Recent Developments in the Calculation of Massive Quark Corrections to Deep Inelastic Scattering at Three Loops

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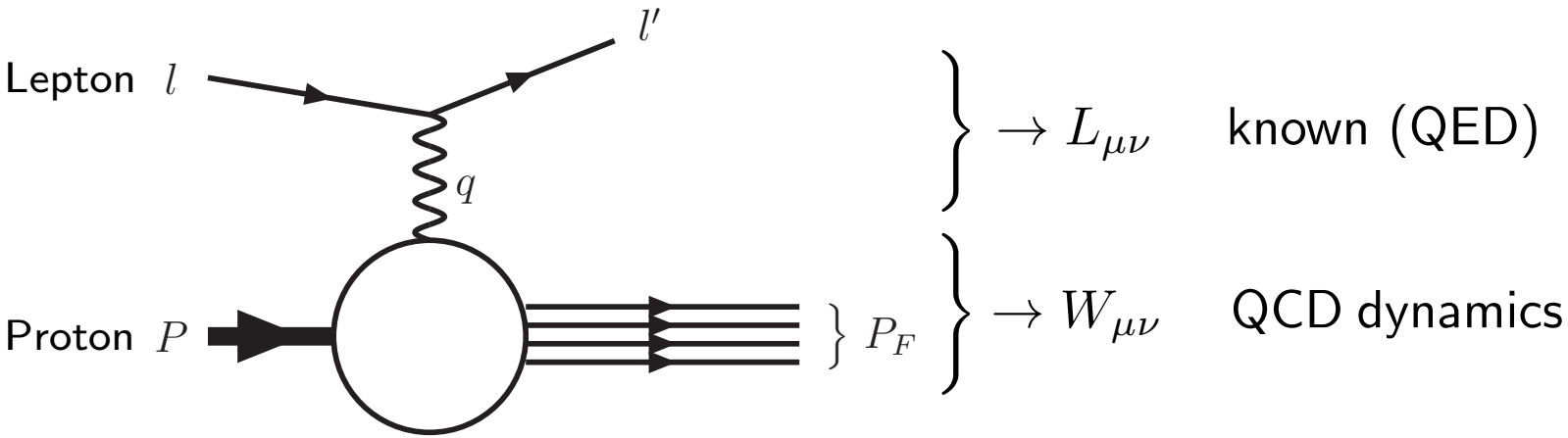
Outline

- Introduction
- Calculation Methods for Massive Graphs with Operator Insertions
- The Results for the Operator Matrix Elements
- Conclusions

Deep-Inelastic Scattering (DIS)

DIS [inclusive, unpolarized, electromagnetic] gives a clean probe of the proton substructure.



kinematic variables: $Q^2 = -q^2$, $x = \dfrac{Q^2}{2P.q}$, $y = \dfrac{P.q}{P.l}$

parametrization of the hadronic tensor with structure functions

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi}\int d^4\xi \exp(iq\xi)\langle P,s \mid [J^{em}_\mu(\xi), J^{em}_\nu(0)] \mid P,s\rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2)$$

$\rightarrow$ contributions of massive and massless quarks

The Heavy Flavor Wilson Coefficients

Use Mellin-space descriptions: $\hat{f}(N) := \int_0^1 dx\, x^{N-1} f(x)$.

At leading twist, the structure functions factorize

$$F_{(2,L)}(N, Q^2) = \sum_j \mathbb{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) f_j(N, \mu^2)$$

into perturbative Wilson coefficients and nonpert. parton densities (PDFs).

Divide the Wilson coefficients into massless and massive parts:

$$\mathbb{C}_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = C_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) + H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)$$

For $Q^2 \gg m^2$ ($Q^2 \gtrsim 10m^2$ für F_2) the massive Wilson coefficients factorize

$$H_{j,(2,L)}\left(N, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right) = \sum_i C_{i,(2,L)}\left(N, \frac{Q^2}{\mu^2}\right) A_{ij}\left(\frac{m^2}{\mu^2}, N\right),$$

[Buza, Matiounine, Smith, van Neerven 1996 NPB]

