FACTORIZATION SIMPLIFIED

Loopfest XIII

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Based on arXiv:1306.6341 with Ilya Feige
and arXiv:1403.6472 with Ilya Feige

Main result:

$$\langle X|\,\mathcal{O}\,|0\rangle \;\cong\; \mathcal{C}(S_{ij})\,\frac{\langle X_1|\,\phi^\star W_1\,|0\rangle}{\langle 0|\,Y_1^\dagger W_1\,|0\rangle}\cdots\frac{\langle X_N|\,W_N^\dagger\phi\,|0\rangle}{\langle 0|\,W_N^\dagger Y_N\,|0\rangle}\;\langle X_s|\,Y_1^\dagger\cdots Y_N\,|0\rangle$$

- Two different amplitudes in QCD are equal at leading power in $\frac{p_i\cdot p_j}{Q^2}$
- We prove this rigorously to all orders in perturbation theory

QCD:

$$\left(\begin{aligned}\mathcal{M}_{\{\pm\}} &\cong \sum_I \mathcal{C}_{I,\{\pm\}}(S_{ij})\\ &\times\cdots\frac{\langle X_i|\,\bar\psi_i W_i\,|0\rangle^{\pm h_i}}{\mathrm{tr}\,\langle 0|\,Y_i^\dagger W_i\,|0\rangle}\cdots\frac{\langle X_j|\,A^\mu\mathcal{W}_j\,|0\rangle^{\pm a_j}}{\mathrm{tr}\,\langle 0|\,\mathcal{Y}_j^\dagger\mathcal{W}_j\,|0\rangle}\cdots\frac{\langle X_k|\,W_k^\dagger\psi_k\,|0\rangle^{\pm h_k}}{\mathrm{tr}\,\langle 0|\,W_k^\dagger Y_k\,|0\rangle}\cdots\\ &\times\langle X_s|\cdots(Y_i^\dagger T_I^i)^{h_i l_i}\cdots(\mathcal{Y}_j^\dagger T_I^j)^{l_{j-1}a_j l_{j+1}}\cdots(T_I^k Y_k)^{l_k h_k}\cdots|0\rangle\end{aligned}\right)$$

Perturbative QCD

- Why is **perturbative QCD** useful at all?

Asymptotic freedom

- α_s is small at high energy
- Perturbation theory works

$$\underbrace{\beta = \mu \frac{d}{d\mu} \alpha_s < 0}$$

Determined by
UV properties of QCD

Factorization

Universal

Perturbative

$$d\sigma = [\text{PDFs}] \times [\text{hard process}] \times [\text{soft/collinear radiation}] \times [\text{hadronization}]$$

(Re)summable

Small

Determined by
IR properties of QCD

Why is proving factorization so hard?

1. Non-perturbative effects
   - To show factorization up to $\frac{m_P}{Q}$ or $\frac{\Lambda_{\mathrm{QCD}}}{Q}$
   - No access to non-perturbative scales in perturbation theory

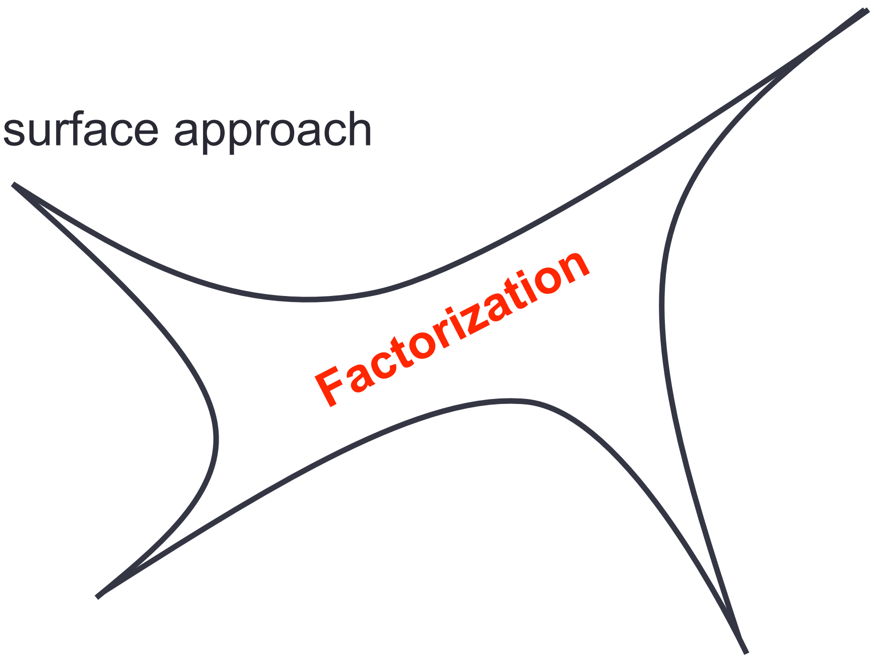
2. Perturbative effects
   - Infrared singularities (pinch surfaces) complicated
   - Gauge dependence subtle
   - Off-shell modes (Glauber gluons)

