One-loop Integrand Reduction with NINJA

Tiziano Peraro

Max-Planck-Institut für Physik
Föhringer Ring 6, D-80805 München, Germany

LoopFest 2014
New York City College of Technology
June 18–20, 2014

MAX-PLANCK-GESELLSCHAFT

Alexander von Humboldt
Stiftung/Foundation

Outline

1. Introduction and motivation
2. The integrand reduction of one-loop amplitudes
3. NINJA: Integrand reduction via Laurent expansion
4. Phenomenological applications of NINJA and GOSAM
5. Summary and Outlook

Introduction and motivation

The goal

Implementation of a reduction algorithm for one-loop amplitudes which

- can be applied to processes with many external legs
- allows the presence of massive external and internal particles
- is reasonably stable and fast
- is suited for automation

NINJA

NINJA is a C++ library which

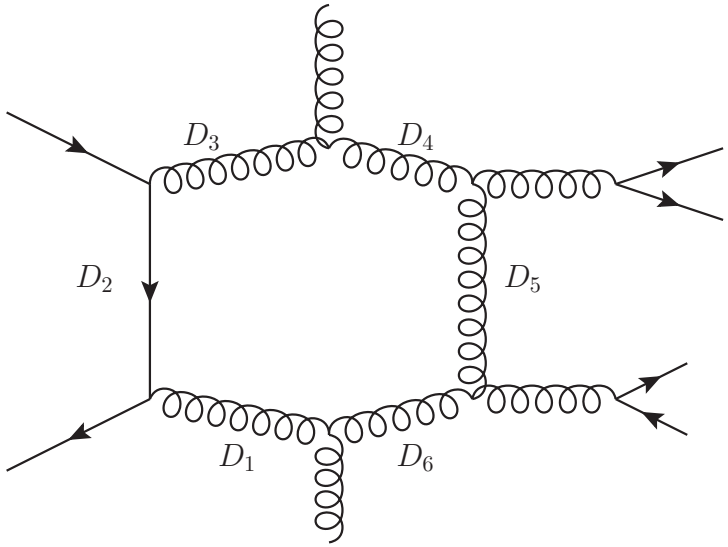
- implements the Larent expansion method for one-loop amplitudes
- is fast, stable and can be applied to any one-loop integrand

The Integrand reduction of one-loop amplitudes

- The integrand of a generic n-point one-loop integral:
  - is a rational function in the components of the loop momentum $\bar{q}$
  - polynomial numerator $\mathcal{N}$

$$\mathcal{M}_n = \int d^d\bar{q}\ \mathcal{I}_n, \qquad \mathcal{I}_n \equiv \frac{\mathcal{N}(\bar{q})}{D_1 \ldots D_n}$$

  - quadratic polynomial denominators D_i
    - they correspond to Feynman loop propagators



$$D_i = (\bar{q} + p_i)^2 - m_i^2$$

The Integrand reduction of one-loop amplitudes

- Every one-loop integrand, can be decomposed as
  [Ossola, Papadopoulos, Pittau (2007); Ellis, Giele, Kunszt, Melnikov (2008)]

$$\mathcal{I}_n = \frac{\mathcal{N}}{D_1 \cdots D_n} = \sum_{j_1 \ldots j_5} \frac{\Delta_{j_1 j_2 j_3 j_4 j_5}}{D_{j_1} D_{j_2} D_{j_3} D_{j_4} D_{j_5}} + \sum_{j_1 j_2 j_3 j_4} \frac{\Delta_{j_1 j_2 j_3 j_4}}{D_{j_1} D_{j_2} D_{j_3} D_{j_4}} + \sum_{j_1 j_2 j_3} \frac{\Delta_{j_1 j_2 j_3}}{D_{j_1} D_{j_2} D_{j_3}} + \sum_{j_1 j_2} \frac{\Delta_{j_1 j_2}}{D_{j_1} D_{j_2}} + \sum_{j_1} \frac{\Delta_{j_1}}{D_{j_1}}$$

- the residues $\Delta_{i_1 \cdots i_k}$
  - are polynomials in the components of $\bar{q}$
  - have a known, universal parametric form
  - are parametrized by unknown, process-dependent coefficients

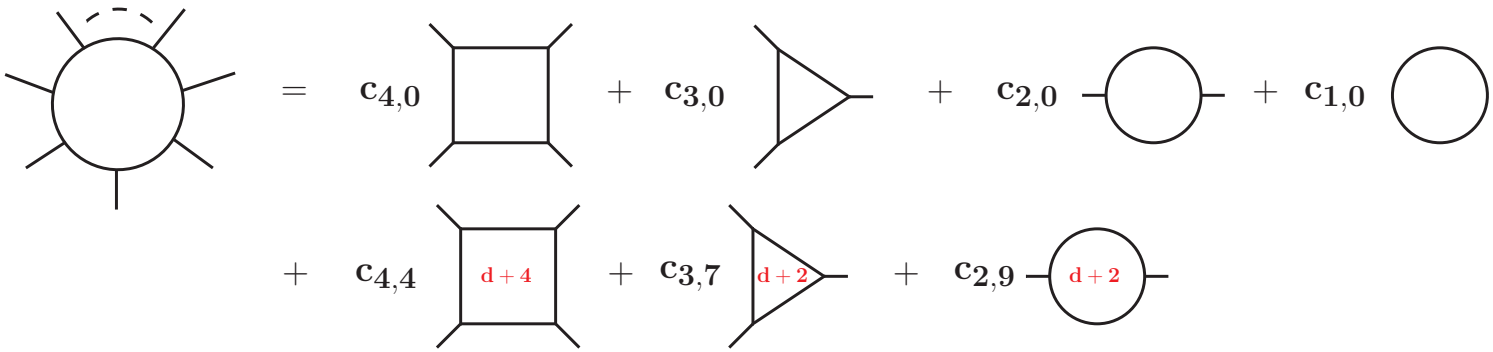
  $\Rightarrow$ can be completely determined with a polynomial fit
- recently extended to higher-loops using multivariate polynomial division techniques [Y. Zhang (2012), P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2012)]
  $\Rightarrow$ see Y. Zhang and P. Mastrolia's talks

The Integrand reduction of one-loop amplitudes

- After integration
  - some terms vanish and do not contribute to the amplitude
    $\Rightarrow$ spurious terms
  - non-vanishing terms give Master Integrals (MIs)
  - the amplitude is a linear combination of known MIs
- The coefficients of this linear combination
  - can be identified with some of the coefficients which parametrize the polynomial residues

  $\Rightarrow$ reduction to MIs $\equiv$ polynomial fit of the residues

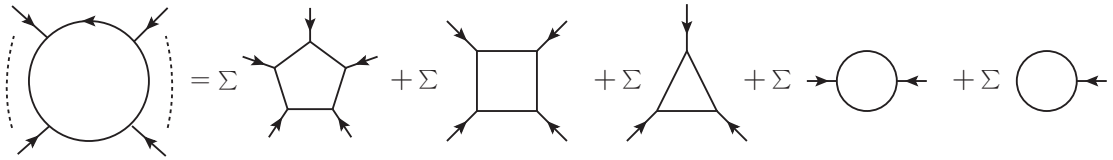
★ any one-loop amplitude can be computed with a polynomial fit