Factorization for $Q^2 \gg m^2$ at 3 Loops

$$L_{q,2}^{(3),\mathrm{NS}}(n_f+1) = A_{qq,Q}^{(3),\mathrm{NS}} + A_{qq,Q}^{(2),\mathrm{NS}} C_{q,2}^{(1),\mathrm{NS}}(n_f+1) + \hat{C}_{q,2}^{(3),\mathrm{NS}}(n_f) \ ,$$

$$L_{q,2}^{(3),\mathrm{PS}}(n_f+1) = A_{qq,Q}^{(3),\mathrm{PS}} + n_f A_{gq,Q}^{(2)} \tilde{C}_{g,2}^{(1)}(n_f+1) + n_f \hat{\tilde{C}}_{q,2}^{(3),\mathrm{PS}}(n_f) \ ,$$

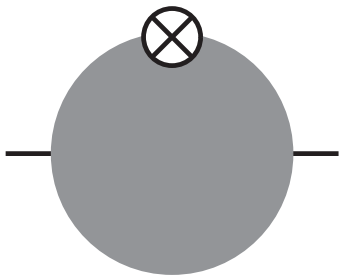
$$\begin{aligned} L_{g,2}^{(3)}(n_f+1) = {} & A_{qg,Q}^{(3)} + n_f A_{gg,Q}^{(1)} \tilde{C}_{g,2}^{(2)}(n_f+1) + n_f A_{gg,Q}^{(2)} \tilde{C}_{g,2}^{(1)}(n_f+1) \\ & + n_f A_{Qg}^{(1)} \tilde{C}_{q,2}^{(2),\mathrm{PS}}(n_f+1) + n_f \hat{\tilde{C}}_{g,2}^{(3)}(n_f) \ , \end{aligned}$$

$$H_{q,2}^{(3),\mathrm{PS}}(n_f+1) = A_{Qq}^{(3),\mathrm{PS}} + \tilde{C}_{q,2}^{(3),\mathrm{PS}}(n_f+1) + A_{gq,Q}^{(2)} \tilde{C}_{g,2}^{(1)}(n_f+1) + A_{Qq}^{(2),\mathrm{PS}} C_{q,2}^{(1),\mathrm{NS}}(n_f+1) \ ,$$

$$\begin{aligned} H_{g,2}^{(3)}(n_f+1) = {} & A_{Qg}^{(3)} + A_{Qg}^{(2)} C_{q,2}^{(1),\mathrm{NS}}(n_f+1) + A_{gg,Q}^{(2)} \tilde{C}_{g,2}^{(1)}(n_f+1) + A_{gg,Q}^{(1)} \tilde{C}_{g,2}^{(2)}(n_f+1) \\ & + A_{Qg}^{(1)} \Big\{ C_{q,2}^{(2),\mathrm{NS}}(n_f+1) + \tilde{C}_{q,2}^{(2),\mathrm{PS}}(n_f+1) \Big\} + \tilde{C}_{g,2}^{(3)}(n_f+1) \ . \end{aligned}$$

OMEs are local operators O_i sandwiched between partonic states $j = q, g$

$$A_{ij} = \langle j \mid O_i \mid j \rangle \quad =$$



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Definition of a Variable Flavor Number Scheme

The matching conditions onto a zero mass description:
[Buza, Matiounine, Smith, van Neerven 1998 Eur.Phys.J.C]$\rightarrow$ NLO
[Bierenbaum, Blümlein, Klein 2009 Nucl.Phys.B] $\rightarrow$ NNLO