3. Hard even to formulate theorem
   - Precisely what is supposed to hold?
   - Gauge-invariant and regulator-independent formulation?

Historically, four approaches

1. OPE approach

2. Pinch surface approach

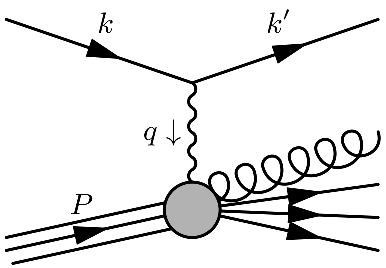


3. Amplitude approach

4. Effective field theory approach

Approach 1: Operator Products

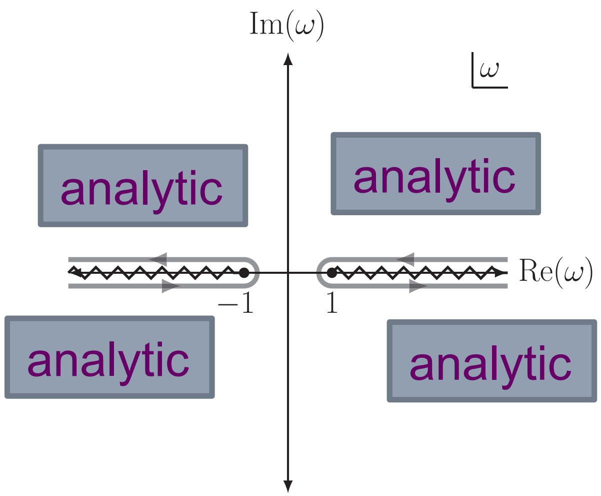
Deep inelastic scattering



$$\omega = \frac{2P\cdot q}{Q^2}$$

Photon momentum

- Use OPE around ω=0 to expand at large Q^2
- Physical region has ω>1



- OPE is possible because we can analytically continue
- We know analytic structure because
  1. **Inclusive** over final states
  2. **Analytic structure** of two-point function $J_\mu(x)J_\nu(0)$ **known exactly**
- Analytic structure for more complicated processes not known exactly

Approach 2: Pinch surfaces

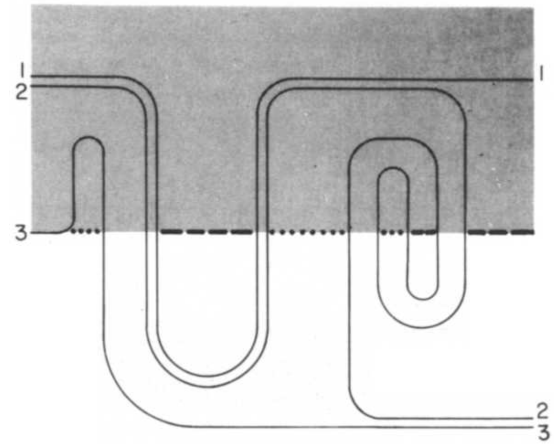
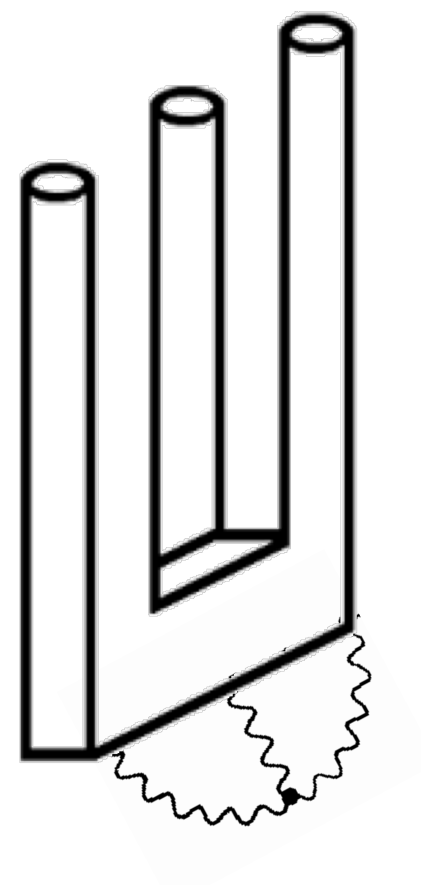