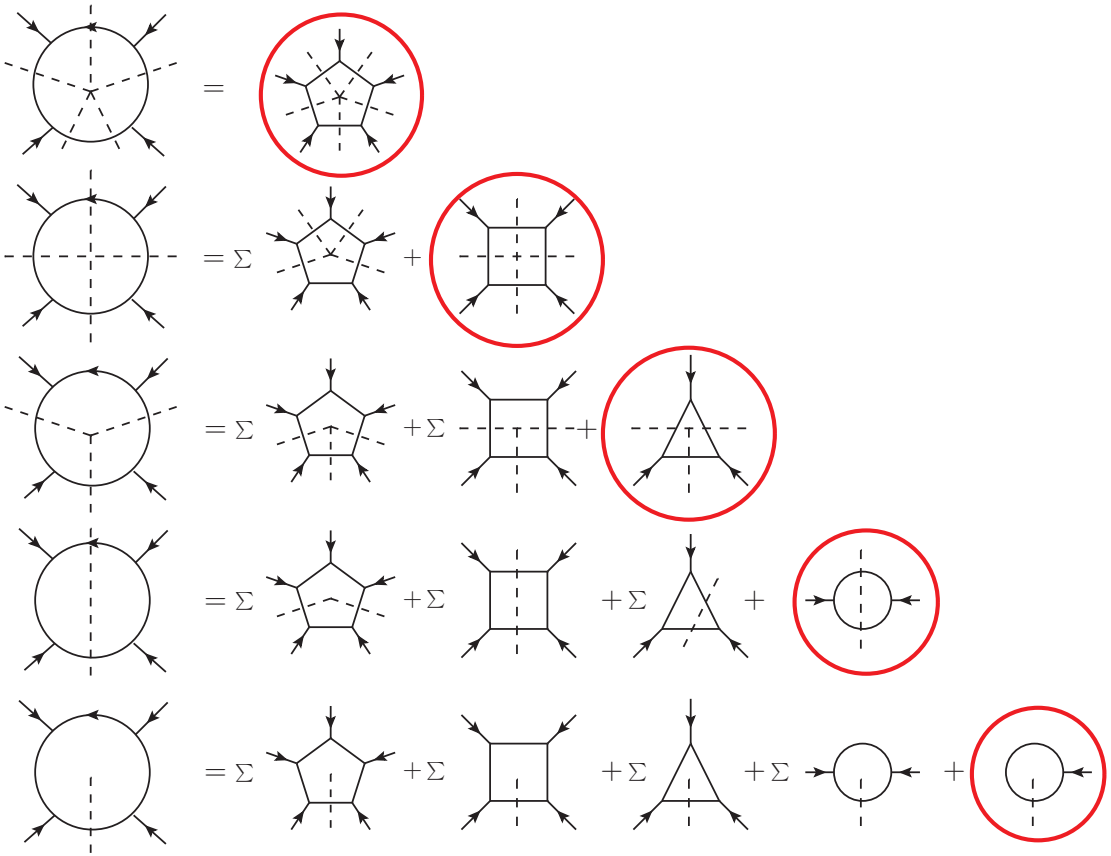


Fit-on-the-cut at one-loop

[Ossola, Papadopoulos, Pittau (2007)]

Integrand decomposition:





Fit-on-the cut

- fit m-point residues on m-ple cuts
- Cutting a loop propagator means

$$\frac{1}{D_i} \to \delta(D_i)$$

i.e. putting it on-shell

Integrand reduction via Laurent expansion (NINJA)

P. Mastrolia, E. Mirabella, T.P. (2012)

The integrand reduction via Laurent expansion:

- fits residues by taking their asymptotic expansions on the cuts
  - elaborating ideas first proposed by Forde and Badger
- simplifications
  - fewer coefficients needed
  - diagonal systems of equations for the coefficients
  - higher-point subtractions as corrections at the coefficient level

Integrand reduction via Laurent expansion (NINJA)

P. Mastrolia, E. Mirabella, T.P. (2012)

The integrand reduction via Laurent expansion:

- fits residues by taking their asymptotic expansions on the cuts
  - elaborating ideas first proposed by Forde and Badger
- simplifications
  - fewer coefficients needed
  - diagonal systems of equations for the coefficients
  - higher-point subtractions as corrections at the coefficient level

★ Implemented in the semi-numerical C++ library NINJA

- Laurent expansions via a simplified polynomial-division algorithm
- interfaced with the package GOSAM
- is a faster and more stable integrand-reduction algorithm

Integrand reduction via Laurent expansion (NINJA)

P. Mastrolia, E. Mirabella, T.P. (2012)

The integrand reduction via Laurent expansion:

- fits residues by taking their asymptotic expansions on the cuts
  - elaborating ideas first proposed by Forde and Badger
- simplifications
  - fewer coefficients needed
  - diagonal systems of equations for the coefficients
  - higher-point subtractions as corrections at the coefficient level

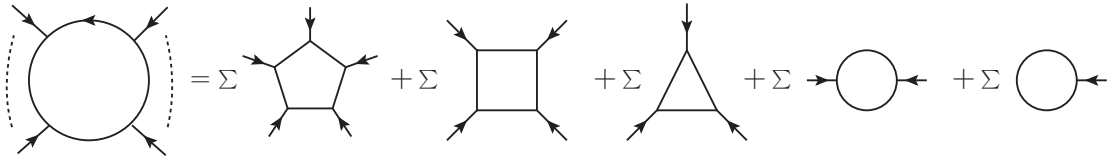
★ Implemented in the semi-numerical C++ library NINJA

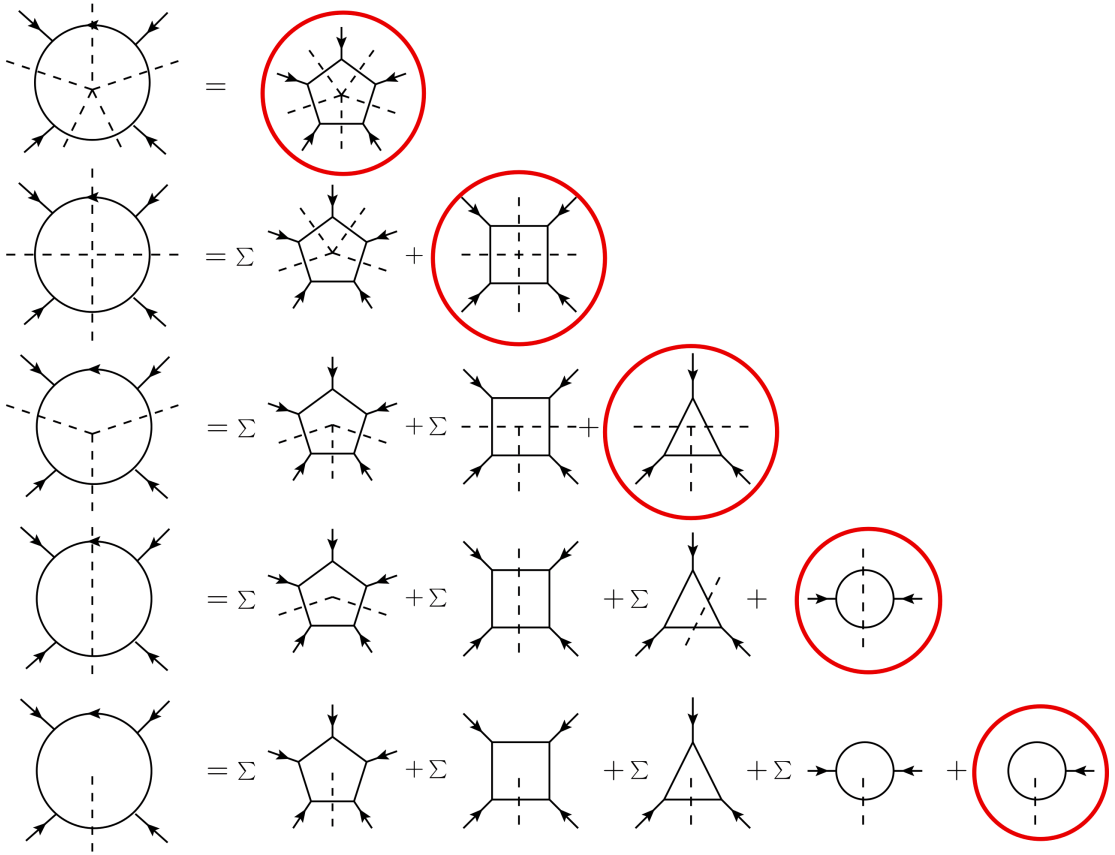
- Laurent expansions via a simplified polynomial-division algorithm
- interfaced with the package GOSAM
- is a faster and more stable integrand-reduction algorithm

★ NINJA is public ⇒ `ninja.hepforge.org`

Integrand reduction via Laurent expansion (NINJA)

Integrand decomposition:

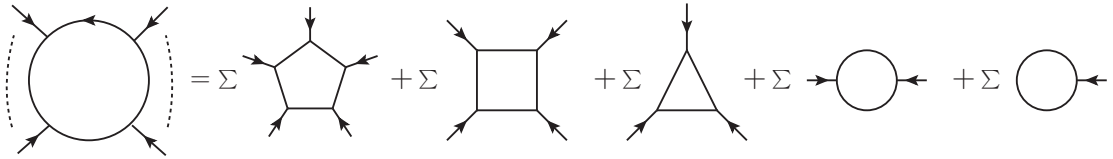


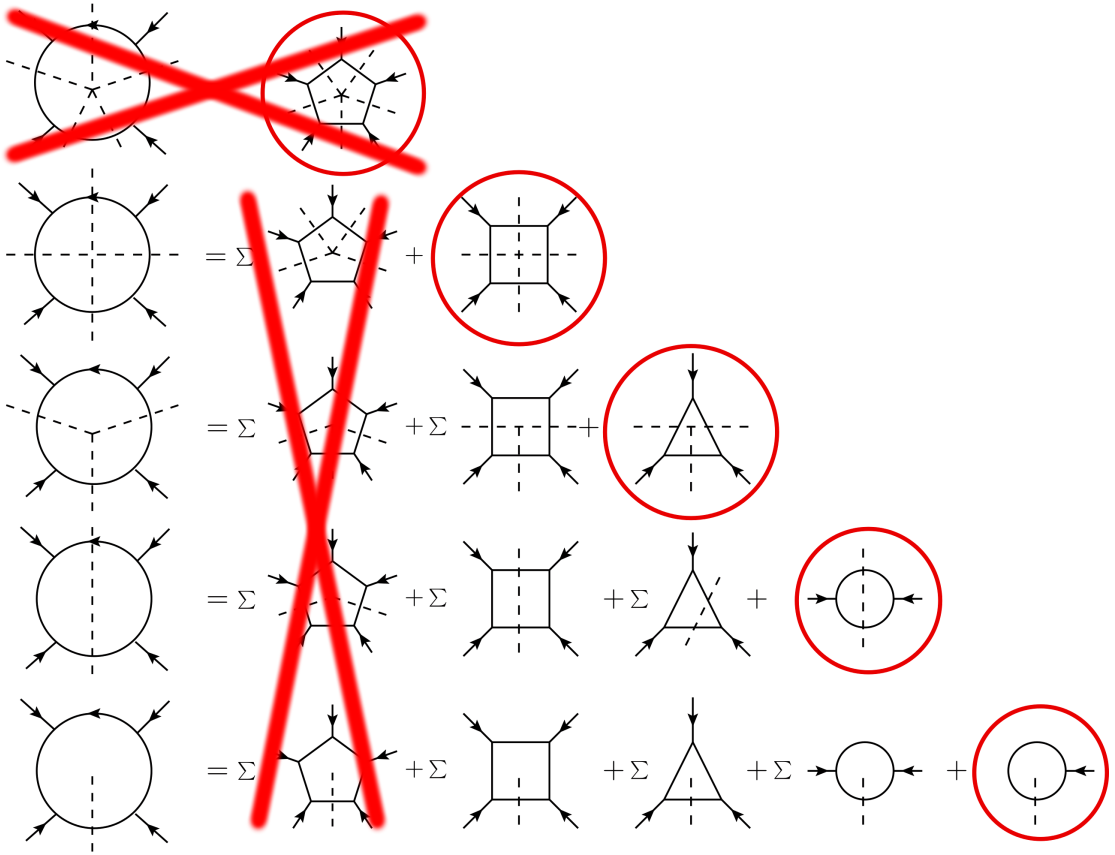


Laurent-expansion method

Integrand reduction via Laurent expansion (NINJA)

Integrand decomposition:



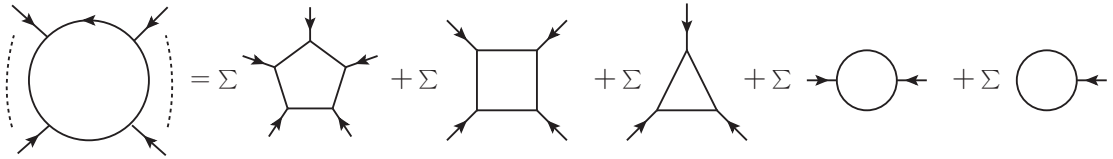


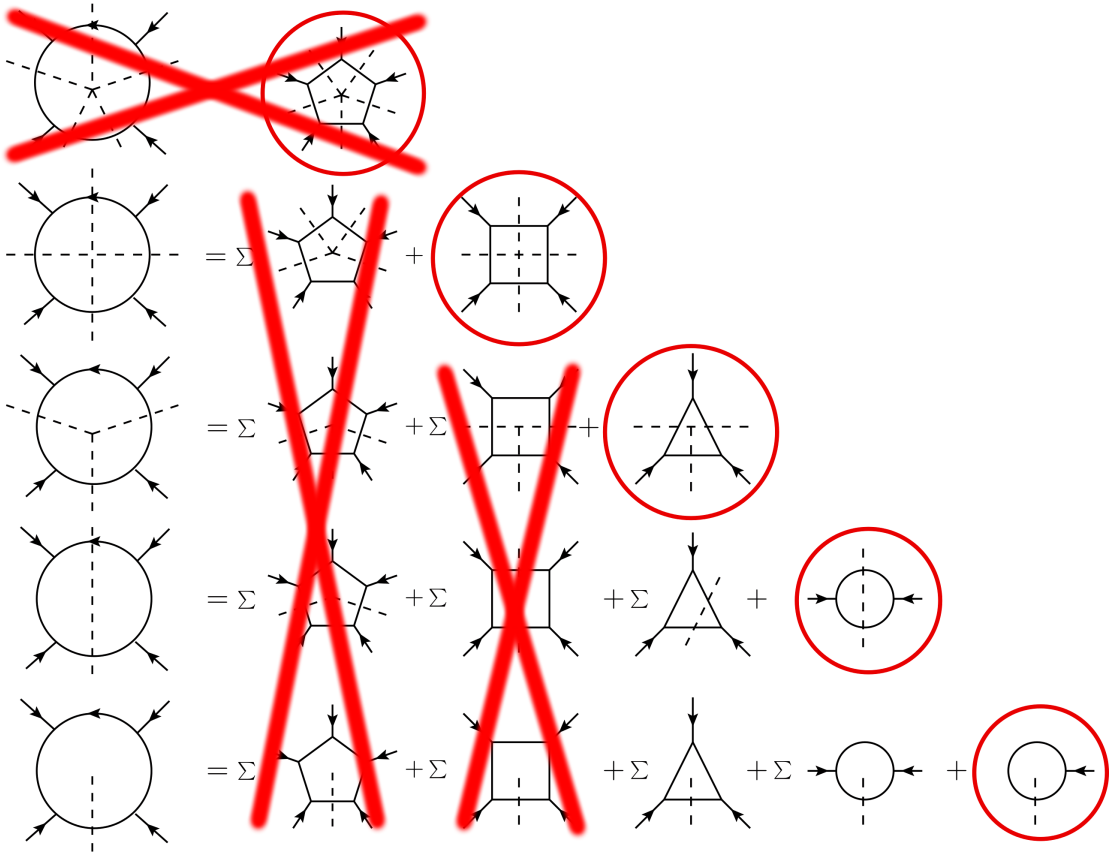
Laurent-expansion method

- pentagons not needed

Integrand reduction via Laurent expansion (NINJA)

Integrand decomposition:



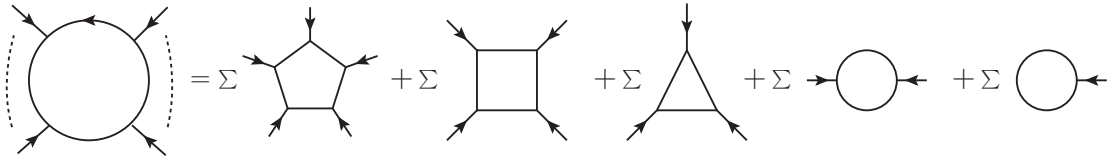


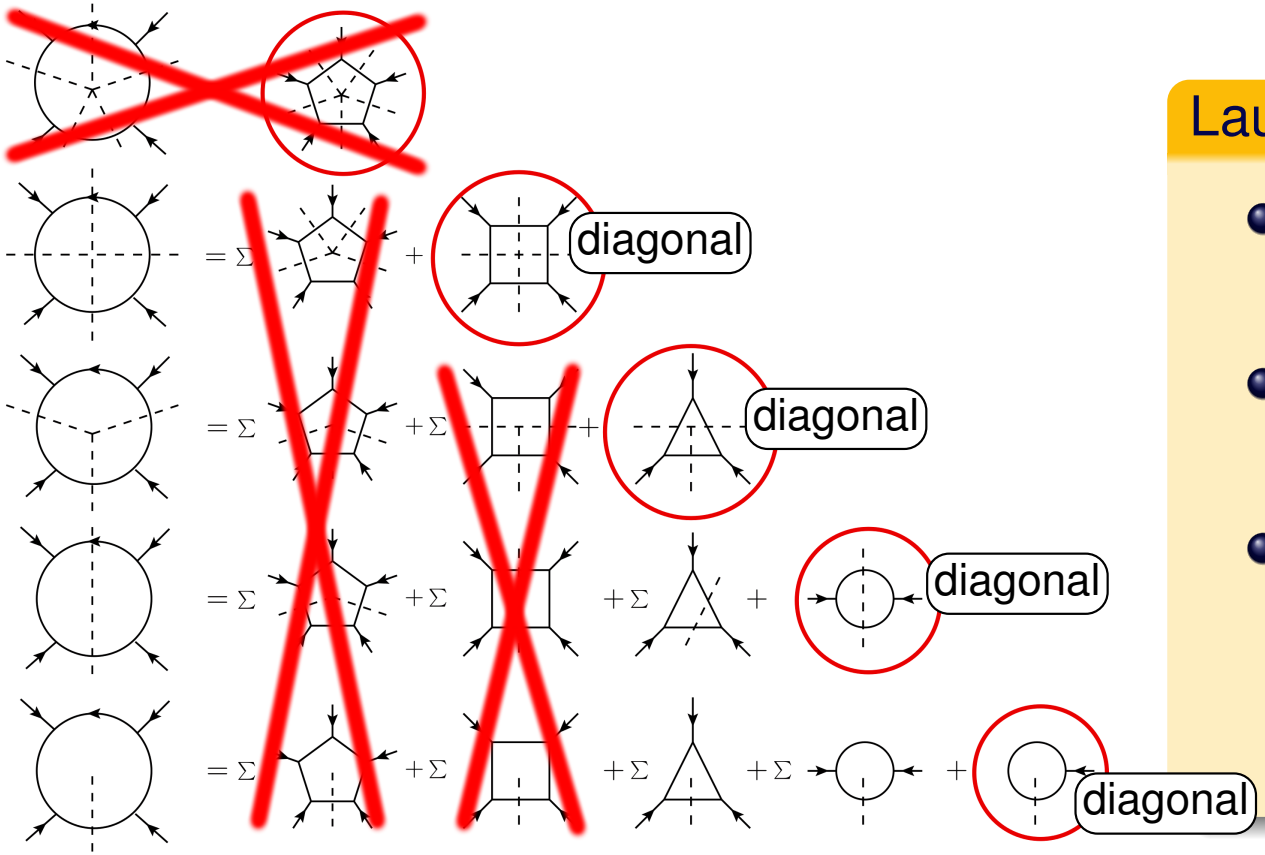
Laurent-expansion method

- pentagons not needed
- boxes never subtracted

Integrand reduction via Laurent expansion (NINJA)