$$
\begin{aligned}
&f_k(N, n_f+1, \mu^2, m^2) + f_{\bar{k}}(N, n_f+1, \mu^2, m^2) \\
&\qquad = A^{\rm NS}_{qq,Q}\left(N, n_f, \frac{\mu^2}{m^2}\right)\left[f_k(N, n_f, \mu^2, m^2) + f_{\bar{k}}(N, n_f, \mu^2, m^2)\right] \\
&\qquad + \frac{1}{n_f} A^{\rm PS}_{qq,Q}\left(N, n_f, \frac{\mu^2}{m^2}\right)\Sigma(N, n_f, \mu^2, x) + \frac{1}{n_f} A_{qg,Q}\left(N, n_f, \frac{\mu^2}{m^2}\right) G(N, n_f, \mu^2, x) \\
&f_Q(N, n_f+1, \mu^2, m^2) + f_{\bar{Q}}(N, n_f+1, \mu^2, m^2) \\
&\qquad = A^{\rm PS}_{Qq}\left(N, n_f, \frac{\mu^2}{m^2}\right)\Sigma(N, n_f, \mu^2, m^2) + A_{Qg}\left(N, n_f, \frac{\mu 2}{m^2}\right) G(N, n_f, \mu^2, m^2) \\
&G(N, n_f+1, \mu^2, m^2) \\
&\qquad = A_{gq,Q}\left(N, n_f, \frac{\mu^2}{m^2}\right)\Sigma(N, n_f, \mu^2, m^2) + A_{gg,Q}\left(N, n_f, \frac{\mu^2}{m^2}\right) G(N, n_f, \mu^2, m^2)
\end{aligned}
$$

where: $\left(\Sigma(N, n_f, ...) = \sum_{k=1}^{n_f}(f_k + f_{\bar{k}}), \quad n_f = 3\right)$

Importance of massive quark contributions

Massive quark contributions:

- amount to 20–30% for small x
- contribute scaling violations which differ in shape from those of massless quarks
- are sensitive to the gluon and sea quark PDFs for small x
- allow for the determination of the strange PDF via charged current DIS
- necessary for the precision determination of α_s at 3 loops
- note: asymptotic representation holds at the 1%-level for F_2

Status of massive quark contributions

Leading Order: [Witten 1976, Babcock, Sivers 1978, Shifman, Vainshtein, Zakharov 1978, Leveille, Weiler 1979, Glück, Reya 1979, Glück, Hoffmann, Reya 1982]

NLO:

[Laenen, van Neerven, Riemersma, Smith 1993]

$Q^2 \gg m^2$: via IBP [Buza, Matiounine, Smith, Migneron, van Neerven 1996]

via ${}_pF_q$'s, more compact [Bierenbaum, Blümlein, Klein, 2007]

$O(\alpha_s^2\varepsilon)$-contributions (for all-N) [Bierenbaum, Blümlein, Klein, Schneider 2008]

[Bierenbaum, Blümlein, Klein 2009]

NNLO: $Q^2 \gg m^2$

moments of F_2: $N = 2...10(14)$ [Bierenbaum, Blümlein, Klein 2009]

all n_f-contributions (all-N): [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2011]

[Blümlein, AH, Klein, Schneider 2012]

all $\log$-contributions: [Behring, Bierenbaum, Blümlein, De Freitas, Klein, Wißbrock 2014]

Known at 3 Loop:

- $A_{qq,Q}^{\rm PS}, A_{qg,Q}, A_{qq,Q}^{\rm NS,(TR)}$, $A_{gq,Q}$, $A_{Qq}^{\rm PS}$: complete
  [Ablinger, Blümlein, De Freitas, AH, von Manteuffel, Round, Schneider, Wißbrock 2014]...
- A_{Qg}, $A_{gg,Q}$: all $O(n_f T_F^2 C_{A/F})$-contributions known
- $A_{gg,Q}$: $O(T_F^2 C_{A/F})$-contributions ($m_1 = m_2$) known
- $A_{gg,Q}$: $O(T_F^2 C_{A/F})$-contributions ($m_1 \neq m_2$) $\longrightarrow$ scalar protoypes

Calculation Methods for Massive Graphs with Operator Insertions

Methods

Graphs are generated with `QGRAF` [Nogueira 1993]
`FORM` is used for the γ-matrix algebra together with the `color` package for the color algebra.

