PAUL SCHERRER INSTITUT

RG-improved fully differential cross sections for top-pair production

Alessandro Broggio

work in progress with Andrew Papanastasiou (DESY) and Adrian Signer (PSI)

LoopFest, 19 June 2014

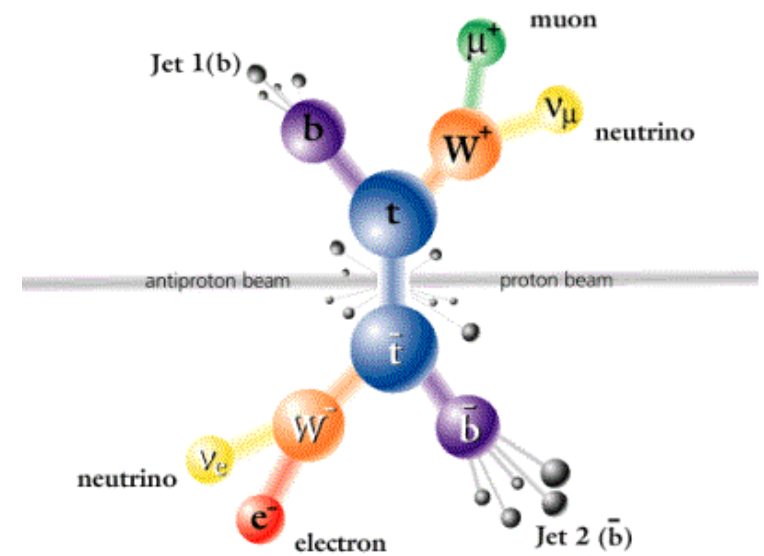
Outline

- Introduction: why fully differential predictions are important?
- What is known: SCET resummation framework for stable tops, PIM and 1PI kinematics
- Include decay and implement known higher order corrections in a parton level MC
- Results: distributions for the LHC at 8 TeV

Why improved differential predictions?

LHC experiments:

- measure of differential cross sections to test theory predictions
- top quarks are not directly detected, but reconstructed from their decay products
- top decays nearly exclusively $t \to W^+ b$
- realistic cuts on leptons-jets-met in the final state



State of the art predictions for top-pair production at hadron colliders:

Stable tops (inclusive):

- NNLO+NNLL (σ_{tot}) [Bärnreuther, Czakon, Fiedler, Mitov '12, '13]
- NLO+NNLL (Approx-NNLO) $\left(\frac{d^2\sigma}{dM_{t\bar{t}}d\cos\theta}, \frac{d^2\sigma}{dp_T dy}\right)$ [Kidonakis, Laenen, Moch, Vogt '01] [Ahrens, Ferroglia, Neubert, Pecjak, Yang '10, '11]

Unstable tops (exclusive):

- NLO: On-shell top-pair production with decay [Bernreuther et al., '04, Melnikov & Schulze, '09, Ellis & Campbell '12]
- NLO $W^+W^-b\bar{b}$ [Bevilacqua et al. '11; Denner et al. '11 '12; Frederix '13; Cascioli et al. '13]

Is it possible to improve fixed-order NLO predictions for unstable top-pair production?

Improvement at the production level

- It is possible to compute higher order contributions in perturbation theory using the knowledge of lower orders by solving RGEs
- These terms capture an important part of the higher order correction

Stable top-pair: approx-NNLO predictions (from NNLL resummation formula) for the $M_{t\bar{t}},\, p_T,\, y$ were obtained by [Ahrens, Ferroglia, Neubert, Pecjak, Yang '10, '11] in PIM and 1PI kinematics (using SCET methods)

Idea: "improve" the weights of the events (in parton-level MC) by including approx-NNLO corrections for the production subprocess and use these to look at other distributions!

- adapt and include these corrections in a fully differential framework
- inclusion of top decay in NWA

PIM & 1PI kinematics

The Pair Invariant Mass kinematics (PIM)

$$N_1(P_1) + N_2(P_2) \rightarrow (t + \bar{t})(p_t + p_{\bar{t}}) + X(p_X)$$

$$M_{t\bar{t}} = (p_t + p_{\bar{t}})^2$$

to study the invariant mass distribution

$$\boxed{\frac{d^2\sigma}{dM\, d\cos\theta}}$$

$$(1-z) = 1 - \frac{M_{t\bar{t}}^2}{s} \rightarrow 0$$

Soft gluon Energy $\quad E_s = \frac{(1-z)M_{t\bar{t}}}{2\sqrt{z}}$

One Particle Inclusive kinematics (1PI)

$$N_1(P_1) + N_2(P_2) \rightarrow t(p_t) + (\bar{t} + X)(p_{\bar{t}} + p_X)$$

$$s_4 = (p_{\bar{t}} + p_X)^2 - m_t^2$$

to study the transverse momentum and rapidity distributions

$$\boxed{\frac{d^2\sigma}{dp_T dy}}$$

$$s_4 \rightarrow 0$$

Soft gluon Energy $\quad E_s = \frac{s_4}{\left(2\sqrt{m_t^2 + s_4}\right)}$

PIM & 1PI factorization

Factorization of the cross sections studied in these limits by
QCD: [Kidonakis, Laenen, Moch, Sterman,…], SCET: [Ahrens, Ferroglia, Neubert, Pecjak, Yang, '10, '11]

PIM

$$\frac{d^2\hat{\sigma}}{dM\,d\cos\theta} = \frac{\pi\beta_t}{sM}\sum_{i,j} C_{\mathrm{PIM},\,ij}(z, M, m_t, \cos\theta, \mu_f)$$

$$C_{\mathrm{PIM},\,ij}(z, M, m_t, \cos\theta, \mu_f) = \mathrm{Tr}\left[\mathbf{H}_{ij}(M, m_t, \cos\theta, \mu_f)\,\mathbf{S}_{\mathrm{PIM},\,ij}(\sqrt{s}(1-z), M, m_t, \cos\theta, \mu_f)\right]$$

$$P_m(z) = \left[\frac{\ln^m(1-z)}{1-z}\right]_+ \;; \quad m = 0, \ldots, 2n-1$$

1PI

$$\frac{d^2\hat{\sigma}}{dp_T dy} = \frac{2\pi p_T}{s}\sum_{i,j} C_{1\mathrm{PI},\,ij}(s_4, s, t_1, u_1, m_t, \mu_f)$$

$$C_{1\mathrm{PI},\,ij}(s_4, s, t_1, u_1, m_t, \mu_f) = \mathrm{Tr}\left[\mathbf{H}_{ij}(s, t_1, u_1, m_t, \mu_f)\,\mathbf{S}_{1\mathrm{PI},\,ij}(s_4, s, t_1, u_1, m_t, \mu_f)\right]$$

$$\bar{P}_m(s_4) = \left[\frac{\ln^m(s_4/m_t^2)}{s_4}\right]_+ = \frac{1}{m_t^2}P_m\left(1 - \frac{s_4}{m_t^2}\right) \;; \quad m = 0, \ldots, 2n-1$$

- **H** and **S** satisfy RG equations
- By knowing **H** and **S** at NLO in both kinematics, we can solve explicitly the RG equations for **H** and **S** at NNLO

Adding the top decay

- On-shell top-quarks decayed in NWA
- Corrections to the decay are included only at fixed order (LO/NLO)

Factorization of amplitudes:

$$\mathcal{M}_{ij}^{\{\lambda\}} = \sum_{\lambda_t,\lambda_{\bar{t}}} \mathcal{M}^P(ij \to t^{\lambda_t}\bar{t}^{\lambda_{\bar{t}}})\mathcal{M}^D(t^{\lambda_t} \to W^+ b)\mathcal{M}^D(\bar{t}^{\lambda_{\bar{t}}} \to W^- \bar{b})$$

- Glue together production/decay using spinor-helicity methods production amps: [Badger, Sattler, Yundin, '11]
- Spin correlations between production and decay included
- Decompose amplitudes in color basis to construct hard functions

$$\mathcal{M}_{ij,\{a\}}^{\{\lambda\}}(p_1,\dots p_8,m_t,\mu_f) = \sum_I \mathcal{M}_{ij,I}^{\{\lambda\}}(p_1,\dots p_8,m_t,\mu_f)(c_I^{ij})_{\{a\}}$$

- W-bosons also decayed to leptons

Approximate NNLO

- Hard functions (NEW): computed 1-loop modified hard functions where the tops are decayed (in NWA)

$$H_{IJ}^{(0)} = \frac{1}{4}\sum_{\{\lambda\}} \left(\mathcal{M}_I^{\text{ren}\,(0)\,\{\lambda\}}\right)^* \left(\mathcal{M}_J^{\text{ren}\,(0)\,\{\lambda\}}\right) ,$$
$$H_{IJ}^{(1)} = \frac{1}{4}\sum_{\{\lambda\}} \left[\left(\mathcal{M}_I^{\text{ren}\,(0)\,\{\lambda\}}\right)^* \left(\mathcal{M}_J^{\text{ren}\,(1)\,\{\lambda\}}\right) + \left(\mathcal{M}_I^{\text{ren}\,(1)\,\{\lambda\}}\right)^* \left(\mathcal{M}_J^{\text{ren}\,(0)\,\{\lambda\}}\right)\right]$$