Integrand decomposition:



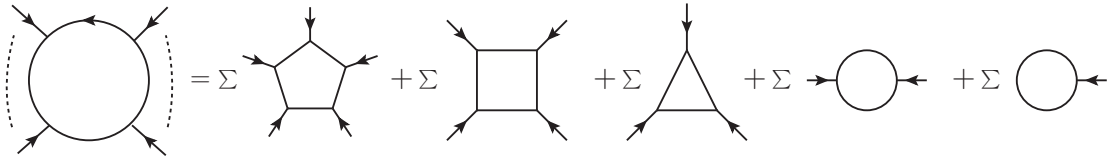


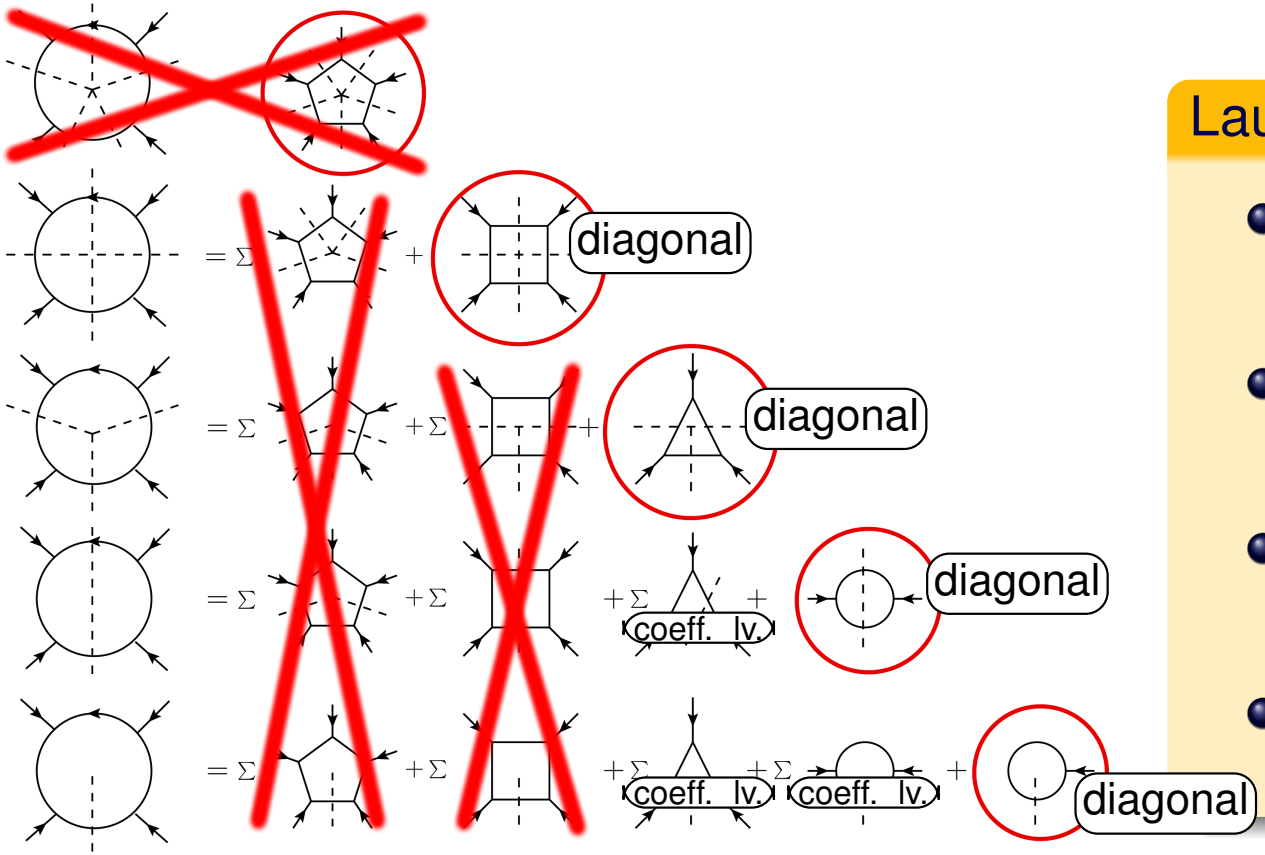
Laurent-expansion method

- pentagons not needed
- boxes never subtracted
- diagonal systems of equations

Integrand reduction via Laurent expansion (NINJA)

Integrand decomposition:





Laurent-expansion method

- pentagons not needed
- boxes never subtracted
- diagonal systems of equations
- subtractions at coefficient level

Semi-numerical implementation in NINJA

- The input is the numerator $\mathcal{N}$ cast in (three or) four different forms
  - leading terms of parametric expansions of the numerator
  - coefficients of the expansion written to an array $\mathcal{N}[\,]$
  - all easily obtained from its analytic expression
- The PYTHON script NINJANUMGEN uses FORM-4 to
  - automatically compute expansions from a FORM expression of $\mathcal{N}$
  - generate optimized source code needed as input for NINJA
- NINJA at run-time
  - computes parametric on-shell solutions
  - performs Laurent expansions via pol. div.
  - implements subtractions at coefficient level
  - multiplies the obtained coefficients with the MIs
- Semi-numeric Laurent expansion via polynomial division
  - expansion of numerator $\mathcal{N}[\,]$ / denominators D_i

Semi-numerical implementation in NINJA

```
// Numerator: can be generated using the script ninjanumgen
class MyNumerator : public ninja::Numerator {
public:

  // evaluates the numerator N(q, μ^2) - same as Samurai
  virtual Complex evaluate(q, μ^2, ...);

  // (optional) expansion for 4-ple cut rational term q^μ → t v_⊥^μ + O(1)
  virtual void muExpansion(v_⊥, ..., Complex N);

  // expansion for triangles and tadpoles q^μ → v_0^μ + t v_3^μ + (β + μ^2)/(2 t) v_4^μ
  virtual void t3Expansion(v_0, v_3, v_4, β, ..., Complex N[]);

  // expansion for bubbles q^μ → v_1^μ + x v_2^μ + t v_3^μ + (β_0 + β_1 x + β_2 x^2 + μ^2)/(2 t) v_4^μ
  virtual void t2Expansion(v_1, v_2, v_3, v_4, β_i, ..., Complex N[]);
};
```

–

note: `t2Expansion` is `t3Expansion` with: $v_0 \to v_1^\mu + x\, v_2^\mu$, $\beta \to \beta_0 + \beta_1 x + \beta_2 x^2$

Semi-numerical implementation in NINJA

Master Integrals:

- are called via a generic interface
  - $\Rightarrow$ any user-defined library of Master Integrals can be used
- the library of MIs to be used can be specified at run time
- NINJA provides the interface for two default libraries
  - ONELOOP library [A. van Hameren] wrapper + caching
    - computed MIs are cached by NINJA
    - constant-time lookup from their arguments
  - LOOPTOOLS library [T. Hahn]
    - an internal cache is already present $\Rightarrow$ interface is a simple wrapper

Higher-rank:

- support for higher-rank $r = n + 1$
- higher-rank MIs (can but) do not need to be provided

Automation of one-loop computation in GOSAM

GOSAM is a PYTHON package which:

- generates analytic integrands
  - using QGRAF [P. Nogueira] and FORM [J. Vermaseren et al.]
- writes them into FORTRAN90 code
- can use different reduction algorithms at run-time
  - SAMURAI (d-dim. integrand reduction)
    - faster than GOLEM95 but numerically less stable
    - former default in GOSAM-1.0
  - GOLEM95 (tensor reduction)
    - slower than SAMURAI but more stable
    - default rescue-system for unstable points
  - NINJA
    - fast (2 to 5 times faster than SAMURAI)
    - stable (in worst cases $\mathcal{O}(1/1000)$ unstable points)
    - current default in GOSAM-2.0 $\leftarrow$ just released (see G. Heinrich's talk)