Fig. 5.11. Cancellations for a complicated garden. The shaded area is the soft subgraph. The solid lines are tulip boundaries. Addition of tulips with new boundary portions along one or more of the dashed or dotted lines produces cancellations.

Collins, Soper, Sterman: pinch surfaces factorize



Collins & Soper, 1981

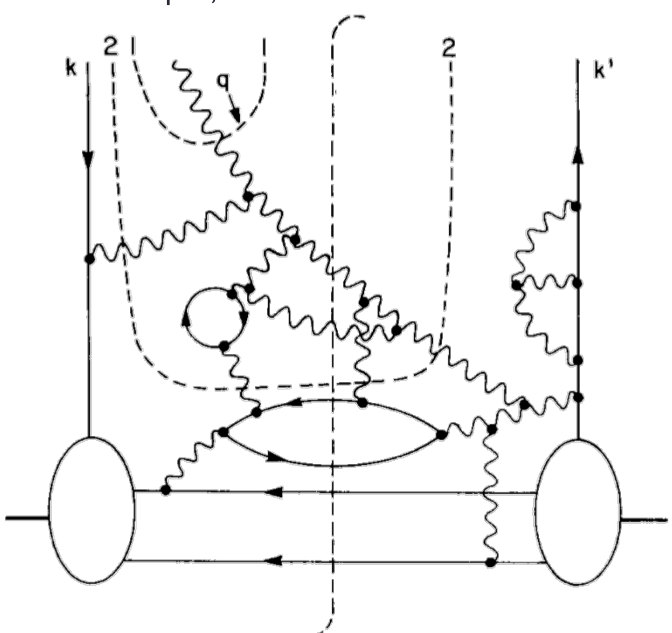
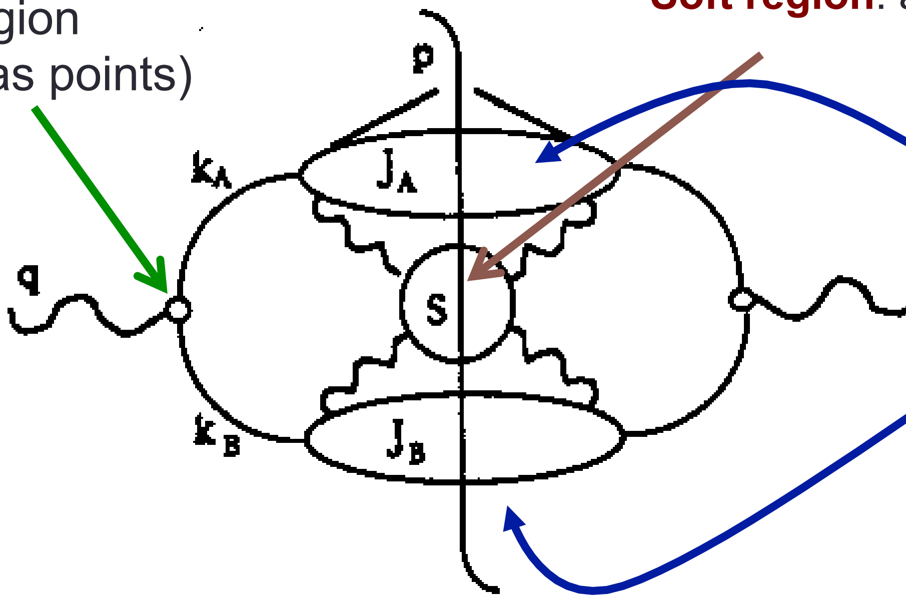


Fig. 5.7. A two-tulip garden.