- Soft functions: 1-loop soft functions in PIM and 1PI do not change (Note: in NWA no soft-gluon connections between production and decay) [Ahrens, Ferroglia, Neubert, Pecjak, Yang, '10, '11]

- RG-equations:

$$\frac{d}{d\ln\mu}\boldsymbol{H} = \boldsymbol{\Gamma}_H \boldsymbol{H} + \boldsymbol{H}\boldsymbol{\Gamma}_H^\dagger$$
$$\frac{d}{d\ln\mu}\tilde{\mathbf{s}}_{\{\text{PIM},1\text{PI}\}} = \boldsymbol{\Gamma}^\dagger{}_{s\,\{\text{PIM},1\text{PI}\}}\tilde{\mathbf{s}}_{\{\text{PIM},1\text{PI}\}} + \tilde{\mathbf{s}}_{\{\text{PIM},1\text{PI}\}}\boldsymbol{\Gamma}_{s\,\{\text{PIM},1\text{PI}\}}$$

  - use two loop anomalous dimensions for massive partons computed by [Ferroglia, Neubert, Pecjak, Yang 09']
  - obtain approximate NNLO contributions by re-expanding resummation formula at fixed-order
  - obtain the correct coefficients of the plus-distributions terms

$$C_{\text{PIM}}^{(2)}(z, p_1, \ldots, p_8, m_t, \mu_f) = \sum_{m=0}^{3} \boxed{D_{\text{PIM},\,m}^{(2)}(z, p_1, \ldots, p_8, m_t, \mu_f)} P_m(z)$$
$$+ Q_{\text{PIM},\,0}^{(2)}(p_1, \ldots, p_8, m_t, \mu_f)\delta(1-z) + R_{\text{PIM}}^{(2)}(z, m_t, \mu_f)$$

Monte Carlo Implementation

$$C_{ij} = \alpha_s^2 \left[C_{ij}^{(0)} + \frac{\alpha_s}{4\pi} C_{ij}^{(1)} + \left(\frac{\alpha_s}{4\pi}\right)^2 C_{ij}^{(2)} + \mathcal{O}(\alpha_s^3) \right],$$

Stable-tops:

$$C_{\rm PIM}^{(2)}(z, M, m_t, \cos\theta, \mu_f) = \sum_{m=0}^{3} \boxed{D_{{\rm PIM},m}^{(2)}(z, M, m_t, \cos\theta, \mu_f)} P_m(z) + Q_{\rm PIM,0}^{(2)}(M, m_t, \cos\theta, \mu_f)\delta(1-z) + R_{\rm PIM}^{(2)}$$

Restore explicit dependence on outgoing particle momenta:

Unstable-tops:

$$C_{\rm PIM}^{(2)}(z, p_1, \ldots, p_8, m_t, \mu_f) = \sum_{m=0}^{3} \boxed{D_{{\rm PIM},m}^{(2)}(z, p_1, \ldots, p_8, m_t, \mu_f)} P_m(z) + Q_{\rm PIM,0}^{(2)}(p_1, \ldots, p_8, m_t, \mu_f)\delta(1-z) + R_{\rm PIM}^{(2)}(z, m_t, \mu_f)$$

- Monte Carlo phase-space integrator:
  - generate phase-space (momentum configurations) $\{p_i\}$
  - we evaluate approximate contributions using momenta $\{p_i\} \rightarrow$ weights
  - bin weights according to observables constructed from final state momenta

Improved predictions

Approximate-NLO
(nLO)

$$d\sigma^{\mathrm{nLO}}_{\mathrm{full}} = (\Gamma_t^{\mathrm{NLO}})^{-2}\Big\{\Big(d\sigma^{(0)}_{t\bar{t}} + d\tilde{\sigma}^{(1)}_{t\bar{t}}\Big) \otimes d\Gamma^{(0)}_{t\to l^+\nu_l b} \otimes d\Gamma^{(0)}_{\bar{t}\to l^-\bar{\nu}_l\bar{b}}$$
$$+ d\sigma^{(0)}_{t\bar{t}} \otimes d\Gamma^{(1)}_{t\to l^+\nu_l b} \otimes d\Gamma^{(0)}_{\bar{t}\to l^-\bar{\nu}_l\bar{b}}$$
$$+ d\sigma^{(0)}_{t\bar{t}} \otimes d\Gamma^{(0)}_{t\to l^+\nu_l b} \otimes d\Gamma^{(1)}_{\bar{t}\to l^-\bar{\nu}_l\bar{b}}\Big\}$$

NLO

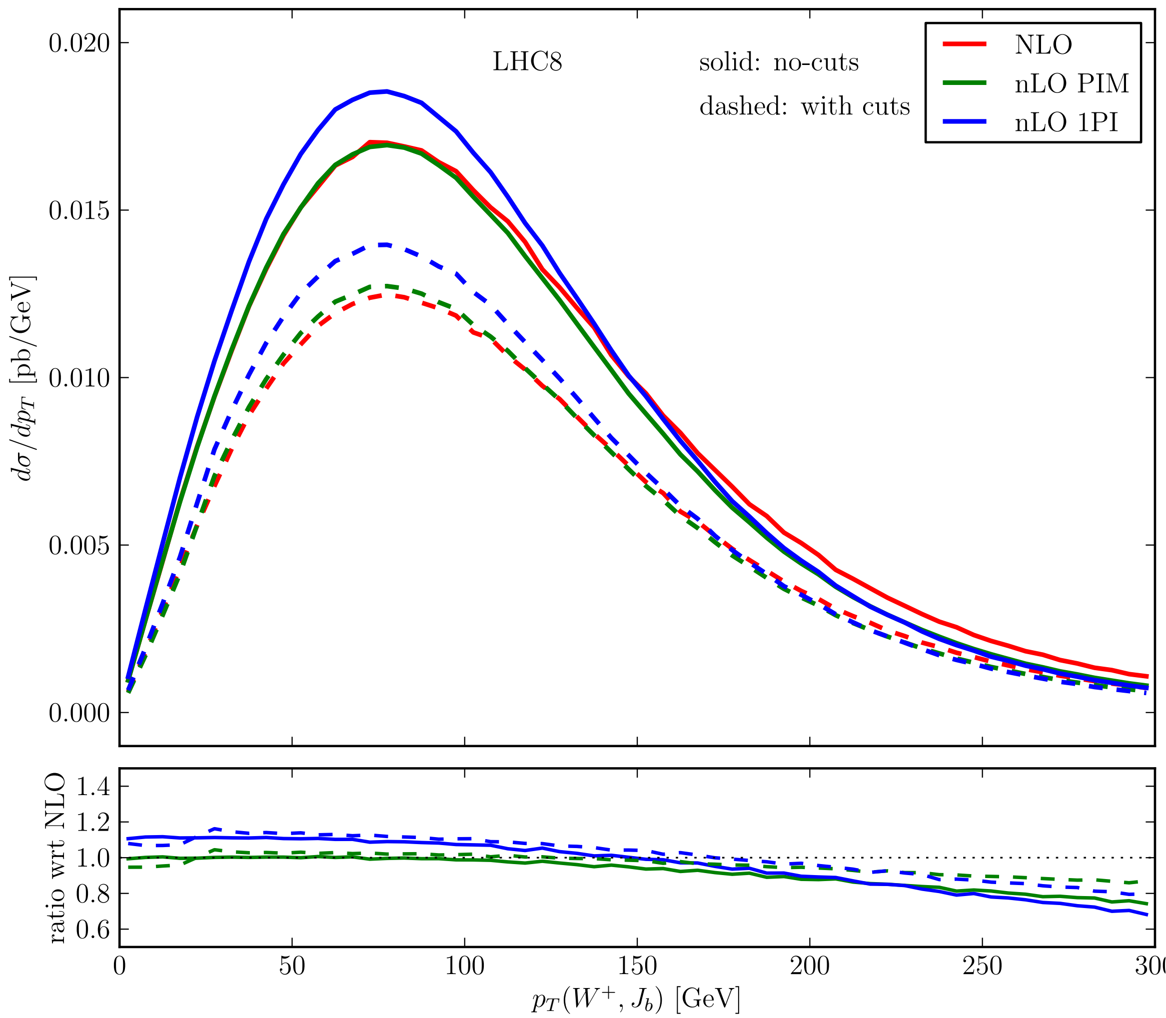
$$d\sigma^{\mathrm{NLO}}_{\mathrm{full}} = (\Gamma_t^{\mathrm{NLO}})^{-2}\Big\{\Big(d\sigma^{(0)}_{t\bar{t}} + d\sigma^{(1)}_{t\bar{t}}\Big) \otimes d\Gamma^{(0)}_{t\to l^+\nu_l b} \otimes d\Gamma^{(0)}_{\bar{t}\to l^-\bar{\nu}_l\bar{b}}$$
$$+ d\sigma^{(0)}_{t\bar{t}} \otimes d\Gamma^{(1)}_{t\to l^+\nu_l b} \otimes d\Gamma^{(0)}_{\bar{t}\to l^-\bar{\nu}_l\bar{b}}$$
$$+ d\sigma^{(0)}_{t\bar{t}} \otimes d\Gamma^{(0)}_{t\to l^+\nu_l b} \otimes d\Gamma^{(1)}_{\bar{t}\to l^-\bar{\nu}_l\bar{b}}\Big\}$$