A first calculation paradigm:

- introduce Feynman parameterizations
- rewrite integrals in terms of hypergeometric functions at 1 ($\rightarrow {}_pF_q$, Appell functions, Kampe de Feriet functions,...)
- represent them in terms of a convergent series ($\rightarrow$ multiple sum expressions)
- solve the sums with `Sigma` [C. Schneider] in indefinite sums and products (see also `EvaluateMultiSums`, `SumProduction` [C. Schneider])
- for special function relations and limits: `HarmonicSums` [J. Ablinger]

$\rightarrow$ successful in all 2-loop contributions to the massive OMEs and large classes at 3-loop order [cf. page 8]

Integration by Parts & Differential Equations

Employ integration by parts relations:

- work on generating functions: $\hat{f}(x) := \sum_{N}^{\infty} x^N f(N)$
- IBP reduction to master integrals (`REDUZE 2` [von Manteuffel, Studerus 2012])
- some masters already solved, or calculable by the former paradigm
- derive system of differential equations from remaining masters
- back to N-space $\rightarrow$ system of difference equations
- uncouple (`OreSys` [S. Gerhold 2002])
- solve (`Sigma` [C. Schneider])
- determine constants, i.e. calculate initial values
  $\rightarrow$ using e.g. `MATAD` [M. Steinhauser 2000] or other methods (Mellin-Barnes, hyperg. series, symb. summation,...)
- combine and simplify (`HarmonicSums`, `EvaluateMultiSums`, `Sigma`)

Generating functions

Feynman rules in space of generating functions

$$(\Delta \cdot k)^N \to \sum_{N=0}^{\infty} x^N (\Delta \cdot k)^N = \frac{1}{1 - x\Delta \cdot k}$$

treated as an additional propagator in Laporta's algorithm

We want to obtain each diagram $D(N)$ as a function of N. So after calculation of the master integrals $M_i(N)$:

1. Construct gen. func. $M_i(x) = \sum_{N=0}^{\infty} x^N M_i(N)$.
2. Insert the gen. func. in $D(x) = \sum_i c_i(x) M_i(x)$.
3. Obtain $D(N)$ by extracting the Nth term in the Taylor expansion of $D(x)$.

Steps 1 to 3 are done using the Mathematica packages `HarmonicSums`, `SumProduction`, `EvaluateMultiSums`, `Sigma` by [J. Ablinger and C. Schneider].

Statistics and Results

After performing γ-matrix algebra, the diagrams are expressed as a linear combination of scalar integrals, wich are then reduced to masters.

	$A_{qq,Q}^{(3),\mathrm{NS}}$	$A_{gq,Q}^{(3)}$	$A_{Qg}^{(3),\mathrm{PS}}$
# diagrams	110	86	123
# scalar ints.	7426	12529	5470
# masters	35	41	66

ramaining: $A_{gg,Q}^{(3)} \to$ 642 diags. $A_{Qg}^{(3)} \to$ 1233 diags.

- all $\log$-contributions known,
- all $n_f T_F^2$-contributions known
- $A_{gg,Q}^{(3)}$: $n_f^0 T_F^2$-contributions known