Approach 2: Pinch surfaces

Hard region
(drawn as points)

Soft region: all particles have $p^\mu = 0$



Jet regions: all particles have
$$p_i^\mu = c_i p^\mu$$

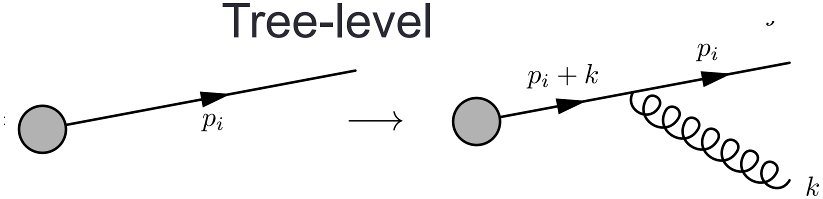
- All momenta zero or exactly proportional to some external momentum
  - Sidesteps soft/collinear overlap region (zero bin)
  - More work needed to factorize finite-momentum amplitudes
- Factorizes hard from jet/soft – does not factorize jet from soft
- Do not provide operator definitions

Approach 3: Amplitudes

Collinear

Primary goal is practical formulas (e.g. for subtractions):

Tree-level



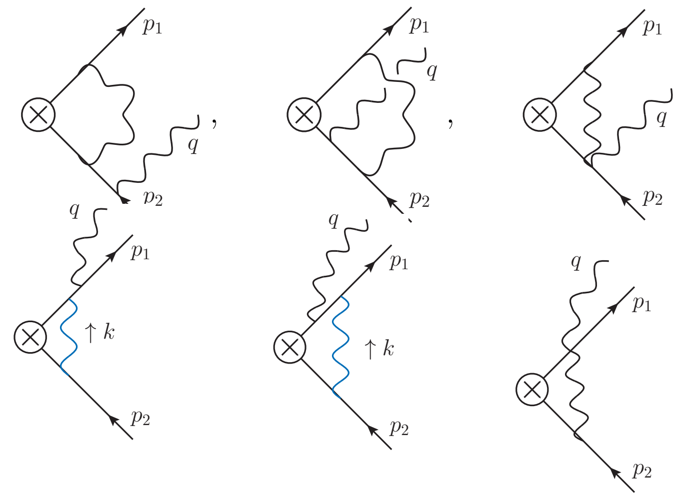
$$\mathcal{M} \to \mathcal{M} \times \mathcal{P}_{ab}$$

DGLAP splitting functions (1977)

$$P_{qq}(z) = C_F \left[(1+z^2) \left[\frac{1}{1-z} \right]_+ + \frac{3}{2} \delta(1-z) \right]$$

- Leading order splitting functions **universal** (process independent)
- Splitting functions for **PDF evolution** defined **to all orders**

One-loop

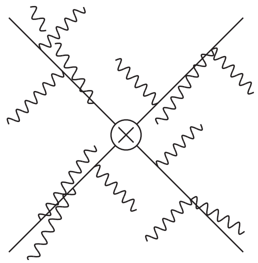


- IR divergent at 1-loop
- Relevant diagrams are gauge and process-dependent

- Bern and Chalmers (1995): collinear universality proven at 1-loop
- Kosower (1999): universality proven to all orders **at leading color (large N)**
- **No all-orders proof** in QCD (until now)

Approach 3: Amplitudes

Soft



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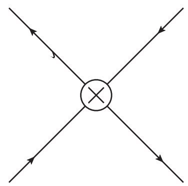


$$\times \; \langle k_1 \cdots k_\ell |\, Y_1^\dagger \cdots Y_N \,|0\rangle$$

Soft gluons see hard particles as classical sources

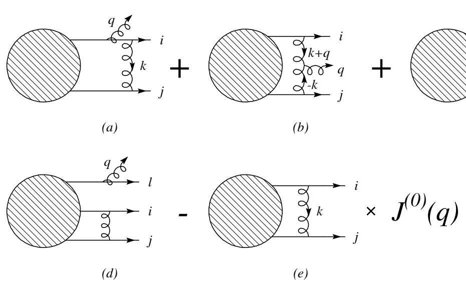
$$Y_j^\dagger(x) = \exp\left(ig \int_0^\infty ds\, n_j \cdot A(x^\mu + s\, n_j^\mu)\, e^{-\varepsilon s} \right)$$