Approximate-NNLO
(nNLO)

$$d\sigma^{\mathrm{nNLO}}_{\mathrm{full}} = (\Gamma_t^{\mathrm{NLO}})^{-2}\Big\{\Big(d\sigma^{(0)}_{t\bar{t}} + d\sigma^{(1)}_{t\bar{t}} + d\tilde{\sigma}^{(2)}_{t\bar{t}}\Big) \otimes d\Gamma^{(0)}_{t\to l^+\nu_l b} \otimes d\Gamma^{(0)}_{\bar{t}\to l^-\bar{\nu}_l\bar{b}}$$
$$+ d\sigma^{(0)}_{t\bar{t}} \otimes d\Gamma^{(1)}_{t\to l^+\nu_l b} \otimes d\Gamma^{(0)}_{\bar{t}\to l^-\bar{\nu}_l\bar{b}}$$
$$+ d\sigma^{(0)}_{t\bar{t}} \otimes d\Gamma^{(0)}_{t\to l^+\nu_l b} \otimes d\Gamma^{(1)}_{\bar{t}\to l^-\bar{\nu}_l\bar{b}}\Big\}$$

Approx-NNLO for the production subprocess and NLO for decay

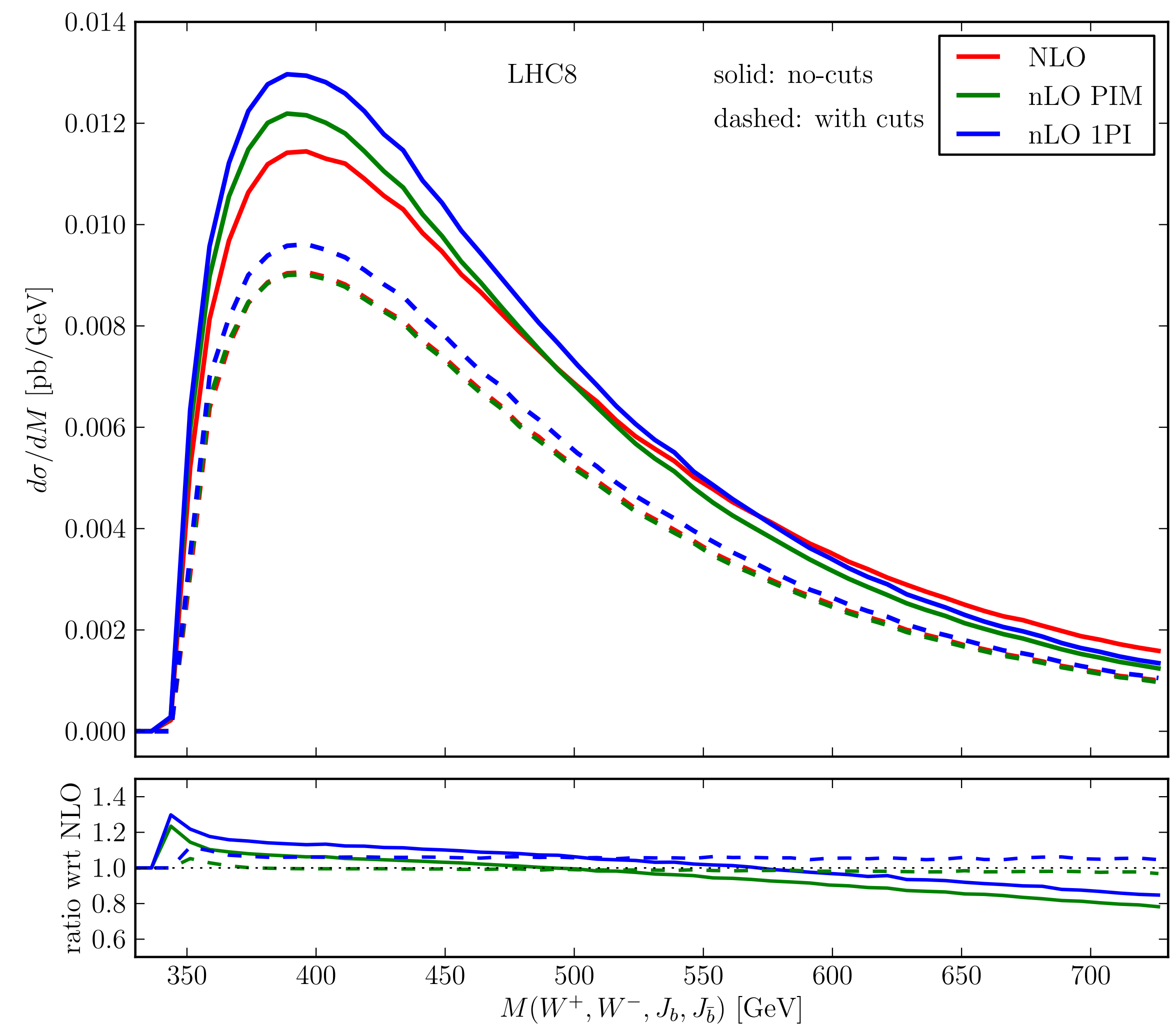
Validation procedure: nLO vs NLO



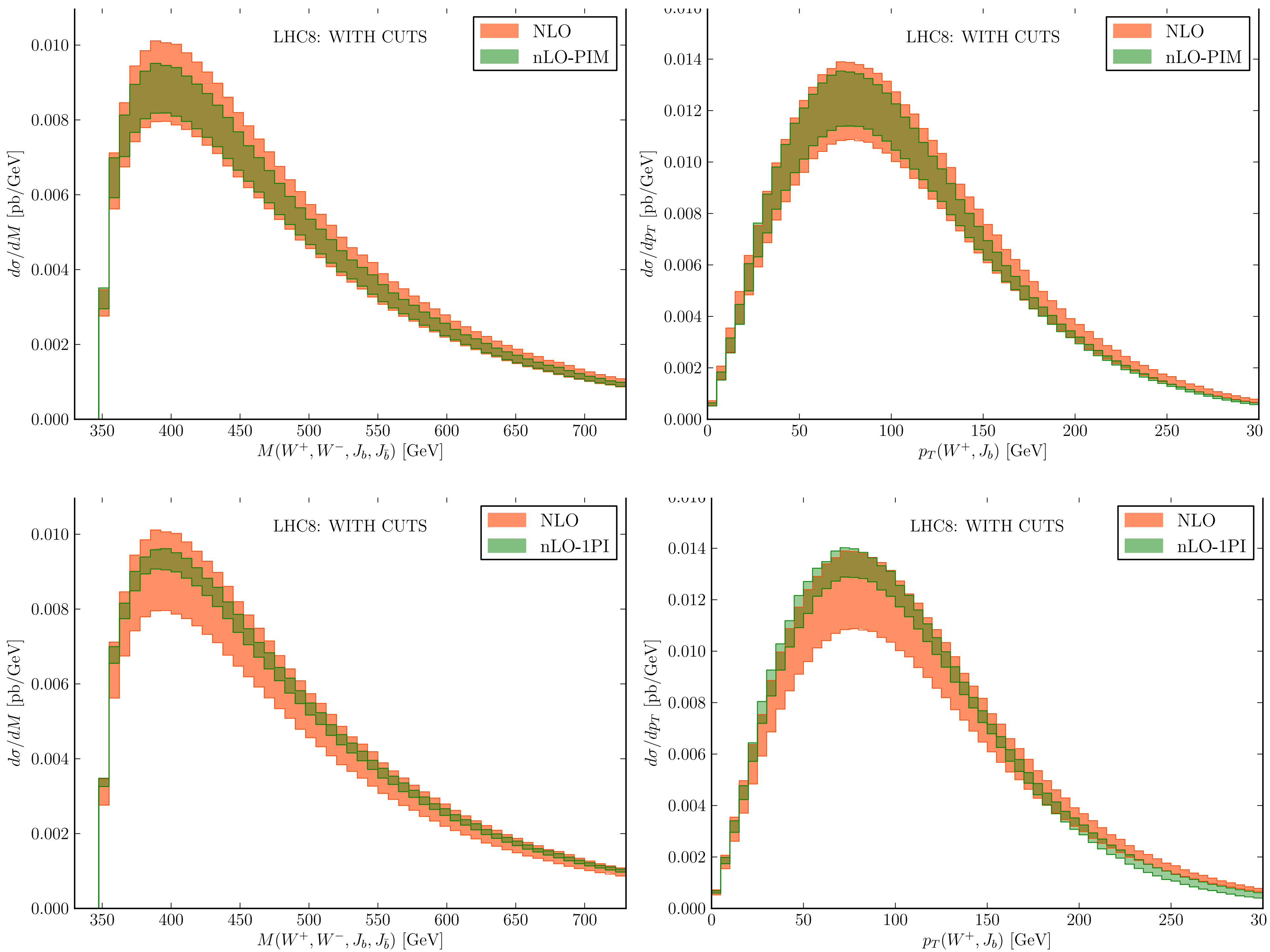
- Reconstructed invariant mass and pT distributions
- Production corrections only (LO decay)
- Validation of the approximation
- Cuts:

$$p_T(J_b) > 15 \text{ GeV} \qquad p_T(J_{\bar{b}}) > 15 \text{ GeV}$$
$$E_T(e^+) > 15 \text{ GeV} \qquad E_T(e^-) > 15 \text{ GeV} \qquad \not{E}_T > 20 \text{ GeV}$$

Surprisingly PIM seems to perform better than 1PI in both cases!