Benchmarks of GOSAM + NINJA

H. van Deurzen, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola and T.P. (2013)

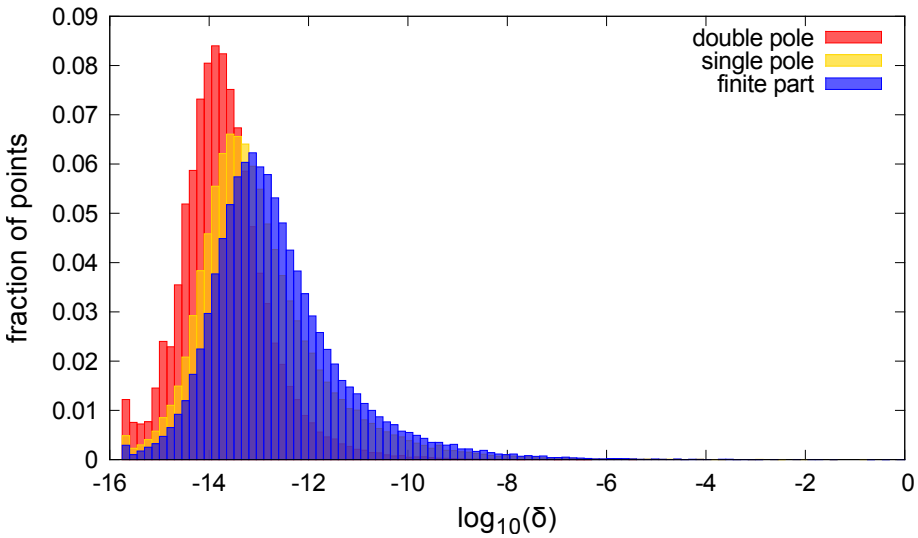
Benchmarks: GOSAM + NINJA			
Process		# NLO diagrams	ms/event[a]
$W+3j$	$d\bar{u} \to \bar{\nu}_e e^- ggg$	1 411	226
$Z+3j$	$d\bar{d} \to e^+ e^- ggg$	2 928	1 911
$t\bar{t}b\bar{b}\ (m_b \neq 0)$	$d\bar{d} \to t\bar{t}b\bar{b}$	275	178
	$gg \to t\bar{t}b\bar{b}$	1 530	5 685
$t\bar{t}+2j$	$gg \to t\bar{t}gg$	4 700	13 827
$W\,b\,\bar{b}+1\,j\ (m_b \neq 0)$	$u\bar{d} \to e^+ \nu_e b\bar{b}g$	312	67
$W\,b\,\bar{b}+2\,j\ (m_b \neq 0)$	$u\bar{d} \to e^+ \nu_e b\bar{b}s\bar{s}$	648	181
	$u\bar{d} \to e^+ \nu_e b\bar{b}d\bar{d}$	1 220	895
	$u\bar{d} \to e^+ \nu_e b\bar{b}gg$	3 923	5 387
$H+3\,j$ in GF	$gg \to Hggg$	9 325	8 961
$t\,\bar{t}\,H+1\,j$	$gg \to t\bar{t}Hg$	1 517	1 505
$H+3\,j$ in VBF	$u\bar{u} \to Hgu\bar{u}$	432	101
$H+4\,j$ in VBF	$u\bar{u} \to Hggu\bar{u}$	1 176	669
$H+5\,j$ in VBF	$u\bar{u} \to Hgggu\bar{u}$	15 036	29 200

more processes in arXiv:1312.6678

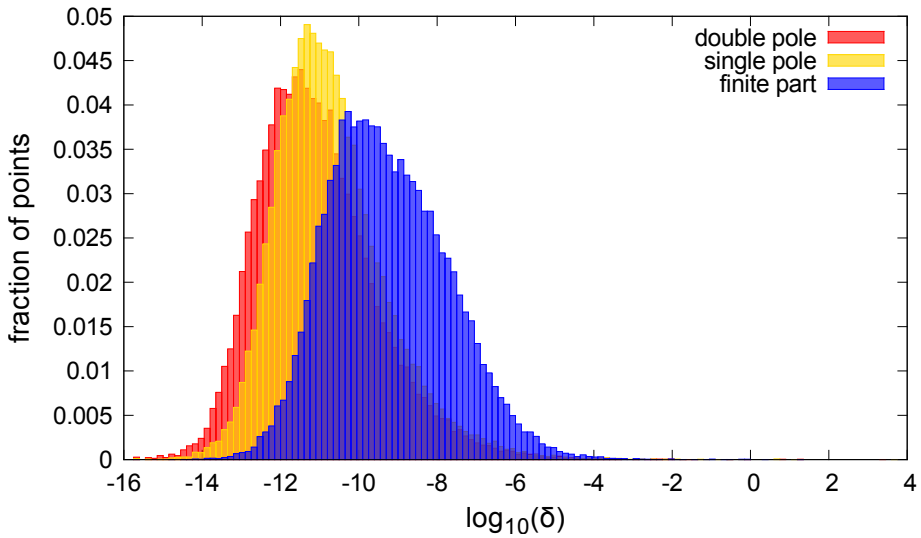
[a] Timings refer to full color- and helicity-summed amplitudes, using an Intel Core i7 CPU @ 3.40GHz, compiled with `ifort`.

Stability of NINJA

- $H + 4j$ in VBF ($u\bar{u} \to Hgg u\bar{u}$)



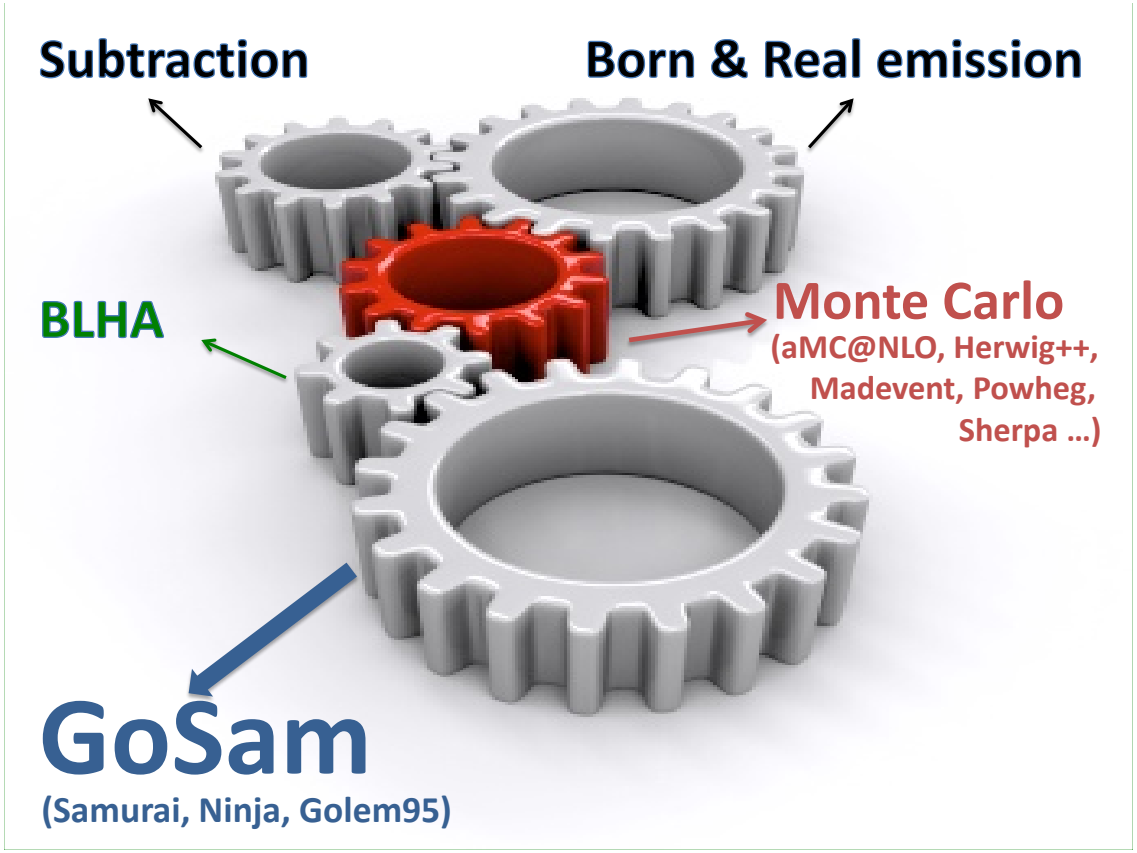
- $t\bar{t}H + 1j$ ($gg \to t\bar{t}Hg$)



Rate of unstable points, i.e. with error $\delta > \delta_{\text{threshold}}$ on the finite part:

$\delta_{\text{threshold}}$	$u\bar{u} \to Hgg u\bar{u}$	$gg \to t\bar{t}Hg$
10^{-3}	0.02%	0.06%
10^{-4}	0.04%	0.16%
10^{-5}	0.08%	0.56%

From amplitudes to observables with GOSAM



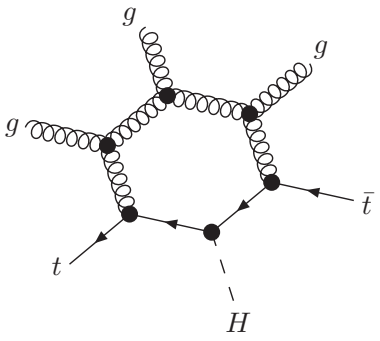
The GOSAM collaboration:

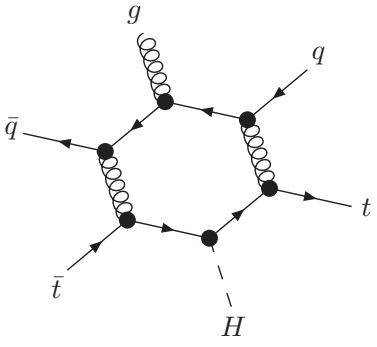
G. Cullen, H. van Deurzen, N. Greiner, G. Heinrich, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, J. Reichel , J. Schlenk, J. F. von Soden-Fraunhofen, T. Reiter, F. Tramontano, T.P.