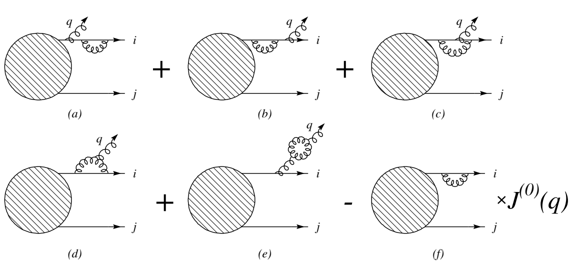
Wilson lines

- Wilson line picture does not disentangle soft from collinear
- **Universal soft current** conjecture (Catani & Grassini 2000)

$$\langle a \,|\, \mathcal{M}(q, p_1, \ldots, p_m)\rangle \simeq \varepsilon^\mu(q)\, J_\mu^a(q,\epsilon)\, |\, \mathcal{M}(p_1,\ldots,p_m)\rangle \; \left[1 + \mathcal{O}(g_S^4)\right] \; ,$$

Computed in dim reg at 1-loop (Catani & Grassini 2000)





- Soft current computed in dim reg at 2-loop (Duhr & Gehrmann 2013, Zhu & Li 2013)
    - Required for NNLO subtractions and automation
- **No operator definition** of J
- **all orders** universality **unproven** (until now)

Approach 4: Soft-Collinear Effective Theory

- Assigns scaling behavior to fields
- Expand Lagrangian to leading power

$$\mathcal{L} = i\bar{\psi}\not{D}\psi \longrightarrow$$

$$\mathcal{L} = \sum_{\tilde{p},\tilde{p}'} e^{i(\tilde{p}'-\tilde{p})\cdot x}\bar{\xi}_{n,\tilde{p}'}\left[in\cdot D_s + \frac{p_\perp^2}{\bar{n}\cdot p}\right]\frac{\not{\bar{n}}}{2}\xi_{n,\tilde{p}}$$
$$+\sum_{\tilde{p},\tilde{p}',\tilde{q}} e^{i(\tilde{p}'-\tilde{p}-\tilde{q})\cdot x}\bar{\xi}_{n,\tilde{p}'}\left[gn\cdot A_{n,\tilde{q}} + g\not{A}^\perp_{n,\tilde{q}}\frac{\not{p}_\perp}{\bar{n}\cdot p} + \frac{\not{p}'_\perp}{\bar{n}\cdot p'}g\not{A}^\perp_{n,\tilde{q}} - \frac{\not{p}'_\perp}{\bar{n}\cdot p'}gn\cdot A_{n,\tilde{q}}\frac{\not{p}_\perp}{\bar{n}\cdot p}\right]\frac{\not{\bar{n}}}{2}\xi_{n,\tilde{p}}$$
$$+\ 2\text{-gluon} + 3\text{-gluon} + \ldots + \mathcal{O}(\lambda)$$

Advantages

- Clarifies universality
- Employs powerful renormalization group methods
- Parameterizes power corrections

Disadvantages

- Feynman rules messy
- Field scaling is gauge-dependent and unphysical
- Zero-bin subtraction frustrates true continuum limit
- How do we know that modes aren't missing?
  - (soft-collinear messenger modes? Glauber modes?)

FACTORIZATION SIMPLIFIED

A precise statement of factorization:

$$\langle X|\,\mathcal{O}\,|0\rangle \;\cong\; \mathcal{C}(S_{ij})\,\frac{\langle X_1|\,\phi^\star W_1\,|0\rangle}{\langle 0|\,Y_1^\dagger W_1\,|0\rangle}\cdots\frac{\langle X_N|\,W_N^\dagger\phi\,|0\rangle}{\langle 0|\,W_N^\dagger Y_N\,|0\rangle}\;\langle X_s|\,Y_1^\dagger\cdots Y_N\,|0\rangle$$

- Two different amplitudes in QCD are equal at leading power in finite kinematic ratios $\frac{p_i\cdot p_j}{Q^2}$
- We prove this rigorously to all orders in perturbation theory