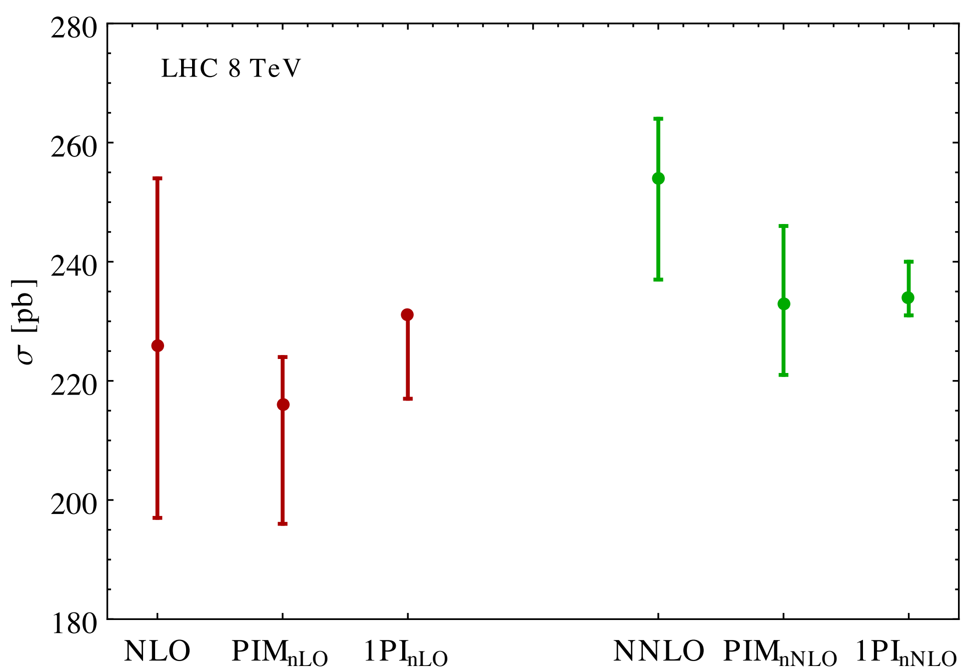
Validation procedure: nLO vs NLO

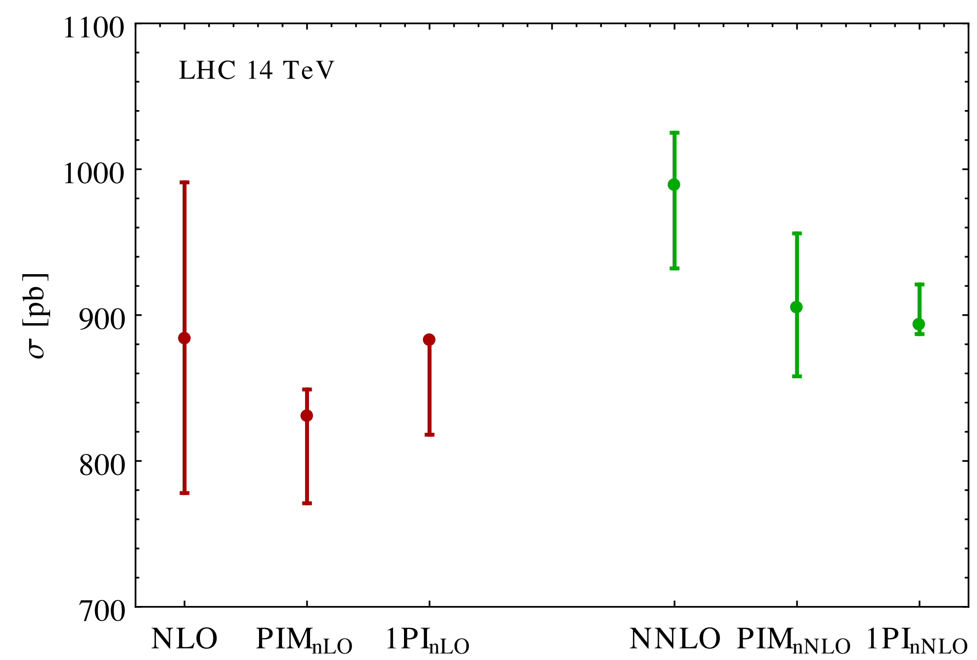


Uncertainty estimate: take envelope of scale variation of {PIM,1PI} for every distribution

Total cross section

- Complete agreement with [Ahrens et al.] for the no-cuts case (consistency-check)
- Compare approx-NNLO (nNLO) corrections with exact NNLO [Top++: Bärnreuther, Czakon, Fiedler, Mitov '12, '13]
- LHC 8 and 14 TeV, MSTW08 NLO PDFs $\mu = \{m_t/2, m_t, 2m_t\}$



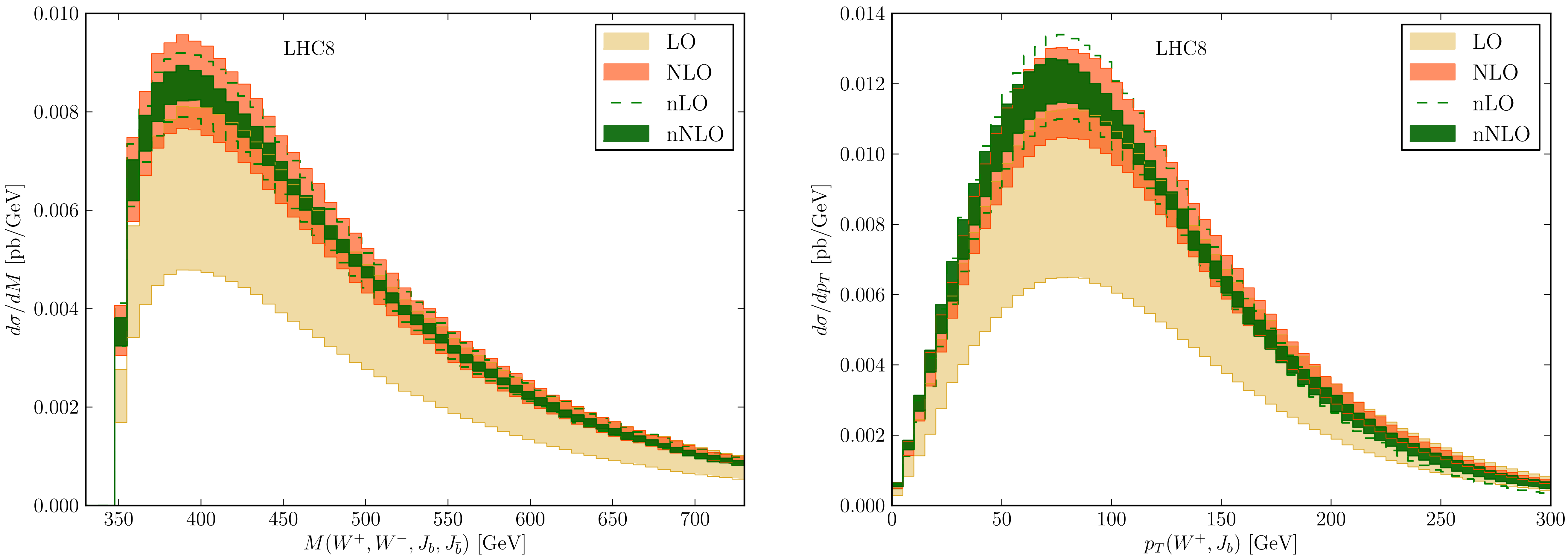


- Incomplete overlap of uncertainty bands at the LHC
- Approximate-NNLO (nNLO) corrections not perfect, but decent approximation (at the Tevatron the situation is a bit worse)
- The approximation can be improved by including 2-loop hard and soft functions