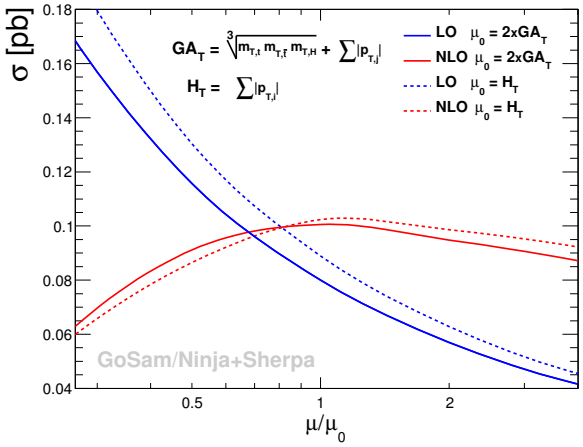
Application: $pp \to t\bar{t}H + jet$ with GOSAM + NINJA

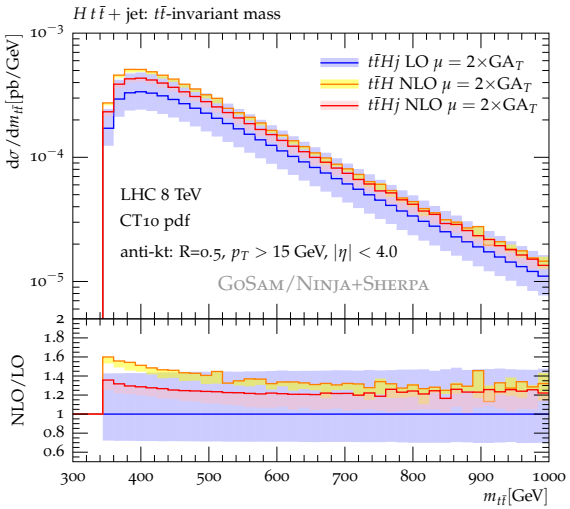
H. van Deurzen, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2013)

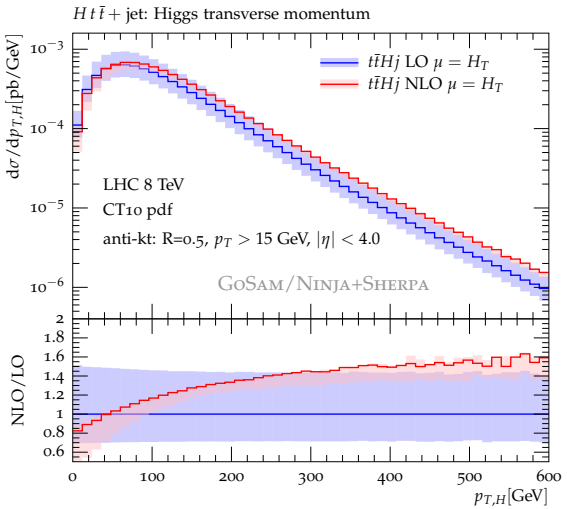
- Interfaced with the Monte Carlo SHERPA

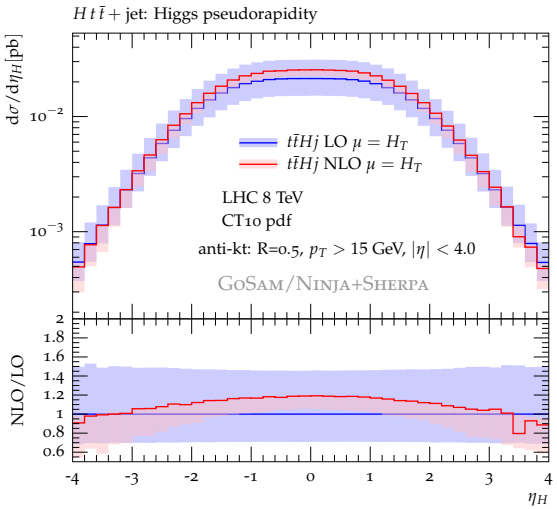






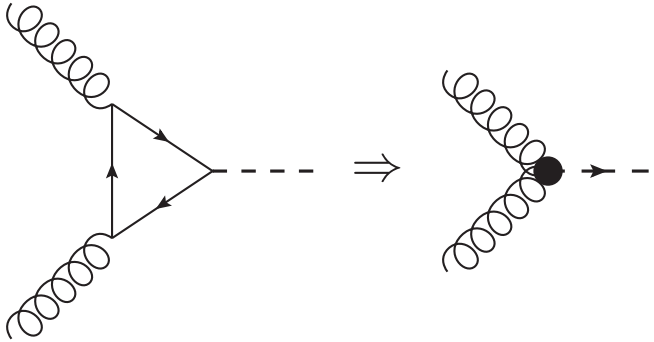






Application: $pp \to H + jets$ in GF with GOSAM + NINJA

- $m_t \to \infty$ approximation



- effective couplings $H + (2,3,4)gl$.
- higher-rank integrands $\Rightarrow$ extension of int. red. methods
  [P. Mastrolia, E. Mirabella,T.P.(2012), H. van Deurzen (2013)]

- $H + 2j$ (GOSAM+SAMURAI+SHERPA)
  [H. van Deurzen, N. Greiner, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, J. F. von Soden-Fraunhofen, F. Tramontano, T.P.(2013)]
- $H + 3j$ (GOSAM+SAMURAI+SHERPA+MADGRAPH4/MADEVENT)
  [G. Cullen, H. van Deurzen, N. Greiner, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, F. Tramontano, T.P.(2013)]
- new analysis with ATLAS-like cuts, using NINJA for the reduction
  [G. Cullen, H. van Deurzen, N. Greiner, J. Huston, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, F. Tramontano, J. Winter, V. Yundin, T.P. (preliminary, 2014), N. Greiner, J. Huston, G. Luisoni, J. Winter, V. Yundin (work in progress)]

Application: $pp \to H + jets$ in GF with GOSAM + NINJA

- new distributions using NINJA (preliminary)
  - better accuracy
  - better performance

$$\mu_F = \mu_R = \frac{\hat{H}_T}{2} = \frac{1}{2}\left(\sqrt{m_H^2 + p_{t,H}^2} + \sum_{jets} |p_{t,jet}|^2\right)$$