QCD:

$$\left(\begin{aligned}\mathcal{M}_{\{\pm\}} &\cong \sum_I \mathcal{C}_{I,\{\pm\}}(S_{ij})\\ &\times\cdots\frac{\langle X_i|\,\bar\psi_i W_i\,|0\rangle^{\pm h_i}}{\mathrm{tr}\,\langle 0|\,Y_i^\dagger W_i\,|0\rangle}\cdots\frac{\langle X_j|\,A^\mu\mathcal{W}_j\,|0\rangle^{\pm a_j}}{\mathrm{tr}\,\langle 0|\,\mathcal{Y}_j^\dagger\mathcal{W}_j\,|0\rangle}\cdots\frac{\langle X_k|\,W_k^\dagger\psi_k\,|0\rangle^{\pm h_k}}{\mathrm{tr}\,\langle 0|\,W_k^\dagger Y_k\,|0\rangle}\cdots\\ &\times\langle X_s|\cdots(Y_i^\dagger T_I^i)^{h_i l_i}\cdots(\mathcal{Y}_j^\dagger T_I^j)^{l_{j-1}a_j l_{j+1}}\cdots(T_I^k Y_k)^{l_k h_k}\cdots|0\rangle\end{aligned}\right)$$

Jet amplitudes

Matrix element in QCD

$$\mathcal{M}_{fi} \cong \mathcal{C}_{c_i h_j}(S_{ij}) \frac{\langle X_1|\,\psi^* W_1\,|0\rangle^{h_1 c_1}}{\langle 0|\,Y_1^\dagger W_1\,|0\rangle} \cdots \frac{\langle 0|\,\overline{W}_N^\dagger \psi\,|X_N\rangle^{h_N c_N}}{\langle 0|\,\overline{W}_N^\dagger \overline{Y}_N\,|0\rangle} \langle X_s^f|\,Y_1^\dagger \cdots \overline{Y}_N\,|X_s^i\rangle^{c_1\cdots c_N}$$

Leading power in momentum scaling

Finite hard amplitude

Operator definition of zero-bin subtraction

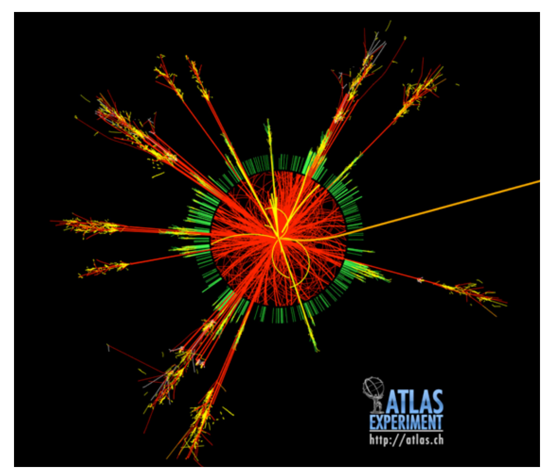
Soft amplitude

Advantages of this approach:

- Gauge and regulator independent
- Soft, Collinear and Soft-Collinear factorization **rigorously proven** at amplitude level
- Combines pinch analysis (reduced diagrams), amplitudes and SCET

Applies to entire amplitude, not just IR divergent regions

Simplifies derivation of SCET



$$\frac{p_i \cdot p_j}{Q^2}$$

- Scaling of external **momenta** is physical
- **No** discussion of **field scaling** is required

Connection to amplitudes

$$\mathcal{M}_{fi} \cong \mathcal{C}_{c_i h_j}(S_{ij}) \frac{\langle X_1 | \psi^* W_1 | 0 \rangle^{h_1 c_1}}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle} \cdots \frac{\langle 0 | \overline{W}_N^\dagger \psi | X_N \rangle^{h_N c_N}}{\langle 0 | \overline{W}_N^\dagger \overline{Y}_N | 0 \rangle} \; \langle X_s^f | Y_1^\dagger \cdots \overline{Y}_N | X_s^i \rangle^{c_1 \cdots c_N}$$

Soft factorization

1 soft emission

Catani-Grassini soft current

$$|\mathcal{M}_{f'i}\rangle \cong J_\mu^a |\mathcal{M}_{fi}\rangle$$

$$\mathbf{J}^\mu = J^\mu_{ahh'} = \frac{\langle \epsilon^\mu(p); a | Y_1^\dagger Y_2 | 0 \rangle^{hh'}}{C_A \mathrm{tr} \langle 0 | Y_1^\dagger Y_2 | 0 \rangle}$$

Normalized to 0 emissions

$$|\mathcal{M}_{fi}\rangle = \mathcal{C}(s_{12}) \frac{\langle p_1 | \psi^* W_1 | 0 \rangle}{\langle 0 | Y_1^\dagger W_1 | 0 \rangle} \frac{\langle p_2 | W_2^\dagger \psi | 0 \rangle}{\langle 0 | W_2^\dagger Y_2 | 0 \rangle}$$