Distributions with final state cuts

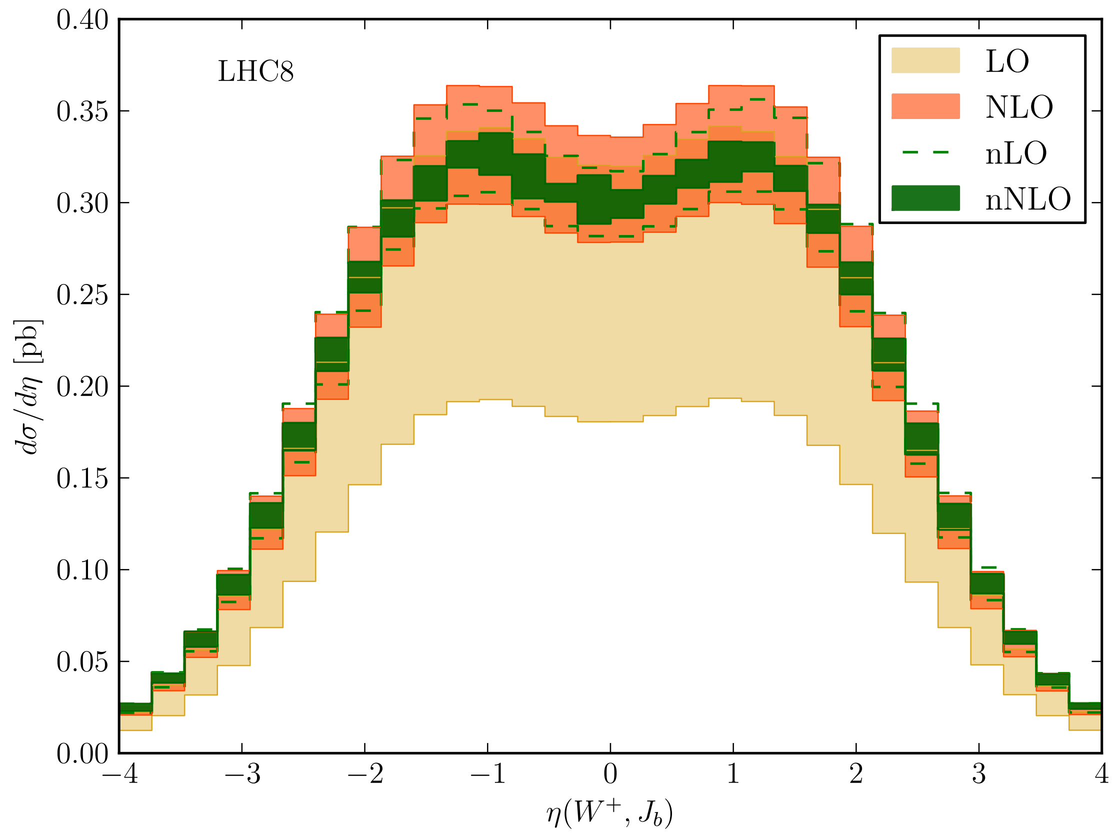
- Use MSTW2008 NLO/NNLO PDFs
- Cuts: $p_T(J_b) > 15$ GeV $\quad p_T(J_{\bar{b}}) > 15$ GeV

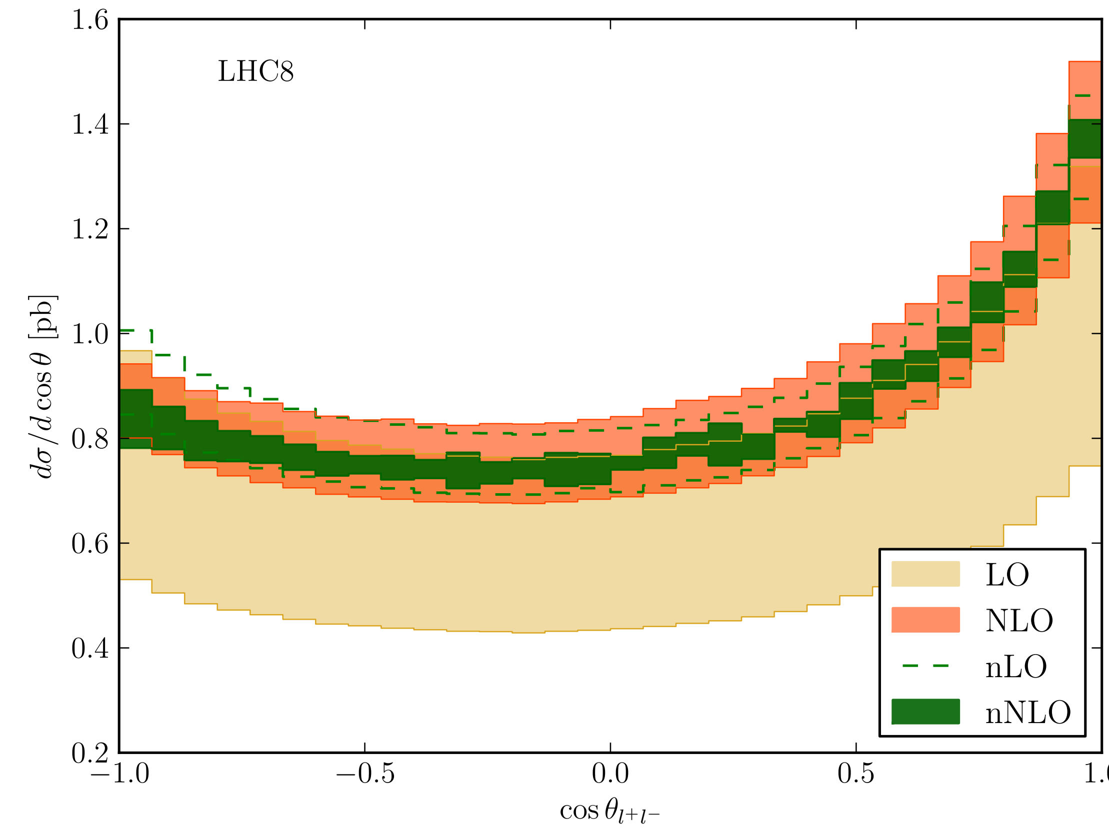
  $E_T(e^+) > 15$ GeV $\quad E_T(e^-) > 15$ GeV $\quad \not{E}_T > 20$ GeV
- Top decay included in NWA at NLO