- ATLAS-like cuts

$$R = 0.4, \qquad p_{t,jet} > 30\text{GeV}, \qquad |\eta_{jet}| < 4.4$$

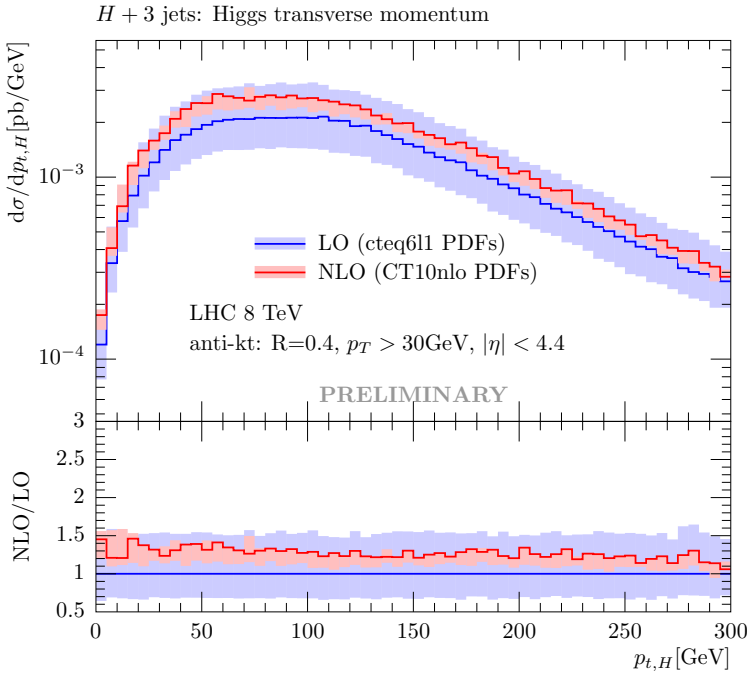
- total cross section

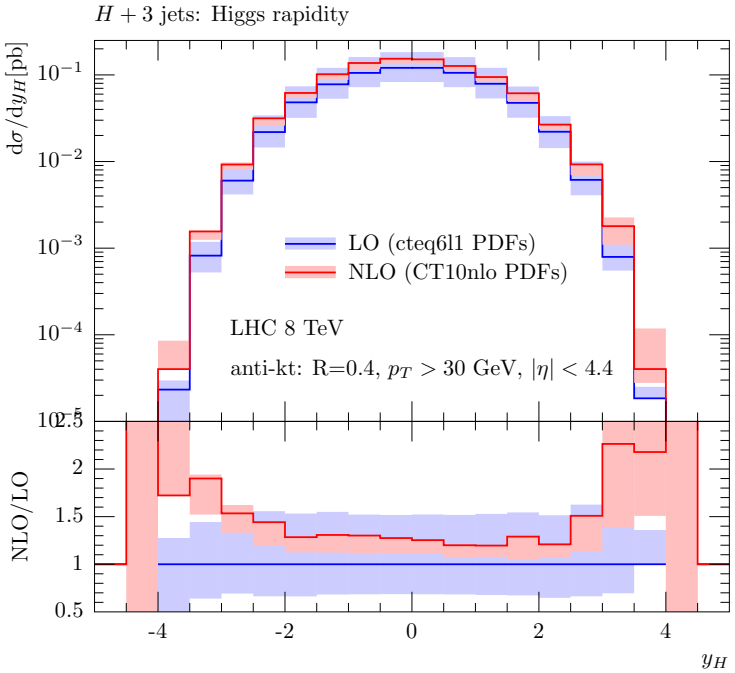
$$\sigma_{LO}^{(H+2j)}([\text{pb}]) = 1.23^{+37\%}_{-24\%}, \qquad \sigma_{LO}^{(H+3j)}([\text{pb}]) = 0.381^{+53\%}_{-32\%}$$
$$\sigma_{NLO}^{(H+2j)}([\text{pb}]) = 1.590^{-4\%}_{-7\%}, \qquad \sigma_{NLO}^{(H+3j)}([\text{pb}]) = 0.485^{-3\%}_{-13\%}$$

Application: $pp \to H + jets$ in GF with GOSAM + NINJA

- new distributions using NINJA (preliminary)
  - better accuracy
  - better performance

$$\mu_F = \mu_R = \frac{\hat{H}_T}{2} = \frac{1}{2}\left(\sqrt{m_H^2 + p_{t,H}^2} + \sum_{jets} |p_{t,jet}|^2\right)$$

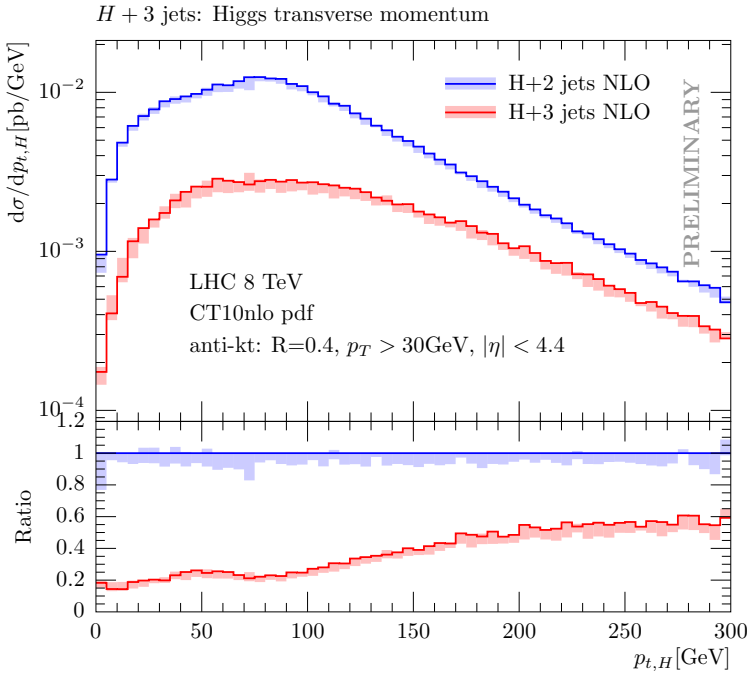


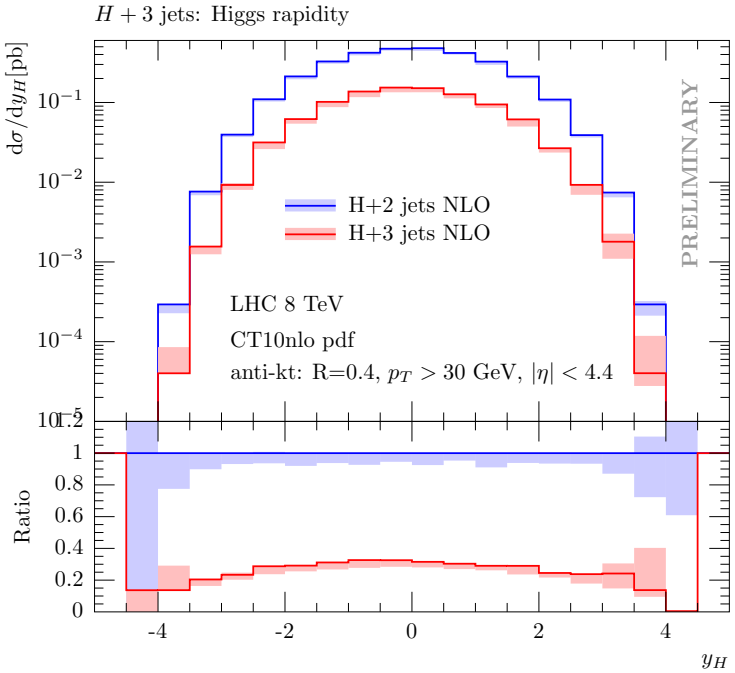


Application: $pp \to H + jets$ in GF with GOSAM + NINJA

- new distributions using NINJA (preliminary)
  - better accuracy
  - better performance

$$\mu_F = \mu_R = \frac{\hat{H}_T}{2} = \frac{1}{2}\left(\sqrt{m_H^2 + p_{t,H}^2} + \sum_{jets} |p_{t,jet}|^2\right)$$





Summary and Outlook

- Summary of NINJA
  - implements the Integrand Reduction via Laurent expansion method
  - has good performance and stability
  - is already producing phenomenological results with GOSAM
  - is public, both as standalone library and within GOSAM-2.0

- Outlook
  - treatment of (few) remaining unstable points within NINJA
  - alternative approach: fully automated algebraic one-loop

THANK YOU
FOR YOUR ATTENTION

BACKUP SLIDES

- Choice of 4-dimensional basis for an m-point residue

$$e_1^2 = e_2^2 = 0\,, \qquad e_1 \cdot e_2 = 1\,, \qquad e_3^2 = e_4^2 = \delta_{m4}\,, \qquad e_3 \cdot e_4 = -(1-\delta_{m4})$$

- Coordinates: $\mathbf{z} = (z_1, z_2, z_3, z_4, z_5) \equiv (x_1, x_2, x_3, x_4, \mu^2)$

$$q^\mu_{\text{4-dim}} = -p^\mu_{i_1} + x_1\, e_1^\mu + x_2\, e_2^\mu + x_3\, e_3^\mu + x_4\, e_4^\mu\,, \qquad \bar{q}^2 = q^2_{\text{4-dim}} - \mu^2$$

- Generic numerator

$$\mathcal{N} = \sum_{j_1,\ldots,j_5} \alpha_{\vec{j}}\, z_1^{j_1}\, z_2^{j_2}\, z_3^{j_3}\, z_4^{j_4}\, z_5^{j_5}\,, \qquad (j_1 \ldots j_5) \quad \text{such that} \quad \text{rank}(\mathcal{N}) \leq \#\text{ loop-denom.}$$

- Residues

$$\begin{aligned}
\Delta_{i_1 i_2 i_3 i_4 i_5} &= c_0\, \mu^2 \\
\Delta_{i_1 i_2 i_3 i_4} &= c_0 + c_1 x_4 + \mu^2 (c_2 + c_3 x_4 + \mu^2 c_4) \\
\Delta_{i_1 i_2 i_3} &= c_0 + c_1 x_3 + c_2 x_3^2 + c_3 x_3^3 + c_4 x_4 + c_5 x_4^2 + c_6 x_4^3 + \mu^2 (c_7 + c_8 x_3 + c_9 x_4) \\
\Delta_{i_1 i_2} &= c_0 + c_1 x_2 + c_2 x_3 + c_3 x_4 + c_4 x_2^2 + c_5 x_3^2 + c_6 x_4^2 + c_7 x_2 x_3 + c_9 x_2 x_4 + c_9 \mu^2 \\
\Delta_{i_1} &= c_0 + c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4
\end{aligned}$$