3-loop contribution to $A_{Qq}^{\rm PS}$

$$
\begin{aligned}
a_{Qq}^{(3),\rm PS}(N) \quad = \quad & C_F^2 T_F \Bigg\{ \frac{64(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} S_{2,2}(2,\tfrac{1}{2}) - \frac{64(N^2+N+2)^2}{(N-1)N^2(N+1)^2(N+2)} S_{3,1}(2,\tfrac{1}{2}) \\
& + 2^N \left[- \frac{32 P_3 S_{2,1}(1,\tfrac{1}{2},N)}{(N-1)^2 N^3 (N+1)^2 (N+2)} - \frac{32 P_3 S_{1,1,1}(\tfrac{1}{2},1,1,N)}{(N-1)^2 N^3 (N+1)^2 (N+2)} + \frac{32 P_4 S_{1,1}(1,\tfrac{1}{2},N)}{(N-1)^3 N^4 (N+1)^2 (N+2)} + \cdots \right] \\
& + 2^{-N} \left[- \frac{64(N^2+N+2)^2 S_{1,2,1}(2,\tfrac{1}{2},1,N)}{(N-1)N^2(N+1)^2(N+2)} + \frac{64(N^2+N+2)^2 S_{1,2,1}(2,1,\tfrac{1}{2},N)}{(N-1)N^2(N+1)^2(N+2)} + \cdots \right] + \cdots \Bigg\} \\
& + C_F T_F^2 N_F \Bigg\{ - \frac{16(N^2+N+2)^2 S_1(N)^3}{27(N-1)N^2(N+1)^2(N+2)} + \frac{16 P_9 S_1(N)^2}{27(N-1)N^3(N+1)^3(N+2)^2} \\
& + \left[- \frac{208(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)} S_2 - \frac{32 P_{23}}{81(N-1)N^4(N+1)^4(N+2)^3} \right] S_1 \\
& + \frac{32 P_{31}}{243(N-1)N^5(N+1)^5(N+2)^4} + \frac{224(N^2+N+2)^2}{9(N-1)N^2(N+1)^2(N+2)} \zeta_3 + \cdots \Bigg\} \\
& + C_F C_A T_F \Bigg\{ \frac{2(N^2+N+2)^2 S_1(N)^4}{9(N-1)N^2(N+1)^2(N+2)} + \frac{4(N^2+N+2) P_6 S_1(N)^3}{27(N-1)^2 N^3 (N+1)^3 (N+2)^2} \\
& + 2^{-N} \left[\frac{16 P_2 S_3(2,N)}{(N-1)N^3(N+1)^2} - \frac{16 P_2 S_{1,2}(2,1,N)}{(N-1)N^3(N+1)^2} + \frac{16 P_2 S_{2,1}(2,1,N)}{(N-1)N^3(N+1)^2} - \frac{16 P_2 S_{1,1,1}(2,1,1,N)}{(N-1)N^3(N+1)^2} \right] \\
& - \frac{32(N^2+N+2)^2 S_{1,1,2}(2,\tfrac{1}{2},1,N)}{(N-1)N^2(N+1)^2(N+2)} + \frac{32(N^2+N+2)^2 S_{1,1,2}(2,1,\tfrac{1}{2},N)}{(N-1)N^2(N+1)^2(N+2)} \\
& + \frac{32(N^2+N+2)^2 S_{1,2,1}(2,\tfrac{1}{2},1,N)}{(N-1)N^2(N+1)^2(N+2)} - \frac{32(N^2+N+2)^2 S_{1,2,1}(2,1,\tfrac{1}{2},N)}{(N-1)N^2(N+1)^2(N+2)} \\
& - \frac{32(N^2+N+2)^2 S_{1,1,1,1}(2,\tfrac{1}{2},1,1,N)}{(N-1)N^2(N+1)^2(N+2)} - \frac{32(N^2+N+2)^2 S_{1,1,1,1}(2,1,\tfrac{1}{2},1,N)}{(N-1)N^2(N+1)^2(N+2)} + \cdots \Bigg\} + \cdots
\end{aligned}
$$

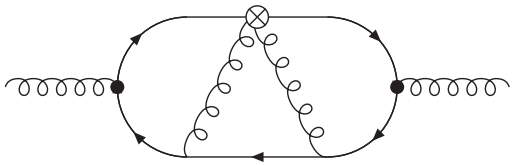
Example: $O(\alpha_s^3 T_F^2)$-Contributions to $A_{gg,Q}$

