\left(M_{0201011}G_{0201011}\right) = \frac{A_3(3\varepsilon-2)(3\varepsilon-1)(-m_1)^{-2\varepsilon}(1-x)^{2\varepsilon-1}x^{-3\varepsilon}(m_1x-m_3)^{2\varepsilon-1}}{2\varepsilon^2} + (2\varepsilon-1)(3\varepsilon-1)(1-x)^{2\varepsilon-1}(m_1x-m_3)^{2\varepsilon-1}f(x)$$

where $f(x) \equiv M_{0101011}G_{0101011}$ and $g(x) \equiv M_{0201011}G_{0201011}$, $M_{0201011} = (1-x)^{2\varepsilon}x^{\varepsilon+1}(m_1x-m_3)^{2\varepsilon}$ and $M_{0101011} = x^{\varepsilon}$

The Simplified Differential Equations Approach

The singularity structure of the right-hand side is now richer. Singularities at $x = 0$ are all proportional to $x^{-1-2\varepsilon}$ and $x^{-1-\varepsilon}$ and can easily be integrated by the following decomposition

$$\int_0^x dt\ t^{-1-2\varepsilon} F(t) = F(0) \int_0^x dt\ t^{-1-2\varepsilon} + \int_0^x dt\ \frac{F(t)-F(0)}{t} t^{-2\varepsilon}$$
$$= F(0)\ \frac{x^{-2\varepsilon}}{(-2\varepsilon)} + \int_0^x dt\ \frac{F(t)-F(0)}{t} \left(1 - 2\varepsilon \log(t) + 2\varepsilon^2 \log^2(t) + \ldots\right)$$

- One-loop up to 5-point at order ϵ
- Two-loop triangles and 4-point MI
- Working/finishing double boxes with two external off-shell legs (more than 100 MI) $\rightarrow$ planar topologies completed!
- Completing the list of all MI with arbitrary off-shell legs ($m = 0$).

The Simplified Differential Equations Approach

- Get DE in one parameter, that always go to the argument of GPs, all weights being independent of x, therefore no limitation on the number of scales (multi-leg).
- Boundary conditions, namely the $x \to 0$ limit, defined by the DE itself
- All coupled systems of DE (up to fivefold) satisfying the decoupling criterion, solvable order by order in ϵ

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NNLO in the future

- The NNLO automation to come
  - In a few years the new "wish list" should be completed
    $pp \to t\bar{t}$, $pp \to W^+W^-$, $pp \to W/Z + nj$, $pp \to H + nj$, ...
  - Virtual amplitudes: Reduction at the integrand level $\oplus$ IBP
  - Master Integrals
  - Virtual-Real A.van Hameren, OneLoop MI
  - Real-Real STRIPPER, M. Czakon, Phys. Lett. B **693** (2010) 259 [arXiv:1005.0274 [hep-ph]].
- A HELAC-NNLO framework ?

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