- It can be easily extended to higher-rank numerators

One-loop boxes via Laurent expansion

- The residue of a box reads

$$\Delta_{ijkl}(q,\mu^2) = d_0 + d_2\mu^2 + d_4\,\mu^4 + (d_1 + d_3\mu^2)(q\cdot v_\perp)$$

- d_0 via 4-dimensional 4ple cuts [Britto, Cachazo, Feng (2004)]
- d_4 from d-dimensional 4-ple cuts in the limit $\mu^2 \to \infty$ [S. Badger (2008)]
  - d-dimensional solutions of a 4-ple cut

$$q_\pm = a^\mu \pm \sqrt{\alpha + \frac{\mu^2}{\beta^2}}\; v_\perp^\mu = \pm\frac{\sqrt{\mu^2}}{\beta}\; v_\perp^\mu + \mathcal{O}(1)$$

  - the integrand in the asymptotic limit $\mu^2 \to \infty$ of the cut-solutions

$$\left.\frac{\mathcal{N}(q,\mu^2)}{\prod_{m\neq i,j,k,l} D_m}\right|_{\text{cut}} = d_4\,\mu^4 + \mathcal{O}(\mu^3)$$

- d_1, d_2, d_3 are spurious and do not need to be computed

One-loop triangles via Laurent expansion

- The residue of a triangle

$$\Delta_{ijk}(q) = c_0 + c_7\,\mu^2 + (c_1 + c_8\mu^2)\,(q\cdot e_3) + c_2\,(q\cdot e_3)^2 + c_3\,(q\cdot e_3)^3 \\ + (c_4 + c_9\mu^2)\,(q\cdot e_4) + c_5\,(q\cdot e_4)^2 + c_6\,(q\cdot e_4)^3$$

- solutions of a triple cut $D_i = D_j = D_k = 0$ parametrized by the free variables t and μ^2

$$q_+^\mu = a^\mu + t\,e_3^\mu + \frac{\alpha+\mu^2}{2\,t}\,e_4^\mu, \qquad q_-^\mu = a^\mu + \frac{\alpha+\mu^2}{2\,t}\,e_3^\mu + t\,e_4^\mu$$

- in the limit $t \to \infty$ [Forde (2007)]

$$\left.\frac{\mathcal{N}(q_\pm)}{\prod_{m\neq i,j,k} D_m}\right|_{\text{cut}} = \Delta_{ijk} + \sum_l \frac{\Delta_{ijkl}}{D_l} + \sum_{lm} \frac{\Delta_{ijklm}}{D_l D_m} \\ = \Delta_{ijk} + d_1^\pm + d_2^\pm\,\mu^2 + \mathcal{O}(1/t)$$

with $d_i^+ + d_i^- = 0$

One-loop triangles via Laurent expansion

- In the asymptotic limit $t \to \infty$

$$\left.\frac{\mathcal{N}(q_\pm)}{\prod_{m\neq i,j,k} D_m}\right|_{\text{cut}} = (d_1^\pm + d_2^\pm\,\mu^2) + \Delta_{ijk} + \mathcal{O}(1/t) \qquad \text{with } d_i^+ + d_i^- = 0$$

  - the integrand

$$\left.\frac{\mathcal{N}(q_\pm)}{\prod_{m\neq i,j,k} D_m}\right|_{\text{cut}} = n_0^\pm + n_4^\pm\,\mu^2 + (n_1^\pm + n_5^\pm\,\mu^2)\,t + n_2^\pm\,t^2 + n_3^\pm\,t^3 + \mathcal{O}(1/t)$$

  - the residue

$$\Delta_{ijk}(q_+) = c_0 + c_7\,\mu^2 - (c_4 + c_9\,\mu^2)\,t + c_5\,t^2 - c_6\,t^3 + \mathcal{O}(1/t)$$
$$\Delta_{ijk}(q_-) = c_0 + c_7\,\mu^2 - (c_1 + c_8\,\mu^2)\,t + c_2\,t^2 - c_3\,t^3 + \mathcal{O}(1/t)$$

  - by comparison we get

$$c_0 = \frac{n_0^+ + n_0^-}{2}, \quad c_1 = -n_1^-, \quad c_2 = n_2^-, \quad c_3 = -n_3^-, \quad \ldots$$

One-loop bubbles via Laurent expansion

- The residue of a bubble

$$\Delta_{ij}(q) = b_0 + b_1\,(q\cdot e_2) + b_2\,(q\cdot e_2)^2 + b_3\,(q\cdot e_3) + b_4\,(q\cdot e_3)^2 + b_5\,(q\cdot e_4) + b_6\,(q\cdot e_4)^2 + b_7\,(q\cdot e_2)(q\cdot e_3) + b_8\,(q\cdot e_2)(q\cdot e_4) + b_9\,\mu^2$$

- solutions of a double cut $D_i = D_j = 0$, parametrized by the free variables t, x and μ^2

$$q_+ = x\,e_1 + (\alpha_0 + x\,\alpha_1)e_2 + t\,e_3 + \frac{\beta_0 + \beta_1 x + \beta_2 x^2 + \mu^2}{2\,t}\,e_4$$

$$q_- = x\,e_1 + (\alpha_0 + x\,\alpha_1)e_2 + \frac{\beta_0 + \beta_1 x + \beta_2 x^2 + \mu^2}{2\,t}\,e_3 + t\,e_4$$

- in the limit $t \to \infty$

$$\left.\frac{\mathcal{N}(q_\pm)}{\prod_{m\neq i,j} D_m}\right|_{\text{cut}} = \Delta_{ij} + \sum_k \frac{\Delta_{ijk}}{D_k} + \sum_{kl} \frac{\Delta_{ijkl}}{D_k D_l} + \sum_{klm} \frac{\Delta_{ijklm}}{D_k D_l D_m}$$

$$= \Delta_{ij} + \sum_k \frac{\Delta_{ijk}}{D_k} + \mathcal{O}(1/t)$$

One-loop bubbles via Laurent expansion

- In the asymptotic limit $t \to \infty$
  - the integrand

$$\left.\frac{\mathcal{N}(q_\pm)}{\prod_{m\neq i,j,k} D_m}\right|_{\text{cut}} = n_0^\pm + n_6^\pm\,\mu^2 + n_1^\pm\,x + n_2^\pm\,x^2 + \left(n_3^\pm + n_4^\pm x\right)t + n_5^\pm\,t^2 + \mathcal{O}(1/t)$$

  - the subtraction term

$$\frac{\Delta_{ijk}(q_\pm)}{D_k} = \tilde{b}_0^{k,\pm} + \tilde{b}_6^{k,\pm}\,\mu^2 + \tilde{b}_1^{k,\pm}\,x + \tilde{b}_2^{k,\pm}\,x^2 + \left(\tilde{b}_3^{k,\pm} + \tilde{b}_4^{k,\pm}x\right)t + \tilde{b}_5^{k,\pm}\,t^2 + \mathcal{O}(1/t)$$

    - $\tilde{b}_i^{k,\pm}$ are known functions of the triangle coefficients
  - the residue

$$\Delta_{ij}(q_+) = b_0 + b_9\,\mu^2 + b_1\,x + b_2x^2 - \left(b_5 + b_8x\right)t + b_6\,t^2 + \mathcal{O}(1/t)$$

$$\Delta_{ij}(q_-) = b_0 + b_9\,\mu^2 + b_1\,x + b_2x^2 - \left(b_3 + b_7x\right)t + b_4\,t^2 + \mathcal{O}(1/t)$$

  - by comparison, applying subtractions at the coefficient level

$$b_0 = n_0^\pm - \sum_k \tilde{b}_0^{k,\pm}, \quad b_1 = n_1^\pm - \sum_k \tilde{b}_1^{k,\pm}, \quad b_3 = -n_3^- + \sum_k \tilde{b}_3^{k,-}, \quad \ldots$$

Rotation method for error estimation

H. van Deurzen, G. Luisoni, P. Mastrolia, E. Mirabella, G. Ossola, T.P. (2013)

- Definitions

$$
\begin{aligned}
A &: \quad \text{numerical result for the amplitude} \\
A_{rot} &: \quad \text{numerical result for the amplitude with rotated kinematics} \\
A_{ex} &: \quad \text{exact result for the amplitude} \sim \text{amplitude in quad. prec.}
\end{aligned}
$$

- the exact error is defined as

$$\delta_{ex} = \left| \frac{A_{ex} - A}{A_{ex}} \right|$$

- the estimated error is defined as

$$\delta_{rot} = 2 \left| \frac{A_{rot} - A}{A_{rot} + A} \right|$$

- one can check that $\delta_{rot} \sim \delta_{ex}$

Rotation method for error estimation

A validation of the rotation method

- example: $W\,b\,\bar{b} + 1\,j$ $(u\bar{d} \to e^+\nu_e b\bar{b}g)$, with $m_b \neq 0$