$$
\begin{aligned}
&A^{(3)}_{gg,Q,\mathrm{T_F^2}}(N) = \\
&T_F^2 \Bigg\{ \Bigg\{ C_F \Bigg[\dots \Bigg] + C_A \Bigg[\dots \Bigg] \Bigg\} \ln^3\left(\frac{m^2}{\mu^2}\right) + \Bigg\{ C_F \Bigg[\dots \Bigg] + C_A \Bigg[\dots \Bigg] \Bigg\} \ln^2\left(\frac{m^2}{\mu^2}\right) + \Bigg\{ C_F \Bigg[\dots \Bigg] + C_A \Bigg[\dots \Bigg] \Bigg\} \ln\left(\frac{m^2}{\mu^2}\right) \\
&- C_F \frac{1}{4^N} \binom{2N}{N} \frac{16 P_5}{3(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} \left[\sum_{j=1}^{N} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3 \right] \\
&- C_A \frac{1}{4^N} \binom{2N}{N} \frac{4 P_{22}}{45(N-1)N(N+1)^2(N+2)(2N-3)(2N-1)} \left[\sum_{j=1}^{N} \frac{4^k S_1(j-1)}{\binom{2j}{j} j^2} - 7\zeta_3 \right] \\
&+ \frac{1}{243} C_F \Bigg[144 F S_1^3 + \Bigg[-1296 F S_2 - \frac{96 P_{10}}{(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \Bigg] S_1 \\
&- \frac{189 P_{16}}{(N-1)N^2(N+1)^2(N+2)} \zeta_3 + \frac{8 P_{26}}{(N-1)N^5(N+1)^5(N+2)(2N-3)(2N-1)} \\
&+ \frac{144 P_4}{(N-1)N^3(N+1)^3(N+2)} (S_1^2 - 3S_2) - 3168 F S_3 + 5184 F S_{2,1} - 10368 \zeta_2 \Bigg] \\
&+ C_A \frac{1}{7290} \Bigg[216 \frac{P_{15}}{(N-1)N^2(N+1)^2(N+2)} S_1^2 + \frac{2 P_{25}}{(N-1)N^4(N+1)^4(N+2)(2N-3)(2N-1)} \\
&- 189 \frac{P_{14}}{(N-1)N(N+1)(N+2)} \zeta_3 + 7290 \left[\frac{8 P_{24}}{3645(N-1)N^3(N+1)^3(N+2)(2N-3)(2N-1)} - \frac{896}{27} \zeta_3 \right] S_1 \\
&+ 216 \frac{P_{17}}{(N-1)N^2(N+1)^2(N+2)} S_2 - \frac{7776(4N^3 + 4N^2 - 7N + 1)}{(N-1)N(N+1)} [S_3 - S_{2,1}] \Bigg] \Bigg\},
\end{aligned}
$$

New nested sums $\leftrightarrow$ iterated integrals

inverse binomial sums $\leftrightarrow$ iterated integrals with square-roots

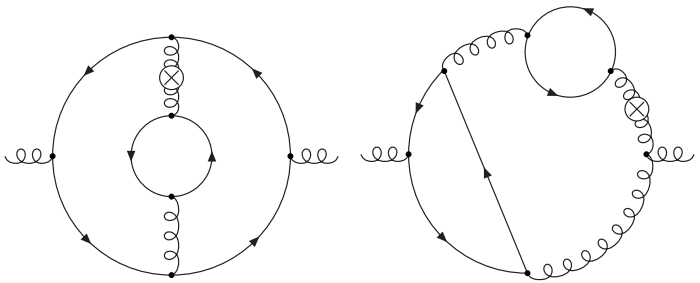
$$\sum_{j=1}^{N} \frac{4^j S_1(j-1)}{\binom{2j}{j} j^2} = \int_0^1 dx \, \frac{x^N - 1}{1-x} \int_x^1 dy \, \frac{1}{y\sqrt{1-y}} \times \Big[\ln(1-y) - \ln(y) + 2\ln(2)\Big]$$

- don't reduce further (cf. independence proof in x-space [C. Raab])
- all alg. relations exploited in the result of $A_{gg,Q}$ (proven by `SIGMA` [C. Schneider])
- occur in more complicated topologies, e.g. 

$\rightarrow$ properties under investigation (Mellin transformation, series expansions, ...)

Graphs with two mass scales

starting from 2-loop there are graphs with two heavy flavors
$\rightarrow$ breaks variable flavor number scheme due to $\eta = \frac{m_c^2}{m_b^2} \approx \frac{1}{10}$
moments $N = 2, 4, 6$ calculated for expansion in η up to $O(\eta^3 \log^3(\eta))$

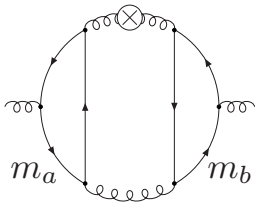
calculate contributions  $\cdots$

as functions of η & N

calculations of scalar prototype graphs show the emergence of generalizations of inverse binomial sums, e.g.

$$\sum_{i=1}^{N} \frac{4^i (1-\eta)^{-i} [S_2(1-\eta, i) - S_{1,1}(1-\eta, 1, i)]}{\binom{2i}{i}}$$