- Gives **operator definition** of soft current and matrix element
- **Gauge invariant and regulator independent**
  - Previous results only in Feynman gauge with dimensional regularization

Collinear factorization

- Generalizes Kosower's large N proof to finite N
- **Gauge invariant and regulator independent**
- Operator definition of splitting functions for any process

Connection to SCET

- Give any state in $|X_j\rangle$ the quantum number "j"
- Give any state in $|X_s\rangle$ the quantum number "s"
- Introduce gluon and quark fields which can create and destroy these states

$$\mathcal{L}_{\text{eff}} \equiv \mathcal{L}_1 + \cdots \mathcal{L}_m + \mathcal{L}_s$$

Identical copies of QCD Lagrangian

Then

$$\langle X_1 \cdots X_m X_s | \bar{\psi}_1 \cdots \psi_m | 0 \rangle_{\mathcal{L}_{\text{QCD}}} \sim \langle X_1 | \bar{\psi}_1 W_1 | 0 \rangle \cdots \langle X_m | W_m^\dagger \psi_m | 0 \rangle \langle X_s | Y_1 \cdots Y_m^\dagger | 0 \rangle_{\mathcal{L}_{\text{QCD}}}$$

$$= \langle X_1 \cdots X_m X_s | \bar{\psi}_1 \underbrace{W_1 Y_1 \cdots Y_m W_m^\dagger \psi_m | 0 \rangle}_{\text{Now a single operator in an effective theory}}{}_{\mathcal{L}_{\text{eff}}}$$

Now a **single operator** in an effective theory

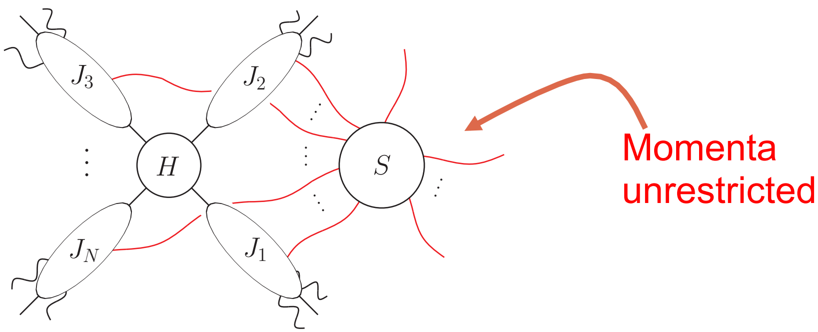
- This formulation is most similar to Luke/Freedman SCET (2011)
- Equivalent to label SCET [Bauer et al 2001] and multipole SCET [Beneke et al 2002] at leading power
- Provides operator definition of zero-bin subtraction

$$\widehat{Z}_i \equiv \frac{1}{N_c} \text{tr} \langle 0 | W_i^\dagger Y_i | 0 \rangle$$

Outline of proof

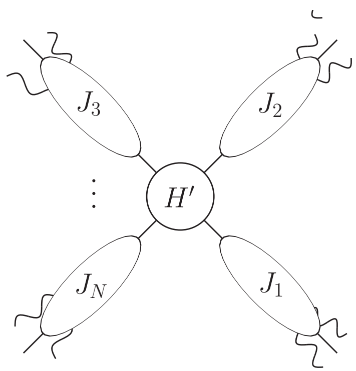
1. Establish power counting
2. Separate soft-sensitive gluons from soft-insensitive ones
3. Prove "reduced diagram" structure at leading power in physical gauges

$$\langle X_1 \cdots X_N; X_s | \, \mathcal{O} \, | 0 \rangle \overset{\text{gen. } r}{\cong} \sum$$



4. Prove soft collinear decoupling

$$\langle X_1 \cdots X_N; X_s | \, \mathcal{O} \, | 0 \rangle \overset{\substack{\text{any } r_s \\ \text{gen. } r_c}}{\cong}$$



$$\times \frac{\langle X_s | \, Y_1^\dagger \cdots Y_N \, | 0 \rangle}{\langle 0 | \, Y_1^\dagger \, | 0 \rangle \cdots \langle 0 | \, Y_N \, | 0 \rangle}$$