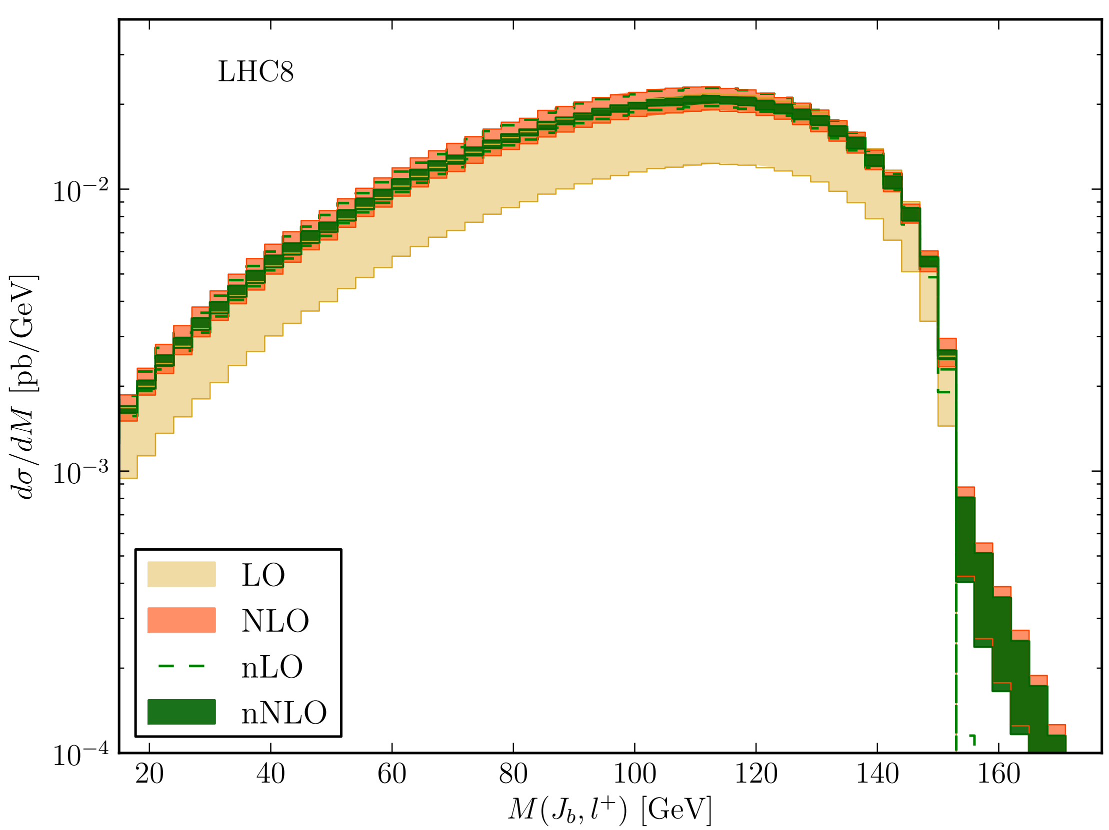


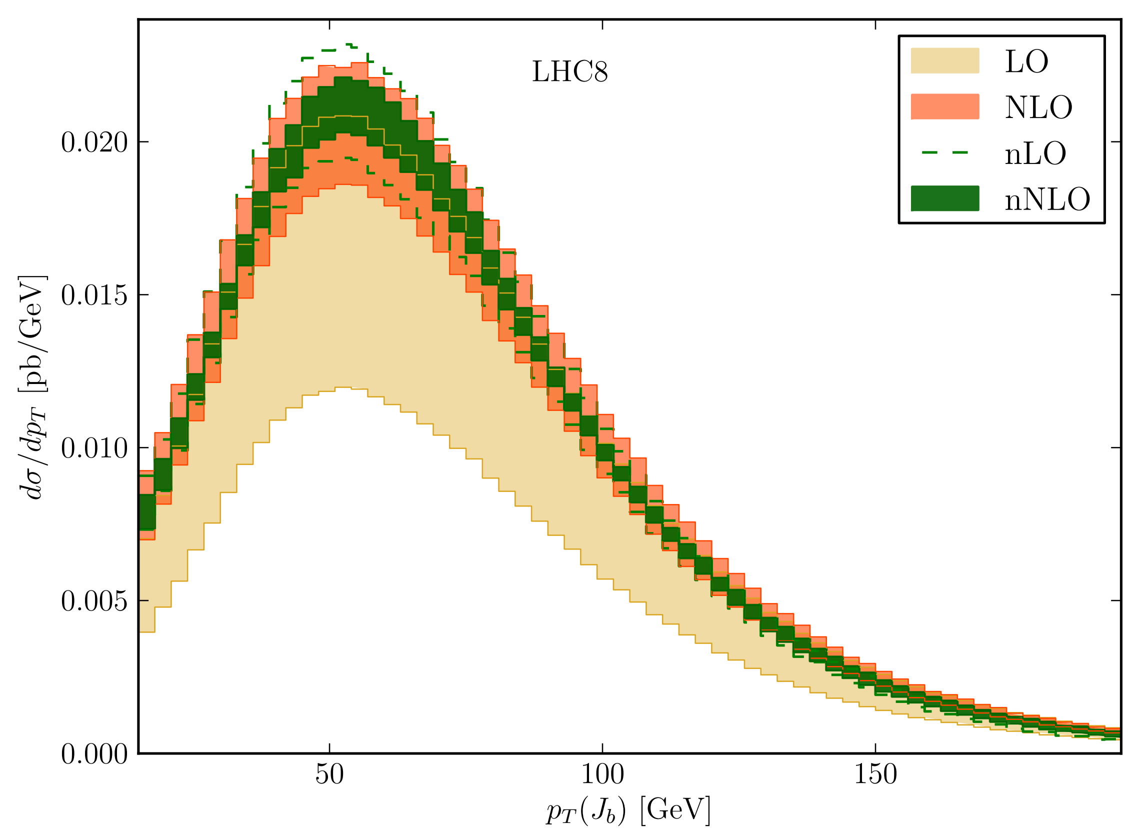
- Uncertainty bands of nNLO: scale variation+kinematics (envelope of PIM and 1PI)
- Good perturbative behaviour, reduction of theoretical uncertainty

Distributions with final state cuts







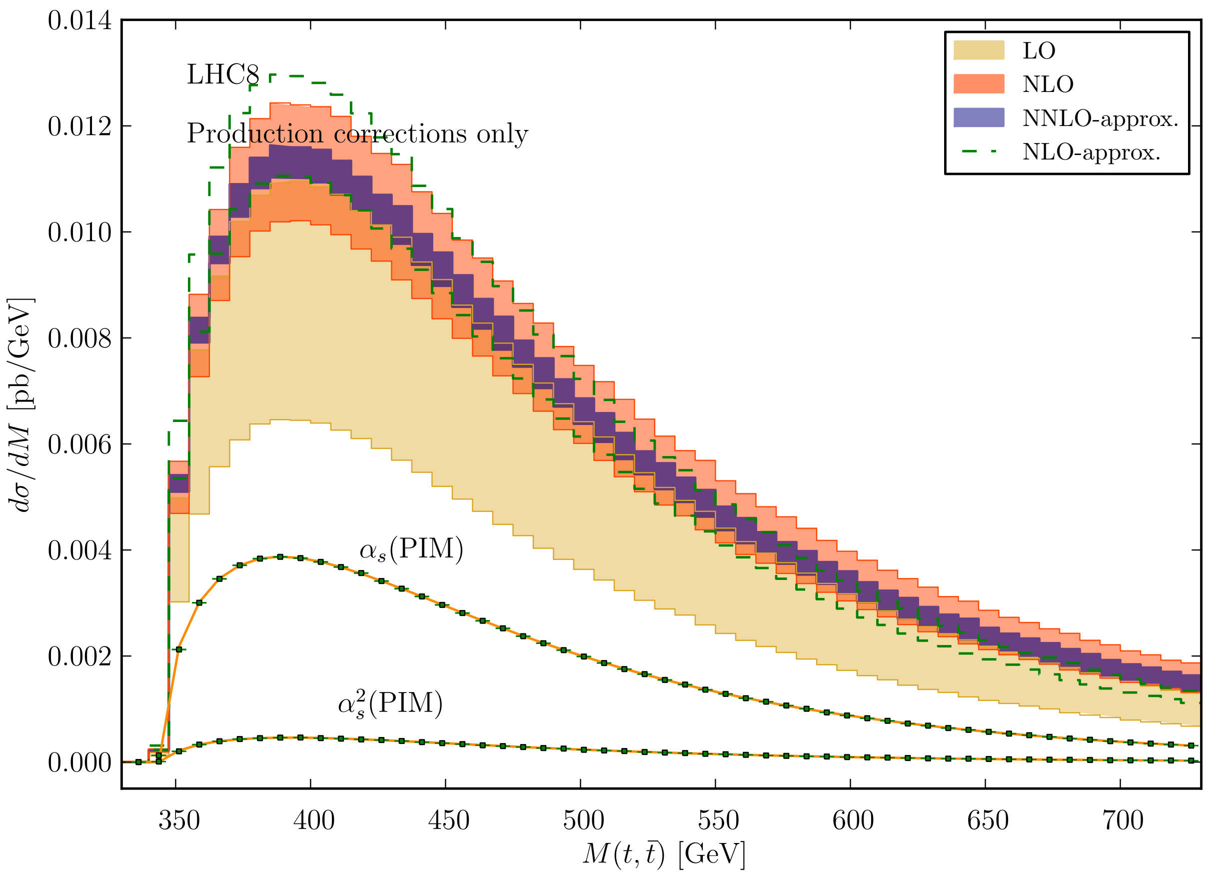


Conclusions & Outlook

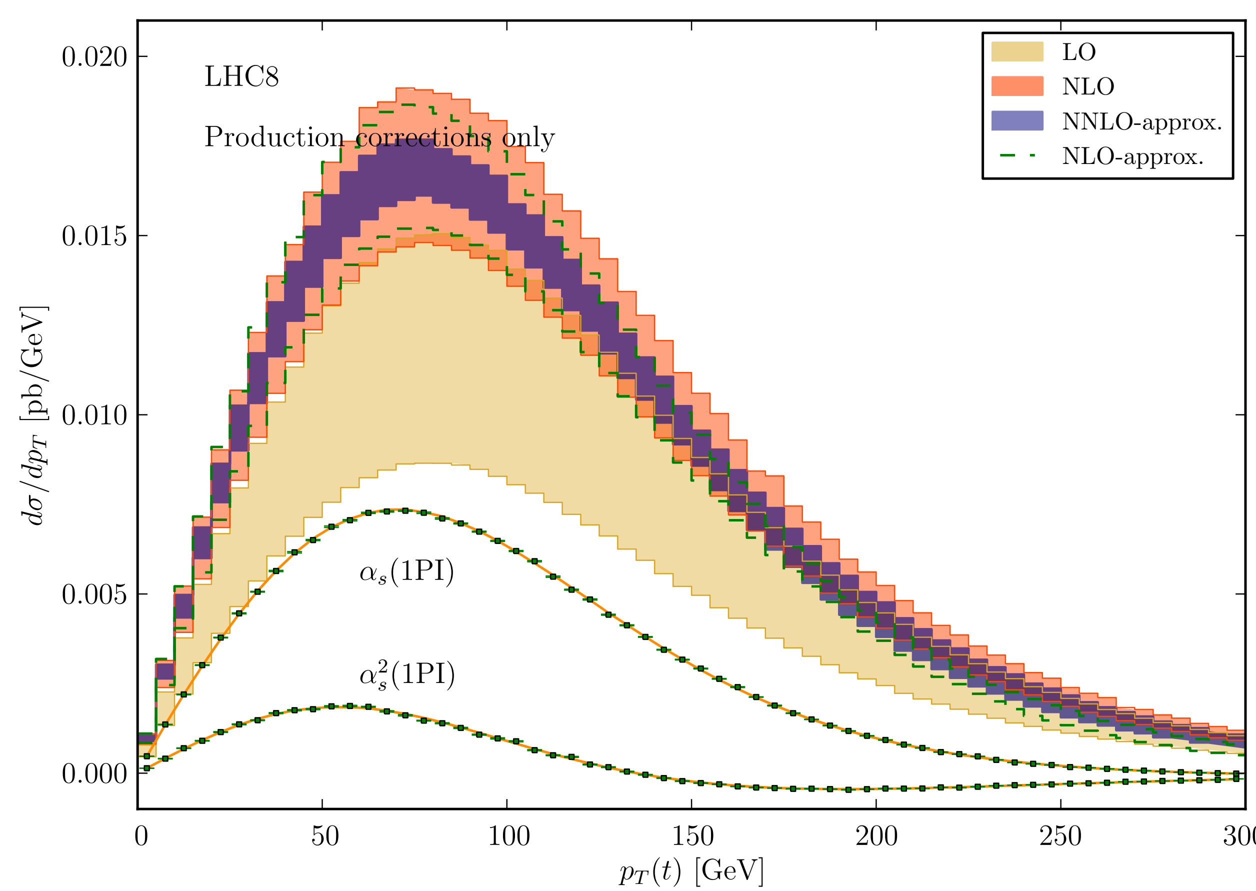
- We have adapted (including decay) and implemented known PIM and 1PI n(N)LO contributions in a fully-differential code, including top decays (at NLO) and spin-correlations
- We studied fully differential distributions
- Reduction of theoretical uncertainty (scale + kinematics)
- Formally we have not proved anything, but it seems to work