The Example in Mellin Space



$$
\begin{aligned}
D_7(N) = \left(m_2^2\right)^{-3+3/2\varepsilon} \left[\frac{1+(-1)^N}{2}\right] \Bigg\{ & \\
& - \frac{\eta+1}{24\varepsilon\eta^2(N+1)} + \Bigg[P_8 \frac{1}{5760\eta^3 N(N+1)^2(N+2)} + \frac{1}{45} \frac{2^{-2N-9}\binom{2N}{N} P_2}{(\eta-1)\eta^3(N+1)^2(N+2)} \sum_{i_1=1}^{N} \frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}}{\binom{2i_1}{i_1}} \\
& - \frac{1}{45} \frac{2^{-2N-8}\binom{2N}{N}}{\eta^2(N+1)^2(N+2)} P_4 + \frac{\left(\frac{\eta}{\eta-1}\right)^N}{11520(\eta-1)\eta^2 N(N+1)^2(N+2)} P_5 + \frac{(1-\eta)^{-N} P_6}{11520(\eta-1)\eta^3 N(N+1)^2(N+2)} \\
& + \frac{S_1\left(\frac{1}{1-\eta}, N\right)}{360(N+1)} + \frac{S_1\left(\frac{\eta}{\eta-1}, N\right)}{360\eta^3(N+1)} - \frac{1}{45} P_{11} \frac{2^{-2N-9}\binom{2N}{N}}{(\eta-1)\eta(N+1)^2(N+2)} \sum_{i_1=1}^{N} \frac{2^{2i_1}(1-\eta)^{-i_1}}{\binom{2i_1}{i_1}} \Bigg] \log^2(\eta) \\
& + \Bigg[- \frac{P_7}{5760\eta^3(N+1)^2(N+2)} + \frac{1}{45}(\eta+1) \frac{2^{-2N-7}\binom{2N}{N}}{\eta^3(N+1)^2(N+2)} P_2 \\
& - \frac{1}{45} \frac{2^{-2N-8}\binom{2N}{N} P_2}{(\eta-1)\eta^3(N+1)^2(N+2)} \sum_{i_1=1}^{N} \frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1} S_1\left(\frac{-1+\eta}{\eta}, i_1\right)}{\binom{2i_1}{i_1}} \\
& + \frac{1}{90} \frac{\left(\eta^3-1\right) S_1(N)}{\eta^3 N(N+1)^2(N+2)} - \frac{\left(\eta^3-1\right) S_2(N)}{180\eta^3(N+1)} + \frac{(1-\eta)^{-N} P_6 S_1(1-\eta, N)}{5760(\eta-1)\eta^3 N(N+1)^2(N+2)} \\
& - \frac{\left(\frac{\eta}{\eta-1}\right)^N P_5 S_1\left(\frac{\eta-1}{\eta}, N\right)}{5760(\eta-1)\eta^2 N(N+1)^2(N+2)} + \frac{S_{1,1}\left(\frac{1}{1-\eta}, 1-\eta, N\right)}{180(N+1)} - \frac{S_{1,1}\left(\frac{\eta}{\eta-1}, \frac{\eta-1}{\eta}, N\right)}{180\eta^3(N+1)}
\end{aligned}
$$

$$
-\frac{1}{45}\frac{2^{-2N-8}\binom{2N}{N}P_{10}}{(\eta-1)\eta(N+1)^2(N+2)}\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}S_1\left(1-\eta,i_1\right)}{\binom{2i_1}{i_1}}\Bigg]\log(\eta)+\Bigg[\frac{\left(27\eta^2+10\eta+27\right)}{5760\eta^{5/2}(N+1)}
$$

$$
-\frac{2^{-2N-8}\binom{2N}{N}P_1}{45\eta^{5/2}(N+1)^2(N+2)}\Bigg]L_1(\eta)-\frac{1}{45}(\eta-1)\frac{2^{-2N-6}\binom{2N}{N}}{\eta^3(N+1)^2(N+2)}P_2+\frac{2^{-2N}\binom{2N}{N}P_2}{11520(\eta-1)\eta^3(N+1)^2(N+2)}
$$