5. Prove gauge-invariant formulation

$$\mathcal{M}_{fi} \cong \mathcal{C}_{c_i h_j}(S_{ij}) \frac{\langle X_1 | \, \psi^* W_1 \, | 0 \rangle^{h_1 c_1}}{\langle 0 | \, Y_1^\dagger W_1 \, | 0 \rangle} \cdots \frac{\langle 0 | \, \overline{W}_N^\dagger \psi \, | X_N \rangle^{h_N c_N}}{\langle 0 | \, \overline{W}_N^\dagger \overline{Y}_N \, | 0 \rangle} \; \langle X_s^f | \, Y_1^\dagger \cdots \overline{Y}_N \, | X_s^i \rangle^{c_1 \cdots c_N}$$

Summary

- Matrix elements of states with only soft and collinear momenta **factorize**:

$$\mathcal{M}_{fi} \cong \mathcal{C}_{c_i h_j}(S_{ij}) \frac{\langle X_1| \psi^* W_1 |0\rangle^{h_1 c_1}}{\langle 0| Y_1^\dagger W_1 |0\rangle} \cdots \frac{\langle 0| \overline{W}_N^\dagger \psi |X_N\rangle^{h_N c_N}}{\langle 0| \overline{W}_N^\dagger \overline{Y}_N |0\rangle} \langle X_s^f| Y_1^\dagger \cdots \overline{Y}_N |X_s^i\rangle^{c_1 \cdots c_N}$$

- Generalizes Collins-Soper-Sterman pinch analysis
  - Works for amplitudes with nonsingular momenta
  - In addition, soft and collinear modes factorized
- Defines and proves factorization of amplitudes
  - gauge-invariant and regulator-independent definition for Catani-Grassini soft current.
  - Collinear factorization proven to all orders
  - Soft-collinear factorization proven to all orders

$$\mathcal{L}_{\text{eff}} \equiv \mathcal{L}_1 + \cdots \mathcal{L}_m + \mathcal{L}_s$$

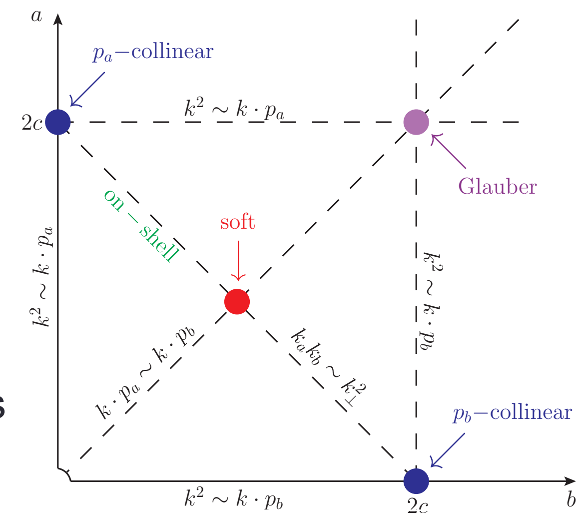
- Easily written with an effective Lagrangian:

$$\langle X_1 \cdots X_m X_s|\bar{\psi}_1 \cdots \psi_m|0\rangle_{\mathcal{L}_{\text{QCD}}} \sim \langle X_1 \cdots X_m X_s|\bar{\psi}_1 W_1 Y_1 \cdots Y_m W_m^\dagger \psi_m|0\rangle_{\mathcal{L}_{\text{eff}}}$$

  - Equivalent to SCET Lagrangian at leading power
  - Avoids having to fix a gauge
  - Avoids having to assign scaling behavior to unphysical fields
  - Operator definition of zero-bin subtraction

$$\widehat{Z}_i \equiv \frac{1}{N_c} \text{tr} \langle 0| W_i^\dagger Y_i |0\rangle$$

Future directions



- **Proofs** of factorization dramatically simpler
  - Can forward scattering be understood the same way?
    - Add Glauber modes to reduced diagrams?
      - Possible with our off-shell reduced diagrams
    - Cleaner understanding of BFKL
      - Leading power derivation, to all orders?
  - More exclusive observables?
    - Universality of PDFs?

- Practical **applications**
  - Jet physics at subleading power?
    - Resummation of subleading power corrections has never been done
  - Universal formulas for coefficients of soft divergences (anomalous dimensions)?
  - Simpler subtraction schemes for NNLO or NNNLO calculations?
    - We have a factorized expression which agrees in all soft or collinear limits