- Compute charge asymmetry at the LHC
- Adapt (including decay) and implement virtual + soft approximation [Ferroglia, Pecjak, Yang' 13; Ferroglia, Marzani, Pecjak, Yang '13]
- Mismatch in production/decay corrections
  - Include NNLO decay corrections [Gao, Li, Zhu '12; Brucherseifer, Caola, Melnikov '13]

Differential checks



- Final state patrons clustered into jets
- Tops reconstructed via b-jet and lepton momenta
- No cuts on final state applied (to recover total cs)
- Tops decays only at LO



- Full agreement at differential level with [Ahrens et al.]
- Good perturbative behaviour

Approximate NNLO formulas

- **H** and **S** satisfy RG equations of the form:

$$\frac{d}{d\ln\mu}\boldsymbol{H} = \boldsymbol{\Gamma}_H \boldsymbol{H} + \boldsymbol{H}\boldsymbol{\Gamma}_H^\dagger$$
$$\frac{d}{d\ln\mu}\tilde{\mathrm{s}}_{\{\mathrm{PIM},1\mathrm{PI}\}} = \boldsymbol{\Gamma}^\dagger{}_{s\,\{\mathrm{PIM},1\mathrm{PI}\}}\tilde{\mathrm{s}}_{\{\mathrm{PIM},1\mathrm{PI}\}} + \tilde{\mathrm{s}}_{\{\mathrm{PIM},1\mathrm{PI}\}}\boldsymbol{\Gamma}_{s\,\{\mathrm{PIM},1\mathrm{PI}\}}$$

- The two loop anomalous dimensions including massive partons were computed by [Ferroglia, Neubert, Pecjak, Yang 09']

- The large logarithms could be resummed to all-orders by solving the RG equations for **H** and **S**, but here we follow a different possibility. The resummed formulas can be re-expanded to obtain fixed-oder formulas.

- The perturbative expansion of the hard-scattering kernels reads

$$\mathsf{A} = \begin{pmatrix} 1 & & & & \\ \alpha\mathsf{L}^2 & \alpha\mathsf{L} & \alpha & & \\ \alpha^2\mathsf{L}^4 & \alpha^2\mathsf{L}^3 & \alpha^2\mathsf{L}^2 & \alpha^2\mathsf{L} & \alpha^2 \\ \alpha^3\mathsf{L}^6 & & \ldots & & \\ \vdots & & & & \end{pmatrix}$$

[Chiu, Kelley, Manohar, 08']

Logarithmic structure of the scattering amplitude for Sudakov problems

- By knowing the analytical expressions for **H** and **S** at NLO in both kinematics, we can solve explicitly the RG equations for **H** and **S** at NNLO

Approximate NNLO formulas

[Ahrens et al. '10,'11]

$$C_{ij} = \alpha_s^2 \left[C_{ij}^{(0)} + \frac{\alpha_s}{4\pi} C_{ij}^{(1)} + \left(\frac{\alpha_s}{4\pi} \right)^2 C_{ij}^{(2)} + \mathcal{O}(\alpha_s^3) \right],$$

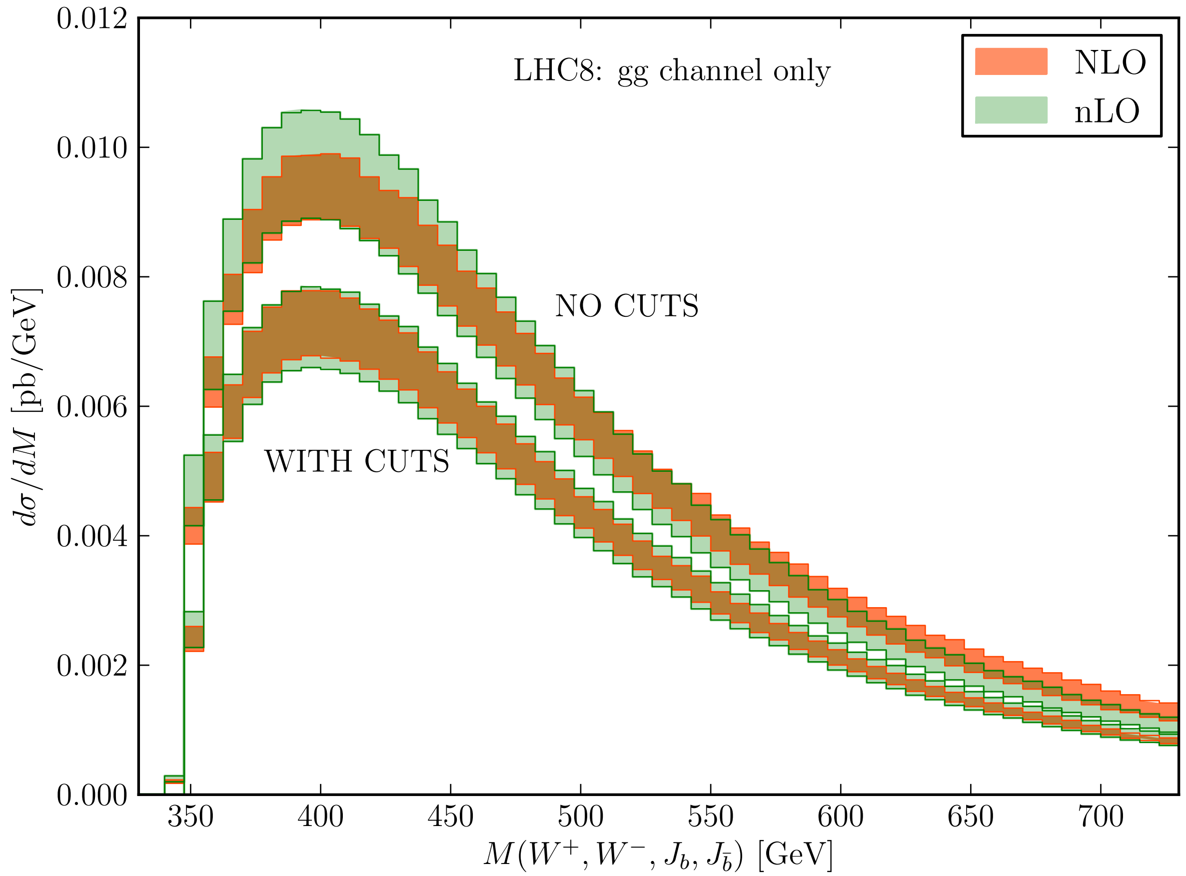
where for PIM

$$C_{\rm PIM}^{(1)} = D_1^{(1,{\rm PIM})} \left[\frac{\ln(1-z)}{1-z} \right]_+ + D_0^{(1,{\rm PIM})} \left[\frac{1}{1-z} \right]_+ + C_0^{(1,{\rm PIM})} \delta(1-z) + R^{(1,{\rm PIM})}(z),$$