$$
\times\sum_{i_1=1}^{N}\frac{2^{2i_1}\left(\frac{\eta}{-1+\eta}\right)^{i_1}\left[S_{1,1}\left(\frac{-1+\eta}{\eta},1,i_1\right)-S_2\left(\frac{-1+\eta}{\eta},i_1\right)\right]}{\binom{2i_1}{i_1}}-\frac{P_9}{2880\eta^3N(N+1)^2(N+2)}
$$

$$
+\frac{(\eta+1)P_3S_1\left(N\right)}{5760\eta^3(N+1)^2(N+2)}+\frac{\left(\eta^3+1\right)}{180\eta^3N(N+1)^2(N+2)}\left[S_2\left(N\right)-S_1\left(N\right)^2\right]+\frac{\left(\eta^3+1\right)S_3(N)}{180\eta^3(N+1)}
$$

$$
+\frac{(1-\eta)^{-N}P_6}{5760(\eta-1)\eta^3N(N+1)^2(N+2)}\left[S_{1,1}\left(1-\eta,1,N\right)-S_2\left(1-\eta,N\right)\right]
$$

$$
+\frac{\left(\frac{\eta}{\eta-1}\right)^{N}P_5}{5760(\eta-1)\eta^2N(N+1)^2(N+2)}\left[S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)-S_2\left(\frac{\eta-1}{\eta},N\right)\right]
$$

$$
+\frac{1}{180(N+1)}\Bigg[S_1\left(\frac{1}{1-\eta},N\right)S_{1,1}\left(1-\eta,1,N\right)-S_{1,2}\left(\frac{1}{1-\eta},1-\eta,N\right)
$$

$$
+S_{1,2}\left(1-\eta,\frac{1}{1-\eta},N\right)-S_{1,1,1}\left(1-\eta,1,\frac{1}{1-\eta},N\right)-S_{1,1,1}\left(1-\eta,\frac{1}{1-\eta},1,N\right)\Bigg]
$$

$$
+\frac{1}{180\eta^3(N+1)}\Bigg[S_1\left(\frac{\eta}{\eta-1},N\right)S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)+S_{1,2}\left(\frac{\eta-1}{\eta},\frac{\eta}{\eta-1},N\right)-S_{1,2}\left(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},N\right)
$$

$$
-S_{1,1,1}\left(\frac{\eta-1}{\eta},1,\frac{\eta}{\eta-1},N\right)-S_{1,1,1}\left(\frac{\eta-1}{\eta},\frac{\eta}{\eta-1},1,N\right)\Bigg]+\frac{2^{-2N}\binom{2N}{N}P_{10}}{11520(\eta-1)\eta(N+1)^2(N+2)}
$$

$$
\times\sum_{i_1=1}^{N}\frac{2^{2i_1}(1-\eta)^{-i_1}\left[S_2\left(1-\eta,i_1\right)-S_{1,1}\left(1-\eta,1,i_1\right)\right]}{\binom{2i_1}{i_1}}\Bigg\}
$$

Conclusions

- For the precise determination of α_s and the PDFs from DIS data, the heavy flavor Wilson coefficients are needed at the 3 loop order.
- The method of IBP (`REDUZE`) was adapted for diagrams with operator insertions.
- The master integrals are solvable with differential/difference equations (combining `HarmonicSums`, `Sigma`, `EvaluateMultiSums`, ...), employing MB-representations, hypergeometric series, etc. for initial values.
- 6 out of 8 massive OMEs have been completed recently ($A_{gg,Q}$ and A_{Qg} remain). [Ablinger, Behring, Blümlein, De Freitas, AH, von Manteuffel, Round, Schneider, Wißbrock 2014]
- For $A_{gg,Q}$,A_{Qg} all $n_F T_F^2$-contributions and all $\log$s are known.
- For $A_{gg,Q}$ we also computed the complete T_F^2-contributions [Ablinger, Blümlein, De Freitas, AH, von Manteuffel, Round, Schneider 2014], as well as all scalar prototypes graphs with two massive flavors.
- New basis functions occur: inverse binomial sums ($\leftrightarrow$ iterated integrals with square-roots)
  $\rightarrow$ algebraic relations, series expansions, Mellin trafos, are being worked out [Ablinger, Blümlein, Raab, Schneider 2014]