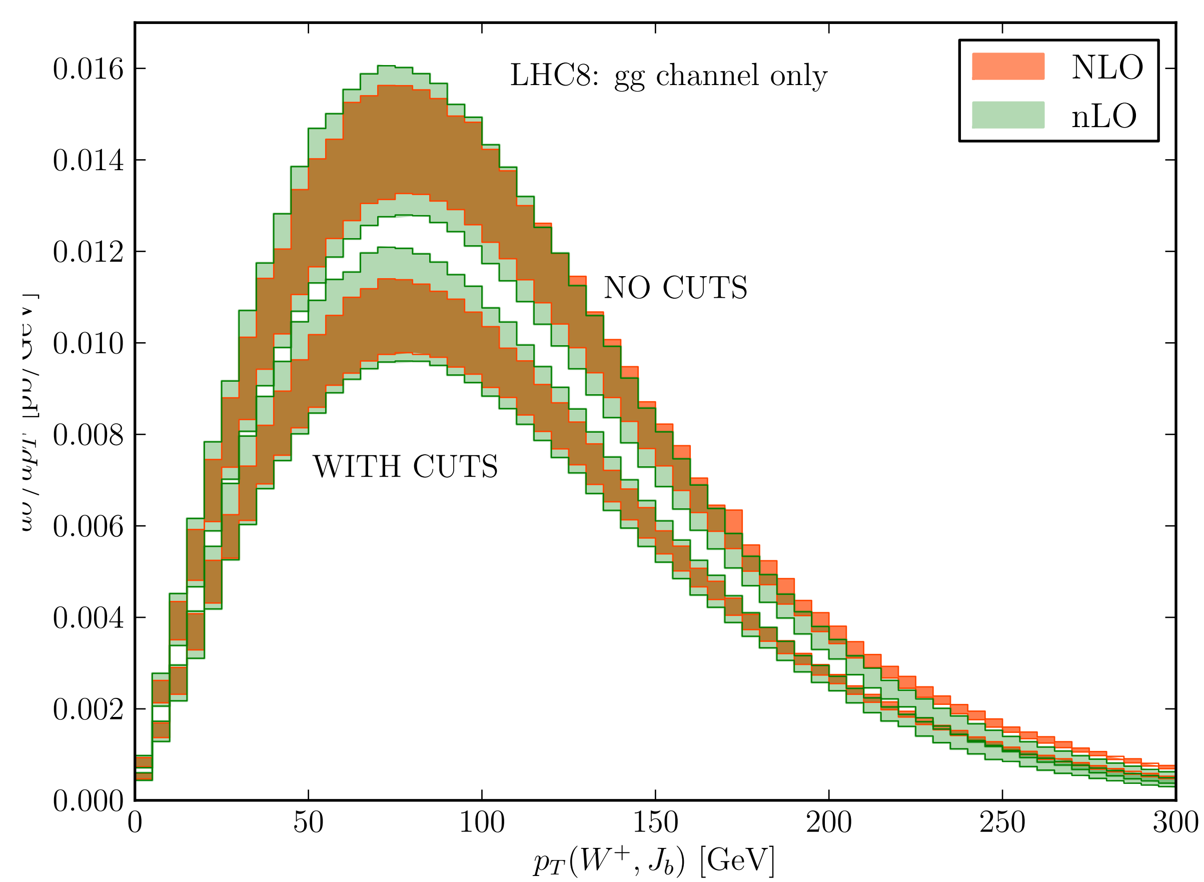
$$C_{\rm PIM}^{(2)} = D_3^{(2,{\rm PIM})} \left[\frac{\ln^3(1-z)}{1-z} \right]_+ + D_2^{(2,{\rm PIM})} \left[\frac{\ln^2(1-z)}{1-z} \right]_+ + D_1^{(2,{\rm PIM})} \left[\frac{\ln(1-z)}{1-z} \right]_+ + D_0^{(2,{\rm PIM})} \left[\frac{1}{1-z} \right]_+ + C_0^{(2,{\rm PIM})} \delta(1-z) + R^{(2,{\rm PIM})}(z).$$

This would require a complete 2 loop calculation

Validation procedure: nLO vs NLO



- Comparison of uncertainty bands between PIM-1PI envelope and full NLO for gg-channel only
- Take the envelope of PIM and 1PI predictions for every distribution



LHC 8 TeV setup

Collider: LHC 8 TeV

Use MSTW2008 NLO/NNLO PDFs

$$m_t^{\text{pole}} = 173.1 \text{ GeV} \qquad M_W = 80.4 \text{ GeV} \qquad \mu_F = \mu_R \in [0.5, 2.0] * m_t$$

$$\Gamma_t^{\text{NLO}}(m_t) = 1.373 \text{ GeV} \qquad \Gamma_W = 2.140 \text{ GeV}$$

Cuts:

$$p_T(J_b),\ p_T(J_{\bar{b}}) > 15 \text{ GeV} \qquad \not{E}_T > 20 \text{ GeV}$$

$$p_T(l^+),\ p_T(l^-) > 15 \text{ GeV} \qquad M_{t\bar{t}} > 350 \text{ GeV}$$

$$M_{t\bar{t}} = M_{t\bar{t}}^{\text{rec.}} = M\left((W^+, J_b), (W^-, J_{\bar{b}})\right)$$

- Top decay included in NWA at NLO

Anomalous dimensions for top production

[arXiv:1103.0550]

$$\boldsymbol{\Gamma}_H^{\rm PIM}\left(M,\cos\theta,\alpha_s\right)=\Gamma_{\rm cusp}(\alpha_s)\left(\ln\frac{M^2}{\mu^2}-i\pi\right)+\boldsymbol{\gamma}^h\left(M,\cos\theta,\alpha_s\right)$$

$$\boldsymbol{\Gamma}_H^{\rm 1PI}\left(s',t_1',u_1',\alpha_s\right)=\Gamma_{\rm cusp}(\alpha_s)\left(\ln\frac{s'}{\mu^2}-i\pi\right)+\boldsymbol{\gamma}^h\left(s',t_1',u_1',\alpha_s\right)\ ,$$

$$\tilde{\boldsymbol{\Gamma}}_{s\rm PIM}=-\left[\Gamma_{\rm cusp}(\alpha_s)\ln\frac{M^2}{\mu^2}+2\gamma^{\phi}\left(\alpha_s\right)\right]\mathbf{1}-\boldsymbol{\gamma}^h\left(M,\cos\theta,\alpha_s\right)$$

$$\tilde{\boldsymbol{\Gamma}}_{s\rm 1PI}=-\left[\Gamma_{\rm cusp}(\alpha_s)\ln\frac{s'}{\mu^2}+2\gamma^{\phi}\left(\alpha_s\right)+\Gamma_{\rm cusp}(\alpha_s)\log\frac{s'm_{\tilde{t}_1}^2}{t_1'u_1'}\right]\mathbf{1}-\boldsymbol{\gamma}^h\left(s',t_1',u_1',m_{\tilde{t}_1},\alpha_s\right)$$

NNLO solution of the RGE for the Soft function

[arXiv:1103.0550]

$$\tilde{s}(L, \alpha_s(\mu)) = 1 + \frac{\alpha_s(\mu)}{4\pi} \sum_{n=0}^{2} s^{(1,n)} L^n + \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \sum_{n=0}^{4} s^{(2,n)} L^n + \dots$$

$$\begin{aligned}\tilde{s}(L, \alpha_s(\mu)) = 1 &+ \frac{\alpha_s(\mu)}{4\pi} \left[\frac{\Gamma_0}{2} L^2 + L\gamma_0^s + s^{(1,0)}\right] \\ &+ \left(\frac{\alpha_s(\mu)}{4\pi}\right)^2 \left[\frac{\Gamma_0^2}{8} L^4 + \left(-\frac{\beta_0 \Gamma_0}{6} + \frac{\Gamma_0 \gamma_0^s}{2}\right) L^3 + \frac{1}{2}\left(\Gamma_1 - \beta_0 \gamma_0^s + (\gamma_0^s)^2 + \Gamma_0 s^{(1,0)}\right) L^2 \right. \\ &\left. + (\gamma_1^s - \beta_0 s^{(1,0)} + \gamma_0^s s^{(1,0)}) L + s^{(2,0)}\right].\end{aligned}$$

Approximation schemes

[Becher, Neubert, Xu, 07']

RG-impr. PT	Log. approx.	Accuracy $\sim \alpha_s^n L^k$	Γ_{cusp}	$\gamma^V,\ \gamma^\phi$	$C_V,\ \widetilde{s}_{\text{DY}}$
—	LL	$k = 2n$	1-loop	tree-level	tree-level
LO	NLL	$2n-1 \le k \le 2n$	2-loop	1-loop	tree-level
NLO	NNLL	$2n-3 \le k \le 2n$	3-loop	2-loop	1-loop
NNLO	NNNLL	$2n-5 \le k \le 2n$	4-loop	3-loop	2-loop

For Sudakov problems the counting of the logarithms is done in the exponent!

The large logarithms count as $1/\alpha_s$, it is always possible to rewrite a log of a ratio of two scales as

$$\ln \frac{\nu}{\mu'} = \int_{\alpha_s(\mu')}^{\alpha_s(\nu)} \frac{d\alpha}{\beta(\alpha)} \qquad \beta(\alpha_s) = -2\alpha_s \left[\beta_0 \left(\frac{\alpha_s}{4\pi}\right) + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3) \right]$$

$$\beta(\alpha_s) \sim \alpha_s^2$$